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SPREADING-VANISHING DICHOTOMY IN THE DIFFUSIVE LOGISTIC MODEL WITH A FREE BOUNDARY*

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Abstract. In this paper we investigate a diffusive logistic model with a free boundary in one space dimension. We aim to use the dynamics of such a problem to describe the spreading of a new or invasive species, with the free boundary representing the expanding front. We prove a spreading-vanishing dichotomy for this model, namely the species either successfully spreads to all the new environment and stabilizes at a positive equilibrium state, or it fails to establish and dies out in the long run. Sharp criteria for spreading and vanishing are given. Moreover, we show that when spreading occurs, for large time, the expanding front moves at a constant speed. This spreading speed is uniquely determined by an elliptic problem induced from the original model.

Key words. diffusive logistic equation, free boundary, spreading-vanishing dichotomy, invasive population

AMS subject classifications. 35K20, 35R35, 35J60, 92B05

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1. Introduction.

The spreading of new or invasive species is a central topic in ecology, and considerable research has been devoted to the better understanding of the nature of such spreading. Some of these efforts are discussed in [26] and [19].

Skellam [27] seems to be the first to observe that the spreading of muskrat in Europe in the early 1900s followed a linear fashion: He calculated the area of the muskrat range from a map obtained from field data, took the square root (which gives the spreading radius) and plotted it against years, and found that the data points lay on a straight line. This phenomenon was also observed in other field data for various animal species, namely the spreading radius eventually exhibits a linear growth curve against time. Several mathematical models have been proposed to describe this phenomenon and many of them can be found in [26]. Later on in this introduction we will briefly explain one of these models and compare it with our research here.

Generally speaking, the mathematical modeling of ecological problems is difficult. First, for most such problems there is a lack of "first principle". As a result, most mathematical models in ecology are established based on heuristic analysis. Second, there are usually very limited empirical data against which to verify such models, since in most cases any useful field data need to cover vast areas and long time spans, and hence are extremely difficult and expensive to obtain. Nevertheless, the modeling of biological invasion has attracted extensive efforts, and remarkable success has been achieved in understanding the spreading of species through the investigation of front propagation; see, for example, [11, 15, 27, 1, 2, 26, 28, 29, 17] and the references therein.

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In this paper, we propose a different approach to the understanding of the spreading of species. This approach is based on the following diffusive logistic problem:

$$(1.1)\qquad \begin{cases} u_t - du_{xx} = u(a-bu), & t>0,\ 0<x<h(t),\\ u_x(t,0)=0,\ u(t,h(t))=0, & t>0,\\ h'(t) = -\mu u_x(t,h(t)), & t>0,\\ h(0)=h_0, u(0,x)=u_0(x), & 0\le x\le h_0, \end{cases}$$

where $x = h(t)$ is the moving boundary to be determined, h_0, μ, d, a, and b are given positive constants, and the initial function $u_0(x)$ satisfies

$$(1.2)\qquad u_0 \in C^2([0,h_0]),\quad u_0'(0) = u_0(h_0) = 0,\quad \text{and } u_0 > 0 \text{ in } [0,h_0).$$

We attempt to use (1.1) to model the spreading of a new or invasive species with population density $u(t,x)$ over a one dimensional habitat. The free boundary $x = h(t)$ represents the spreading front, while the homogeneous Neumann boundary condition at $x = 0$ indicates that the left boundary is fixed, with the population confined to its right. The coefficient a represents the intrinsic growth rate of the species, b measures its intraspecific competition, and d is the dispersal rate.

The equation governing the free boundary, $h'(t) = -\mu u_x(t,h(t))$, is a special case of the well-known Stefan condition, which has been used in the modeling of a number of applied problems. For example, it was used to describe the melting of ice in contact with water [25], in the modeling of oxygen in the muscle [8], and in wound healing [7], to mention but a few. There is a vast literature on the Stefan problem, and some important recent theoretical advances can be found in [5]. In [10, 12], the authors studied a problem very similar to (1.1), but their reaction term has the form u^p $(p > 1)$, and so the dynamical behavior of their problem is completely different from (1.1).

This paper may be the first attempt to use the Stefan condition in the study of the spreading of populations. Here the initial function $u_0(x)$ stands for the population of a new or invasive species in the very early stage of its introduction, which occupies an initial region $[0,h_0]$. We assume that the species can only invade further into the environment from the right end of the initial region, and the spreading front expands at a speed that is proportional to the population gradient at the front, which gives rise to the Stefan condition $h'(t) = -\mu u_x(t,h(t))$. We will show that (1.1) has a unique solution $(u(t,x),h(t))$ defined for all $t > 0$, with $u(t,x) > 0$ and $h'(t) > 0$. Moreover, a spreading-vanishing dichotomy holds for (1.1), namely, as time $t \to \infty$, the population $u(t,x)$ either successfully establishes itself in the new environment (henceforth called spreading), in the sense that $h(t) \to \infty$ and $u(t,x) \to a/b$, or the population fails to establish and vanishes eventually (called vanishing), namely $h(t) \to h_\infty \le \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and $u(t,x) \to 0$. Furthermore, when spreading occurs, for large time, the spreading speed approaches a positive constant k_0, i.e., $h(t) = [k_0 + o(1)]t$ as $t \to \infty$. The asymptotic spreading speed k_0 is uniquely determined by an auxiliary elliptic problem induced from (1.1) (see Proposition 4.1), and is independent of the initial population size u_0. The criteria for spreading or vanishing are the following: If the initial occupying area $[0,h_0]$ is beyond a critical size, namely $h_0 \ge \frac{\pi}{2}\sqrt{\frac{d}{a}}$, then regardless of the initial population size $u_0(x)$ (satisfying (1.2)), spreading always happens. On the other hand, if $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$, then whether spreading or vanishing occurs is determined by the initial population size u_0 and the coefficient μ in the Stefan condition (assuming the other parameters are fixed). We will show that for such h_0, with each given u_0, there exists a critical $\mu^* > 0$ depending on u_0, such that spreading occurs if $\mu > \mu^*$ and vanishing happens when $\mu \le \mu^*$.

The above spreading-vanishing dichotomy indicates that the number $\frac{\pi}{2}\sqrt{\frac{d}{a}}$ serves as a barrier for the spreading process: Either the spreading front $x = h(t)$ breaks through this barrier at some finite time $t \ge 0$, and the population subsequently spreads to the entire available space $[0, \infty)$ and establishes, or the front $x = h(t)$ never breaks through this barrier and the population dies out at the end.

If the left boundary in (1.1) is replaced by a free boundary $x = g(t)$ governed by $g'(t) = -\mu u_x(t, g(t))$, we will show that a similar spreading-vanishing dichotomy holds, and in the case of spreading, both the left front $x = g(t)$ and the right front $x = h(t)$ expand to infinity at the same asymptotic speed k_0 (determined as before). This double fronts case can be handled by simple modifications of the techniques developed for treating (1.1). The details are given in section 5.

A great deal of previous mathematical investigation on the spreading of population has been based on the diffusive logistic equation over the entire space $\mathbb{R}^N$:

$$u_t - d\Delta u = u(a - bu), \quad t > 0, \quad x \in \mathbb{R}^N. \tag{1.3}$$

In the pioneering works of Fisher [11] and Kolmogorov, Petrovsky, and Piskunov [15], for space dimension $N = 1$, traveling wave solutions have been found for (1.3): For any $|c| \ge c^* := 2\sqrt{ad}$, there exists a solution $u(t, x) := W(x - ct)$ with the property that

$$W'(y) < 0 \text{ for } y \in \mathbb{R}^1, \quad W(-\infty) = a/b, \quad W(+\infty) = 0;$$

no such solution exists if $|c| < c^*$. The number c^* is called the minimal speed of the traveling waves. c^* is also known (see [1, 27, 26, 19]) as the spreading speed of a new population $u(t, x)$ (governed by the above logistic equation) with initial distribution $u(0, x)$ confined to a compact set of x (i.e., $u(0, x) = 0$ outside a compact set), since it can be shown that for such $u(t, x)$ (see section 4 in [1]),

$$\lim_{t\to\infty,\ |x|\le(c^*-\epsilon)t} u(t, x) = a/b, \qquad \lim_{t\to\infty,\ |x|\ge(c^*+\epsilon)t} u(t, x) = 0$$

for any small $\epsilon > 0$. These results have been extended to higher dimensions in [2], and extensive further development on traveling wave solutions and the spreading speed has been achieved in several directions; we refer to [30, 13, 28, 3, 4, 17, 29] and the references therein for more details.

A striking difference between (1.1) and (1.3) is that the spreading front in (1.1) is given explicitly by a function $x = h(t)$, beyond which the population density is 0, while in (1.3), the population $u(t, x)$ becomes positive for all x once t is positive. Second, (1.3) guarantees successful spreading of the species for any nontrivial initial population $u(0, x)$ (namely $u(t, x) \to a/b$ as $t \to \infty$), regardless of its initial size and supporting area, but the dynamics of (1.1) exhibits a spreading-vanishing dichotomy. The phenomenon exhibited by this dichotomy seems closer to the reality, and is supported by numerous empirical evidences; for example, the introduction of several bird species from Europe to North America in the 1900s was successful only after many initial attempts. Third, while (1.3) gives an asymptotic spreading speed of $2\sqrt{ad}$ (for large time), which is independent of b and is increasing with the dispersal rate d, we will show that the asymptotic spreading speed k_0 of (1.1) depends on all the parameters in (1.1), and in sharp contrast, it is not increasing with respect to d (at least for large d): k_0 approaches 0 if either d increases to ∞ or d decreases to 0; thus the maximal speed is reached at some finite optimal dispersal rate d. Further discussions of this and several other points in biological terms can be found in the last section of this paper, where more ecological evidences are provided to support the biological predictions drawn from the mathematical results here.

In a forthcoming paper, we will investigate (1.1) in higher space dimensions with a heterogeneous environment and compare our results with those corresponding to analogous extensions of (1.3).

Similar free boundary conditions to the one in (1.1) have been used in ecological models over *bounded* spatial domains in several earlier papers. In [21, 22, 23], Mimura, Yamada, and Yotsutani studied the existence, uniqueness, and asymptotic behavior of the solution to the problem

$$\begin{cases} u_{1t} - k_1 u_{1xx} = u_1 f_1(u_1), & t > 0,\ 0 < x < h(t), \\ u_{2t} - k_2 u_{2xx} = u_2 f_2(u_2), & t > 0,\ h(t) < x < l, \\ u_1 = u_2 = 0, \quad h'(t) = -\alpha_1 \frac{\partial u_1}{\partial x} - \alpha_2 \frac{\partial u_2}{\partial x}, & x = h(t), \\ u_1(t,0) = M_1, \quad u_2(t,l) = M_2, & t > 0, \\ u(0,x) = u_0(x) \geq 0, & 0 \leq x \leq l, \\ h(0) = h_0 \quad (0 < h_0 < l). \end{cases}$$

The multidimensional case of this system was studied in [14]. Recently Lin [18] studied a predator-prey ecological model over a bounded one dimensional domain, with the predator population satisfying a free boundary condition as in (1.1). He showed that the predator species disperses to all the domain in finite time.

In section 2, we first use a contraction mapping argument to prove the local existence and uniqueness of the solution to (1.1). This largely follows some existing techniques in [7]. We then make use of suitable estimates on the solution to show that it exists for all time $t \in (0,\infty)$.

Section 3 is devoted to the proof of the spreading-vanishing dichotomy. Our arguments are based on the comparison principle and the construction of suitable upper and lower solutions of (1.1).

In section 4, we estimate the spreading speed. A key tool in our approach here is an auxiliary elliptic equation (see (4.1)) which determines the spreading speed. Such an equation arises naturally from an intuitive analysis and it turns out that the solution of this equation can be suitably modified to construct sharp upper and lower solutions to (1.1), which provide rather precise estimates for the spreading speed. We also examine the dependence of the spreading speed k_0 on the parameters in (1.1). We will show that k_0 increases in μ and a, decreases in b, but it does not depend on d in a monotone fashion. If all the other parameters are fixed with k_0 viewed as a function of d, namely $k_0 = k_0(d)$, we will show that

$$\lim_{d\to\infty} k_0(d)\sqrt{d} = \sigma_0 \in (0,\infty), \quad \lim_{d\to 0} k_0(d)/\sqrt{d} = \infty, \quad \lim_{d\to 0} k_0(d)/(\sqrt{d}|\ln d|) = 0.$$

In section 5, we explain how the techniques for (1.1) can be modified to study the following double fronts free boundary problem:

$$(1.4) \quad \begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ g(t) < x < h(t), \\ u(t,g(t)) = 0,\ g'(t) = -\mu u_x(t,g(t)), & t > 0, \\ u(t,h(t)) = 0,\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ g(0) = -h_0,\ h(0) = h_0,\ u(0,x) = u_0(x), & -h_0 \leq x \leq h_0, \end{cases}$$

where both $x = g(t)$ and $x = h(t)$ are to be determined, $h_0 > 0$, and u_0 satisfies

$$\begin{cases} u_0 \in C^2([-h_0, h_0]), \\ u_0(-h_0) = u_0(h_0) = 0, \quad \text{and } u_0 > 0 \text{ in } (-h_0, h_0). \end{cases} \tag{1.5}$$

It turns out that all the results for (1.1) can be extended to (1.4).

In section 6, we compare our results in biological terms with some documented ecological observations and those revealed by (1.3).

Finally, we want to mention that our results can be easily extended to cover a more general reaction term $f(u)$ which behaves like $au - bu^2$. We leave this to the interested reader.

2. Existence and uniqueness. In this section, we first prove the following local existence and uniqueness result by the contraction mapping theorem. We then use suitable estimates to show that the solution is defined for all $t > 0$.

THEOREM 2.1. *For any given u_0 satisfying* (1.2) *and any $\alpha \in (0, 1)$, there is a $T > 0$ such that problem* (1.1) *admits a unique solution*

$$(u, h) \in C^{(1+\alpha)/2, 1+\alpha}(D_T) \times C^{1+\alpha/2}([0, T]);$$

moreover,

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(D_T)} + \|h\|_{C^{1+\alpha/2}([0,T])} \le C, \tag{2.1}$$

where $D_T = \{(t, x) \in \mathbb{R}^2 : x \in [0, h(t)], t \in [0, T]\}$, C and T only depend on h_0, α, and $\|u_0\|_{C^2([0,h_0])}$.

Proof. As in [7], we first straighten the free boundary. Let $\zeta(y)$ be a function in $C^3[0, \infty)$ satisfying

$$\zeta(y) = 1 \text{ if } |y - h_0| < \frac{h_0}{4}, \quad \zeta(y) = 0 \text{ if } |y - h_0| > \frac{h_0}{2}, \quad |\zeta'(y)| < \frac{6}{h_0} \ \forall y.$$

Consider the transformation

$$(t, y) \to (t, x), \text{ where } x = y + \zeta(y)(h(t) - h_0), \quad 0 \le y < \infty.$$

As long as

$$|h(t) - h_0| \le \frac{h_0}{8},$$

the above transformation is a diffeomorphism from $[0, +\infty)$ onto $[0, +\infty)$. Moreover, it changes the free boundary $x = h(t)$ to the line $y = h_0$. Now, direct calculations show that

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{1}{1 + \zeta'(y)(h(t) - h_0)} \equiv \sqrt{A(h(t), y)}, \\ \frac{\partial^2 y}{\partial x^2} &= -\frac{\zeta''(y)(h(t) - h_0)}{[1 + \zeta'(y)(h(t) - h_0)]^3} \equiv B(h(t), y), \\ -\frac{1}{h'(t)} \frac{\partial y}{\partial t} &= \frac{\zeta(y)}{1 + \zeta'(y)(h(t) - h_0)} \equiv C(h(t), y). \end{aligned}$$

If we set

$$u(t, x) = u(t, y + \zeta(y)(h(t) - h_0)) = w(t, y),$$

then

$$u_t = w_t - h'(t)C(h(t), y)w_y, \quad u_x = \sqrt{A(h(t), y)}w_y,$$

$$u_{xx} = A(h(t), y)w_{yy} + B(h(t), y)w_y$$

and the free boundary problem (1.1) becomes

$$\text{(2.2)} \qquad \begin{cases} w_t - Adw_{yy} - (Bd + h'C)w_y = w(a - bw), & t > 0,\ 0 < y < h_0, \\ w = 0, \quad h'(t) = -\mu\frac{\partial w}{\partial y}, & t > 0,\ y = h_0, \\ \frac{\partial w}{\partial y}(t, 0) = 0, & t > 0, \\ h(0) = h_0, \quad w(0, y) = u_0(y), & 0 \le y \le h_0, \end{cases}$$

where $A = A(h(t), y)$, $B = B(h(t), y)$, and $C = C(h(t), y)$.

Denote $h_1 = -\mu u_0'(h_0)$, and for $0 < T \le \frac{h_0}{8(1+h_1)}$, define $\Delta_T = [0, T] \times [0, h_0]$,

$$\mathcal{D}_{1T} = \{w \in C(\Delta_T) : w(0, y) = u_0(y),\ ||w - u_0||_{C(\Delta_T)} \le 1\},$$
$$\mathcal{D}_{2T} = \{h \in C^1([0, T]) :\ h(0) = h_0,\ h'(0) = h_1,\ \|h' - h_1\|_{C([0,T])} \le 1\}.$$

It is easily seen that $\mathcal{D} := \mathcal{D}_{1T} \times \mathcal{D}_{2T}$ is a complete metric space with the metric

$$d((w_1, h_1), (w_2, h_2)) = ||w_1 - w_2||_{C(\Delta_T)} + ||h_1' - h_2'||_{C([0,T])}.$$

Let us note that for $h_1, h_2 \in \mathcal{D}_{2T}$, due to $h_1(0) = h_2(0) = h_0$,

$$\text{(2.3)} \qquad \|h_1 - h_2\|_{C([0,T])} \le T||h_1' - h_2'||_{C([0,T])}.$$

Next, we shall prove the existence and uniqueness result by using the contraction mapping theorem. First, we observe that due to our choice of T, for any given $(w, h) \in \mathcal{D}_{1T} \times \mathcal{D}_{2T}$, we have

$$|h(t) - h_0| \le T(1 + h_1) \le \frac{h_0}{8}.$$

Therefore the transformation $(t, y) \to (t, x)$ introduced at the beginning of the proof is well defined. Applying standard L^p theory and then the Sobolev imbedding theorem [16], we find that for any $(w, h) \in \mathcal{D}$ we have the following initial boundary value problem:

$$\text{(2.4)} \qquad \begin{cases} \overline{w}_t - Ad\overline{w}_{yy} - (Bd + h'C)\overline{w}_y = w(a - bw), & t > 0, 0 < y < h_0, \\ \frac{\partial \overline{w}}{\partial y}(t, 0) = 0, \quad \overline{w}(t, h_0) = 0, & t > 0, \\ \overline{w}(0, y) = u_0(y), & 0 \le y \le h_0, \end{cases}$$

admits a unique solution $\overline{w} \in C^{(1+\alpha)/2,1+\alpha}(\Delta_T)$, and

$$\text{(2.5)} \qquad ||\overline{w}||_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} \le C_1,$$

where C_1 is a constant dependent on h_0, α, and $||u_0||_{C^2[0,h_0]}$.

Defining

$$\text{(2.6)} \qquad \overline{h}(t) = h_0 - \int_0^t \mu\overline{w}_y(\tau, h_0)d\tau,$$

we have

$$\overline{h}'(t) = -\mu \overline{w}_y(t, h_0), \quad \overline{h}(0) = h_0, \quad \overline{h}'(0) = -\mu \overline{w}_y(0, h_0) = h_1,$$

and hence $\overline{h}' \in C^{\alpha/2}([0, T])$ with

$$||\overline{h}'||_{C^{\alpha/2}([0,T])} \le C_2 := \mu C_1. \tag{2.7}$$

We now define $\mathcal{F} : \mathcal{D} \to C(\Delta_T) \times C^1([0, T])$ by

$$\mathcal{F}(w, h) = (\overline{w}, \overline{h}).$$

Clearly $(w, h) \in \mathcal{D}$ is a fixed point of $\mathcal{F}$ if and only if it solves (2.2).

By (2.5) and (2.7), we have

$$||\overline{h}' - h_1||_{C([0,T])} \le ||\overline{h}'||_{C^{\alpha/2}([0,T])} T^{\alpha/2} \le \mu C_1 T^{\alpha/2},$$

$$||\overline{w} - u_0||_{C(\Delta_T)} \le ||\overline{w} - u_0||_{C^{(1+\alpha)/2,0}(\Delta_T)} T^{(1+\alpha)/2} \le C_1 T^{(1+\alpha)/2}.$$

Therefore if we take $T \le \min\{(\mu C_1)^{-2/\alpha}, C_1^{-2/(1+\alpha)}\}$, then $\mathcal{F}$ maps $\mathcal{D}$ into itself.

Next we prove that $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$ for $T > 0$ sufficiently small. Indeed, let $(w_i, h_i) \in \mathcal{D}$ $(i = 1, 2)$ and denote $(\overline{w}_i, \overline{h}_i) = \mathcal{F}(w_i, h_i)$. Then it follows from (2.5) and (2.7) that

$$||\overline{w}_i||_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} \le C_1, \quad ||\overline{h}_i'(t)||_{C^{\alpha/2}([0,T])} \le C_2.$$

Setting $U = \overline{w}_1 - \overline{w}_2$, we find that $U(y, t)$ satisfies

$$\begin{aligned}
&U_t - A(h_2, y) dU_{yy} - (B(h_2, y)d + h_2' C(h_2, y))U_y \\
&\quad = [A(h_1, y) - A(h_2, y)] d\overline{w}_{1,yy} + [B(h_1, y) - B(h_2, y)] d\overline{w}_{1,y} \\
&\qquad + [h_1' C(h_1, y) - h_2' C(h_2, y)] w_{1,y} + (w_1 - w_2)(a - bw_1 - bw_2), \ t > 0, \ 0 < y < h_0, \\
&\frac{\partial U}{\partial y}(t, 0) = 0, \quad U(t, h_0) = 0, \quad t > 0, \\
&U(0, y) = 0, \quad 0 \le y \le h_0.
\end{aligned}$$

Using the L^p estimates for parabolic equations and Sobolev's imbedding theorem, we obtain

$$||\overline{w}_1 - \overline{w}_2||_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} \le C_3(||w_1 - w_2||_{C(\Delta_T)} + ||h_1 - h_2||_{C^1([0,T])}), \tag{2.8}$$

where C_3 depends on C_1, C_2 and the functions A, B, and C in the definition of the transformation $(t, y) \to (t, x)$. Taking the difference of the equations for $\overline{h}_1$, $\overline{h}_2$ results in

$$||\overline{h}_1' - \overline{h}_2'||_{C^{\alpha/2}([0,T])} \le \mu\Big(||\overline{w}_{1,y} - \overline{w}_{2,y}||_{C^{\alpha/2,0}(\Delta_T)}\Big). \tag{2.9}$$

Combining (2.3), (2.8), and (2.9), and assuming $T \le 1$, we obtain

$$\begin{aligned}
&||\overline{w}_1 - \overline{w}_2||_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} + ||\overline{h}_1' - \overline{h}_2'||_{C^{\alpha/2}([0,T])} \\
&\quad \le C_4(||w_1 - w_2||_{C(\Delta_T)} + ||h_1' - h_2'||_{C[0,T]}),
\end{aligned}$$

with C_4 depending on C_3 and μ. Hence for

$$T := \min\left\{1, \left(\frac{1}{2C_4}\right)^{2/\alpha},\ (\mu C_1)^{-2/\alpha},\ C_1^{-2/(1+\alpha)},\ \frac{h_0}{8(1+h_1)}\right\},$$

we have

$$\begin{aligned}
&||\overline{w}_1 - \overline{w}_2||_{C(\Delta_T)} + ||\overline{h}_1' - \overline{h}_2'||_{C([0,T])} \\
&\quad \le T^{(1+\alpha)/2}||\overline{w}_1 - \overline{w}_2||_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} + T^{\alpha/2}||\overline{h}_1' - \overline{h}_2'||_{C^{\alpha/2}([0,T])} \\
&\quad \le C_4 T^{\alpha/2}(||w_1 - w_2||_{C(\Delta_T)} + ||h_1' - h_2'||_{C([0,T])}) \\
&\quad \le \frac{1}{2}(||w_1 - w_2||_{C(\Delta_T)} + ||h_1' - h_2'||_{C([0,T])}).
\end{aligned}$$

This shows that for this T, $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$. It now follows from the contraction mapping theorem that $\mathcal{F}$ has a unique fixed point (w, h) in $\mathcal{D}$. Moreover, by the Schauder estimates, we have additional regularity for (w, h) as a solution of (2.2), namely, $h \in C^{1+\alpha/2}(0, T]$ and $w \in C^{1+\alpha/2,2+\alpha}((0, T] \times [0, h_0])$, and (2.5), (2.7) hold. In other words, $(w(t, y), h(t))$ is a unique local classical solution of the problem (2.2). ∎

To show that the local solution obtained in Theorem 2.1 can be extended to all $t > 0$, we need the following estimate.

LEMMA 2.2. *Let (u, h) be a solution to problem* (1.1) *defined for $t \in (0, T_0)$ for some $T_0 \in (0, +\infty]$. Then there exist constants C_1 and C_2 independent of T_0 such that*

$$0 < u(t, x) \le C_1, \quad 0 < h'(t) \le C_2 \quad \textit{for } 0 \le x < h(t), \quad t \in (0, T_0).$$

Proof. Using the strong maximum principle to the equation of u we immediately obtain

$$u(t, x) > 0, \quad u_x(t, h(t)) < 0 \quad \text{for } 0 < t < T_0, \quad 0 \le x < h(t).$$

Hence $h'(t) > 0$ for $t \in (0, T_0)$.

It follows from the comparison principle that $u(t, x) \le \overline{u}(t)$ for $t \in (0, T_0)$ and $x \in [0, h(t)]$, where

$$\overline{u}(t) := \frac{a}{b} e^{\frac{a}{b}t}\left(e^{\frac{a}{b}t} - 1 + \frac{a}{b||u_0||_\infty}\right)^{-1},$$

which is the solution of the problem

$$\frac{d\overline{u}}{dt} = \overline{u}(a - b\overline{u}), \quad t > 0;\ \overline{u}(0) = ||u_0||_\infty. \tag{2.10}$$

Thus we have

$$u(t, x) \le C_1 := \sup_{t \ge 0} \overline{u}(t).$$

It remains to show that $h'(t) \le C_2$ for all $t \in (0, T_0)$ with some C_2 independent of T_0. To this end, we define

$$\Omega = \Omega_M := \{(t, x) : 0 < t < T_0,\ h(t) - M^{-1} < x < h(t)\}$$

and construct an auxiliary function

$$w(t,x) := C_1[2M(h(t)-x) - M^2(h(t)-x)^2].$$

We will choose M so that $w(t,x) \geq u(t,x)$ holds over Ω.

Direct calculations show that, for $(t,x) \in \Omega$,

$$w_t = 2C_1Mh'(t)(1 - M(h(t)-x)) \geq 0,$$

$$-w_{xx} = 2C_1M^2, \quad u(a-bu) \leq aC_1.$$

It follows that

$$w_t - dw_{xx} \geq 2dC_1M^2 \geq au \text{ in } \Omega$$

if $M^2 \geq \frac{a}{2d}$. On the other hand,

$$w(t,h(t)-M^{-1}) = C_1 \geq u(t,h(t)-M^{-1}), \quad w(t,h(t)) = 0 = u(t,h(t)).$$

Thus, if we can choose M such that $u_0(x) \leq w(0,x)$ for $x \in [h_0 - M^{-1}, h_0]$, then we can apply the maximum principle to $w-u$ over Ω to deduce that $u(t,x) \leq w(t,x)$ for $(t,x) \in \Omega$. It would then follow that

$$u_x(t,h(t)) \geq w_x(t,h(t)) = -2MC_1, \quad h'(t) = -\mu u_x(t,h(t)) \leq C_2 := 2MC_1\mu.$$

To complete the proof, we need only find some M independent of T_0 such that $u_0(x) \leq w(0,x)$ for $x \in [h_0 - M^{-1}, h_0]$. We calculate

$$w_x(0,x) = -2C_1M[1 - M(h_0 - x)] \leq -C_1M \text{ for } x \in [h_0 - (2M)^{-1}, h_0].$$

Therefore upon choosing

$$M := \max\left\{\sqrt{\frac{a}{2d}}, \frac{4\|u_0\|_{C^1([0,h_0])}}{3C_1}\right\},$$

we will have

$$w_x(0,x) \leq u_0'(x) \text{ for } x \in [h_0 - (2M)^{-1}, h_0].$$

Since $w(0,h_0) = u_0(h_0) = 0$, the above inequality implies

$$w(0,x) \geq u_0(x) \text{ for } x \in [h_0 - (2M)^{-1}, h_0].$$

Moreover, for $x \in [h_0 - M^{-1}, h_0 - (2M)^{-1}]$, we have

$$w(0,x) \geq \frac{3}{4}C_1, \quad u_0(x) \leq \|u_0\|_{C^1([0,h_0])}M^{-1} \leq \frac{3}{4}C_1.$$

Therefore $u_0(x) \leq w(0,x)$ for $x \in [h_0 - M^{-1}, h_0]$. This completes the proof. ∎

THEOREM 2.3. *The solution of problem* (1.1) *exists and is unique for all* $t \in (0,\infty)$.

Proof. Let $[0, T_{max})$ be the maximal time interval in which the solution exists. By Theorem 2.1, $T_{max} > 0$. It remains to show that $T_{max} = \infty$. Arguing indirectly, we assume that $T_{max} < \infty$. By Lemma 2.2, there exist C_1 and C_2 independent of T_{max} such that for $t \in [0, T_{\max})$ and $x \in [0, h(t)]$,

$$0 \le u(t,x) \le C_1, \quad h_0 \le h(t) \le h_0 + C_2 t, \quad 0 \le h'(t) \le C_2.$$

We now fix $\delta_0 \in (0, T_{max})$ and $M > T_{max}$. By standard L^p estimates, the Sobolev embedding theorem, and the Hölder estimates for parabolic equations, we can find $C_3 > 0$ depending only on δ_0, M, C_1, and C_2 such that $||u(t,\cdot)||_{C^2([0,h(t)])} \le C_3$ for $t \in [\delta_0, T_{\max})$. It then follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending only on C_3, C_2, and C_1 such that the solution of problem (1.1) with initial time $T_{max} - \tau/2$ can be extended uniquely to the time $T_{max} - \tau/2 + \tau$. But this contradicts the assumption. The proof is now complete. ∎

Remark 2.4. It follows from the uniqueness of the solution to (1.1) and some standard compactness argument that the unique solution (u, h) depends continuously on the parameters appearing in (1.1). This fact will be used in the sections below.

3. The spreading-vanishing dichotomy. This section is devoted to both the proof of the spreading-vanishing dichotomy described in the introduction, and the proof of the criteria governing spreading and vanishing.

It follows from Lemma 2.2 that $x = h(t)$ is monotonic increasing and, therefore, there exists $h_\infty \in (0, +\infty]$ such that $\lim_{t\to+\infty} h(t) = h_\infty$. The spreading-vanishing dichotomy is a consequence of the following two lemmas.

LEMMA 3.1. *If* $h_\infty < \infty$*, then* $h_\infty \le \frac{\pi}{2}\sqrt{\frac{d}{a}}$*, and* $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([0,h(t)])} = 0$.

Proof. We first prove that $h_\infty \le \frac{\pi}{2}\sqrt{\frac{d}{a}}$. Otherwise $h_\infty > \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and there exists $T > 0$ such that $l := h(T) > \frac{\pi}{2}\sqrt{\frac{d}{a}}$. This implies that $a > \lambda_1$, where λ_1 denotes the first eigenvalue of the problem

$$-d\phi'' = \lambda\phi \ \text{ in } \ (-l, l), \quad \phi(\pm l) = 0.$$

It follows that for all small $\varepsilon > 0$, the first eigenvalue λ_1^ε of the problem

$$-d\phi'' - \varepsilon\phi' = \lambda\phi \ \text{ in } \ (-l, l), \quad \phi(\pm l) = 0$$

satisfies $\lambda_1^\varepsilon < a$. Fix such an $\varepsilon > 0$ and consider the problem

$$L_\varepsilon v = av - bv^2 \ \text{ in } \ (-l, l), \quad v(\pm l) = 0, \tag{3.1}$$

where $L_\varepsilon v = -dv'' - \varepsilon v'$. This is a logistic problem with $a > \lambda_1^\varepsilon$. It is well known (see, for example, Proposition 3.3 in [6]) that (3.1) admits a unique positive solution $v = v_\varepsilon$. By the moving plane method, one easily sees that $v(x)$ is symmetric about $x = 0$ with $v'(x) < 0$ for $x \in (0, l]$. Moreover, by the comparison principle, $v < \frac{a}{b}$ in $[-l, l]$. We now define

$$w(t,x) = v\left(\frac{l}{h(t)}x\right),$$

and calculate

$$\begin{aligned} w_t - dw_{xx} &= -\frac{lx}{h^2(t)}h'(t)v' - d\frac{l^2}{h^2(t)}v'' \\ &= \frac{l^2}{h^2(t)}\left[-dv'' - \frac{xh'(t)}{l}v'\right]. \end{aligned}$$

Since $h'(t) \to 0$ as $t \to +\infty$, we can find $T_0 > T$ such that $h'(t) < \varepsilon \frac{l}{h_\infty}$ for $t \geq T_0$, and hence for $t \geq T_0$ and $x \in [0, h(t)]$, we have $\frac{xh'(t)}{l} \leq \varepsilon$. It follows that for such t and x,

$$\begin{aligned} w_t - dw_{xx} &\leq \frac{l^2}{h^2(t)}(-dv'' - \varepsilon v') \\ &= \frac{l^2}{h^2(t)}(av - bv^2). \end{aligned}$$

Since $0 \leq v < \frac{a}{b}$, we have $av - bv^2 \geq 0$, and hence from $\frac{l}{h(t)} \leq 1$, we deduce

$$w_t - dw_{xx} \leq av - bv^2 = aw - bw^2 \quad \text{for} \quad t \geq T_0, \quad x \in [0, h(t)].$$

We now choose $\delta \in (0,1)$ small so that $\delta w(T_0, x) \leq u(T_0, x)$. Then $\underline{u}(t,x) := \delta w(t,x)$ satisfies

$$\begin{cases} \underline{u}_t - d\underline{u}_{xx} \leq a\underline{u} - b\underline{u}^2, & t \geq T_0,\ x \in [0, h(t)], \\ \underline{u}_x(t,0) = 0, \quad \underline{u}(t, h(t)) = 0, & t \geq T_0, \\ \underline{u}(T_0, x) \leq u(T_0, x), & 0 \leq x \leq h_0. \end{cases}$$

Hence we can apply the comparison principle to conclude that

$$\underline{u}(t,x) \leq u(t,x) \quad \text{for } t \geq T_0, \quad x \in [0, h(t)].$$

It follows that

$$u_x(t, h(t)) \leq \underline{u}_x(t, h(t)) = \delta \frac{l}{h(t)} v'(l) \to \delta \frac{l}{h_\infty} v'(l) < 0.$$

On the other hand, we have

$$u_x(t, h(t)) = -\frac{1}{\mu} h'(t) \to 0 \ \text{ as } t \to \infty.$$

This contradiction proves that $h_\infty \leq \frac{\pi}{2}\sqrt{\frac{d}{a}}$.

We are now ready to show that $||u(t,\cdot)||_{C([0,h(t)])} \to 0$ as $t \to \infty$. Let $\overline{u}(t,x)$ denote the unique solution of the problem

$$(3.2) \qquad \begin{cases} \overline{u}_t - d\overline{u}_{xx} = a\overline{u} - b\overline{u}^2, & t > 0,\ 0 < x < h_\infty, \\ \overline{u}_x(t,0) = 0, \quad \overline{u}(t, h_\infty) = 0, & t > 0, \\ \overline{u}(0,x) = \tilde{u}_0(x), & 0 < x < h_\infty, \end{cases}$$

where

$$\tilde{u}_0(x) = \begin{cases} u_0(x), & 0 \leq x \leq h_0, \\ 0, & x \geq h_0. \end{cases}$$

The comparison principle gives $0 \leq u(t,x) \leq \overline{u}(t,x)$ for $t > 0$ and $x \in [0, h(t)]$. Since $h_\infty \leq \frac{\pi}{2}\sqrt{\frac{d}{a}}$, we have $a \leq d(\frac{\pi}{2h_\infty})^2$ and it follows from a well-known conclusion on the logistic problem (3.2) that $\overline{u}(t,x) \to 0$ uniformly for $x \in [0, h_\infty]$ as $t \to \infty$ (see, for example, Corollary 3.4 in [6]). Thus $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([0,h(t)])} = 0$. ∎

LEMMA 3.2. *If* $h_\infty = \infty$, *then* $\lim_{t\to+\infty}\ u(t,x) = \frac{a}{b}$ *uniformly in any bounded subset of* $[0, \infty)$.

Proof. First, we recall that the comparison principle gives $u(t,x) \le \overline{u}(t)$ for $t > 0$ and $x \in [0, h(t)]$, where

$$\overline{u}(t) = \frac{a}{b} e^{\frac{a}{b}t} \left(e^{\frac{a}{b}t} - 1 + \frac{a}{b||u_0||_\infty} \right)^{-1}$$

is the solution of (2.10). Since $\lim_{t\to\infty} \overline{u}(t) = \frac{a}{b}$, we deduce

$$\limsup_{t\to+\infty} u(t,x) \le \tfrac{a}{b} \text{ uniformly for } x \in [0,\infty).$$

On the other hand, for any $l > \max\{h_0, \frac{\pi}{2}\sqrt{\frac{d}{a}}\}$, there exists $t_l > 0$ such that $h(t_l) = l$. By the comparison principle we have $u(t,x) \ge \underline{u}_l(t,x)$ in $(t_l,\infty) \times (0,l)$, where $\underline{u}_l$ is the solution of the following problem with fix boundary:

$$\begin{cases} (\underline{u}_l)_t - d(\underline{u}_l)_{xx} = \underline{u}_l(a - b\underline{u}_l), & t > t_l,\ 0 < x < l, \\ (\underline{u}_l)_x(t,0) = \underline{u}_l(t,l) = 0, & t > t_l, \\ \underline{u}_l(t_l,x) = u(t_l,x), & 0 \le x \le l. \end{cases} \tag{3.3}$$

Since $a > d(\frac{\pi}{2l})^2$, it is well known that $\underline{u}_l(t,x) \to u_l^*(x)$ as $t \to \infty$ uniformly in compact subset of $[0,l)$, where u_l^* is the unique positive solution of

$$\begin{cases} -d(u_l^*)_{xx} = u_l^*(a - bu_l^*), & -l < x < l, \\ u_l^*(-l) = u_l^*(l) = 0. \end{cases}$$

It follows that $\liminf_{t\to+\infty} u(t,x) \ge u_l^*(x)$ uniformly in compact subsets of $[0,l)$. Using Lemma 2.2 of [9], we easily see that $u_l^*(x) \to \frac{a}{b}$ as $l \to +\infty$ uniformly in any compact subset of $[0,\infty)$. Therefore, $\liminf_{t\to+\infty} u(t,x) \ge \frac{a}{b}$ uniformly in any compact subset of $[0,\infty)$. In view of our earlier conclusion on $\limsup u(t,x)$, this completes the proof of the desired result. ∎

Combing Lemmas 3.1 and 3.2, we immediately obtain the following spreading-vanishing dichotomy.

THEOREM 3.3. *Let $(u(t,x), h(t))$ be the solution of the free boundary problem* (1.1)*. Then the following alternative holds:*

Either

(i) *spreading: $h_\infty = +\infty$ and $\lim_{t\to+\infty} u(t,x) = \frac{a}{b}$ uniformly for x in any bounded set of $[0,\infty)$;*

or

(ii) *vanishing: $h_\infty \le \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([0,h(t)])} = 0$.*

We next decide exactly when each of the two alternatives occurs. We need to divide our discussion into two cases:

$$\text{(a)}\ \ h_0 \ge \frac{\pi}{2}\sqrt{\frac{d}{a}}\,, \quad \text{(b)}\ \ h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}.$$

In case (a), due to $h'(t) > 0$ for $t > 0$, we must have $h_\infty > \frac{\pi}{2}\sqrt{\frac{d}{a}}$. Hence Lemma 3.1 implies the following result.

THEOREM 3.4. *If $h_0 \ge \frac{\pi}{2}\sqrt{\frac{d}{a}}$, then $h_\infty = +\infty$.*

In order to study case (b), and also for later applications, we now present a comparison principle which can be used to estimate both $u(t,x)$ and the free boundary $x = h(t)$.

LEMMA 3.5. *Suppose that* $T \in (0,\infty)$, $\overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T^*) \cap C^{1,2}(D_T^*)$ *with* $D_T^* = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, 0 < x < \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t - d\overline{u}_{xx} \ge \overline{u}(a - b\overline{u}), & t > 0,\ 0 < x < \overline{h}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu \overline{u}_x, & t > 0,\ x = \overline{h}(t), \\ \overline{u}_x(t,0) \le 0, & t > 0. \end{cases}$$

If

$$h_0 \le \overline{h}(0) \quad \text{and} \quad u_0(x) \le \overline{u}(0,x) \text{ in } [0,h_0],$$

then the solution (u,h) *of the free boundary problem* (1.1) *satisfies*

$$h(t) \le \overline{h}(t) \text{ in } (0,T],\ u(x,t) \le \overline{u}(x,t) \text{ for } t \in (0,T] \text{ and } x \in (0,h(t)).$$

Proof. For small $\epsilon > 0$, let (u_ϵ, h_ϵ) denote the unique solution of (1.1) with h_0 replaced by $h_0^\epsilon := h_0(1-\epsilon)$, with μ replaced by $\mu_\epsilon := \mu(1-\epsilon)$, and with u_0 replaced by some $u_0^\epsilon \in C^2([0,h_0^\epsilon])$ satisfying

$$0 < u_0^\epsilon(x) \le u_0(x) \text{ in } [0,h_0^\epsilon],\ u_0^\epsilon(h_0^\epsilon) = 0,$$

and as $\epsilon \to 0$,

$$u_0^\epsilon\left(\frac{h_0}{h_0^\epsilon}x\right) \to u_0(x)$$

in the $C^2([0,h_0])$ norm.

We claim that $h_\epsilon(t) < \overline{h}(t)$ for all $t \in (0,T]$. Clearly, this is true for small $t > 0$. If our claim does not hold, then we can find a first $t^* \le T$ such that $h_\epsilon(t) < \overline{h}(t)$ for $t \in (0,t^*)$ and $h_\epsilon(t^*) = \overline{h}(t^*)$. It follows that

$$h_\epsilon'(t^*) \ge \overline{h}'(t^*). \tag{3.4}$$

We now compare u_ϵ and $\overline{u}$ over the region

$$\Omega_{t^*} := \{(t,x) \in \mathbb{R}^2 : 0 < t \le t^*, 0 \le x < h_\epsilon(t)\}.$$

The strong maximum principle yields $u_\epsilon(t,x) < \overline{u}(t,x)$ in Ω_{t^*}. Hence $w(t,x) := \overline{u}(t,x) - u_\epsilon(t,x) > 0$ in Ω_{t^*} with $w(t^*, h_\epsilon(t^*)) = 0$. It follows that $w_x(t^*, h_\epsilon(t^*)) \le 0$, from which we deduce, in view of $(u_\epsilon)_x(t^*, h(t^*)) < 0$ and $\mu_\epsilon < \mu$, that $h_\epsilon'(t^*) < \overline{h}'(t^*)$. But this contradicts (3.4), which proves our claim that $h_\epsilon(t) < \overline{h}(t)$ for all $t \in (0,T]$. We may now apply the usual comparison principle over Ω_T to conclude that $u_\epsilon < \overline{u}$ in Ω_T.

Since the unique solution of (1.1) depends continuously on the parameters in (1.1), as $\epsilon \to 0$, (u_ϵ, h_ϵ) converges to (u,h), the unique solution of (1.1). The desired result then follows by letting $\epsilon \to 0$ in the inequalities $u_\epsilon < \overline{u}$ and $h_\epsilon < \overline{h}$. ☐

Remark 3.6. The pair $(\overline{u}, \overline{h})$ in Lemma 3.5 is usually called an upper solution of the problem (1.1). We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 3.5 for lower solutions.

We are now ready to consider case (b), where $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$. We first examine the case that μ is large, then we look at the case $\mu > 0$ is small, and finally we use

Lemma 3.5 and Remark 3.6 to prove the existence of a critical μ^* so that spreading occurs when $\mu > \mu^*$ and vanishing happens if $\mu \in (0, \mu^*]$.

LEMMA 3.7. *Suppose* $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$. *If*

$$\mu \geq \mu_0 := \max\left\{1, \frac{b}{a}\|u_0\|_\infty\right\} d\left(\frac{\pi}{2}\sqrt{\frac{d}{a}} - h_0\right)\left(\int_0^{h_0} u_0(x)dx\right)^{-1},$$

then $h_\infty = +\infty$.

Proof. We first consider the case $||u_0||_\infty \leq \frac{a}{b}$. In this case the solution $\overline{u}(t)$ of (2.10) satisfies $\overline{u}(t) \leq a/b$ for all $t > 0$. It follows that $u(t,x) < \overline{u}(t) \leq \frac{a}{b}$ for $t > 0$ and $x \in [0, h(t)]$.

Direct calculation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{h(t)} u(t,x)\mathrm{d}x &= \int_0^{h(t)} u_t(t,x)\mathrm{d}x + h'(t)u(t,h(t))\\ &= \int_0^{h(t)} du_{xx}\mathrm{d}x + \int_0^{h(t)}(au - bu^2)\mathrm{d}x\\ &= -\frac{d}{\mu}h'(t) + \int_0^{h(t)}(au - bu^2)\mathrm{d}x.\end{aligned}$$

Integrating from 0 to t yields

$$\begin{aligned}\int_0^{h(t)} u(t,x)\mathrm{d}x = &\int_0^{h_0} u_0(x)dx + \frac{d}{\mu}h_0 - \frac{d}{\mu}h(t)\\ &+ \int_0^t\int_0^{h(s)}(au - bu^2)dxds, \quad t \geq 0.\end{aligned} \tag{3.5}$$

Since $0 < u(t,x) < \frac{a}{b}$ for $t > 0$ and $x \in [0, h(t))$, we have

$$\int_0^t\int_0^{h(s)}(au - bu^2)dxds \geq \int_0^1\int_0^{h(s)}(au - bu^2)dxds > 0 \text{ for } t \geq 1.$$

If $h_\infty \neq \infty$, then $h_\infty \leq \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and $\lim_{t\to+\infty} ||u(t,\cdot)||_\infty = 0$. So letting $t \to +\infty$ in (3.5) gives

$$\int_0^{h_0} u_0(x)dx < \frac{d}{\mu}\frac{\pi}{2}\sqrt{\frac{d}{a}} - \frac{d}{\mu}h_0,$$

which is a contradiction to the assumption $\mu \geq \mu_0$.

For the case $||u_0||_\infty > \frac{a}{b}$, we take $\underline{u}_0 = \frac{a}{b||u_0||_\infty}u_0(x)$. The solution $(\underline{u}, \underline{h})$ of (1.1) with u_0 replaced by $\underline{u}_0$ is a lower solution to (1.1), and by Remark 3.6 we have $h(t) \geq \underline{h}(t)$ for $t > 0$. But from what was proved above for the first case, due to $\|\underline{u}_0\|_\infty = a/b$ and our assumption on μ, we have $\lim_{t\to\infty}\underline{h}(t) = \infty$. Thus we also have $h_\infty = \infty$. This completes the proof. ∎

LEMMA 3.8. *Suppose* $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$. *Then there exists* $\overline{\mu} > 0$ *depending on* u_0 *such that* $h_\infty < +\infty$ *if* $\mu \leq \overline{\mu}$.

Proof. We are going to construct a suitable upper solution to (1.1) and then apply Lemma 3.5. Inspired by [24], we define

$$\sigma(t) = h_0\left(1 + \delta - \frac{\delta}{2}e^{-\gamma t}\right),\ t \geq 0; \quad V(y) = \cos\left(\frac{\pi}{2}y\right),\ 0 \leq y \leq 1,$$

and

$$w(t,x) = Me^{-\alpha t}V(x/\sigma(t)), \quad t \geq 0, \quad 0 \leq x \leq \sigma(t),$$

where δ, γ, α, and M are positive constants to be chosen later.

Direct computations yield

$$\begin{aligned} & w_t - dw_{xx} - w(a - bw) \\ & = Me^{-\alpha t}[-\alpha V - x\sigma'\sigma^{-2}V' - d\sigma^{-2}V'' - V(a - bMe^{-\alpha t}V)] \\ & \geq Me^{-\alpha t}V\left[-\alpha + \left(\frac{\pi}{2}\right)^2 \frac{d}{(1+\delta)^2 h_0^2} - a + bMe^{-\alpha t}V\right] \end{aligned}$$

for all $t > 0$ and $0 < x < \sigma(t)$. On the other hand, we have $\sigma'(t) = \gamma h_0 \frac{\delta}{2} e^{-\gamma t}$ and $-w_x(t,\sigma(t)) = \frac{\pi}{2}\varepsilon\sigma^{-1}(t)e^{-\alpha t}$. Noting that $a < d(\frac{\pi}{2h_0})^2$, we can find $\delta > 0$ such that

$$\left(\frac{\pi}{2}\right)^2 \frac{d}{(1+\delta)^2 h_0^2} - a = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 \frac{d}{h_0^2} - a\right].$$

We now choose M sufficiently large such that $u_0(x) \leq M\cos(\frac{\pi}{2}\frac{x}{h_0(1+\delta/2)})$ for $x \in [0, h_0]$, and take

$$\overline{\mu} = \frac{\delta\gamma h_0^2}{4M}, \quad \alpha = \gamma = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 \frac{d}{h_0^2} - a\right].$$

Then for any $0 < \mu \leq \overline{\mu}$, we have

$$\begin{cases} w_t - dw_{xx} \geq w(a - bw), & t > 0,\ 0 < x < \sigma(t), \\ w = 0, \quad \sigma'(t) > -\mu\frac{\partial w}{\partial x}, & t > 0,\ x = \sigma(t),\ t > 0, \\ \frac{\partial w}{\partial x}(t,0) = 0, & t > 0, \\ \sigma(0) = (1 + \frac{\delta}{2})h_0 > h_0. & \end{cases}$$

Hence we can apply Lemma 3.5 to conclude that $h(t) \leq \sigma(t)$ and $u(t,x) \leq w(t,x)$ for $0 \leq x \leq h(t)$ and $t > 0$. It follows that $h_\infty \leq \lim_{t\to\infty}\sigma(t) = h_0(1+\delta) < \infty$. ◻

We are now ready to apply Lemma 3.5 to prove the existence of a threshold $\mu^* > 0$ that governs the alternatives in the spreading-vanishing dichotomy for the case $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$.

THEOREM 3.9. *If $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$, then there exists $\mu^* > 0$ depending on u_0 such that $h_\infty \leq \frac{\pi}{2}\sqrt{\frac{d}{a}}$ if $\mu \leq \mu^*$, and $h_\infty = +\infty$ if $\mu > \mu^*$.*

Proof. Define $\Sigma := \{\mu > 0 : h_\infty \leq \frac{\pi}{2}\sqrt{\frac{d}{a}}\}$. By Lemmas 3.8 and 3.1 we have $\Sigma \supset (0, \overline{\mu}]$. Using Lemma 3.7 we find on the other hand that $\Sigma \cap [\mu_0, \infty) = \emptyset$. Therefore, $\mu^* := \sup\Sigma \in [\overline{\mu}, \mu_0]$. By this definition and Lemma 3.1, we find that $h_\infty = +\infty$ when $\mu > \mu^*$.

We claim that $\mu^* \in \Sigma$. Otherwise $h_\infty = \infty$ for $\mu = \mu^*$. Hence we can find $T > 0$ such that $h(T) > \frac{\pi}{2}\sqrt{\frac{d}{a}}$. To stress the dependence of the solution (u,h) of (1.1) on μ, we now write (u_μ, h_μ) instead of (u,h). So we have $h_{\mu^*}(T) > \frac{\pi}{2}\sqrt{\frac{d}{a}}$. By the continuous dependence of (u_μ, h_μ) on μ, we can find $\epsilon > 0$ small so that $h_\mu(T) > \frac{\pi}{2}\sqrt{\frac{d}{a}}$ for all $\mu \in [\mu^* - \epsilon, \mu^* + \epsilon]$. It follows that for all such μ,

$$\lim_{t\to\infty} h_\mu(t) > h_\mu(T) > \frac{\pi}{2}\sqrt{\frac{d}{a}}.$$

This implies that $[\mu^* - \epsilon, \mu^* + \epsilon] \cap \Sigma = \emptyset$, and $\sup \Sigma \le \mu^* - \epsilon$, contradicting the definition of μ^*. This proves our claim that $\mu^* \in \Sigma$.

For $\mu \in (0, \mu^*)$, (u_{μ^*}, h_{μ^*}) is an upper solution of (1.1). Hence we can use Lemma 3.5 to deduce that $h_\mu(t) \le h_{\mu^*}(t)$ for $t > 0$. It follows that

$$\lim_{t\to\infty} h_\mu(t) \le \lim_{t\to\infty} h_{\mu^*}(t) \le \frac{\pi}{2}\sqrt{\frac{d}{a}}.$$

Hence $\mu \in \Sigma$. Thus we have proved that $\Sigma = (0, \mu^*]$. The proof is complete. ◻

4. Spreading speed. The main purpose of this section is to show that when spreading occurs, the expanding front $x = h(t)$ moves at a constant speed for large time, namely

$$h(t) = \big(k_0 + o(1)\big)t \text{ as } t \to \infty.$$

The constant k_0 will be called the asymptotic spreading speed, and it is determined in Proposition 4.1 below. The fact $\lim_{t\to\infty} h(t)/t = k_0$ will be proved by using modifications of the solution of the following elliptic problem (4.1). We will also discuss how k_0 changes as the parameters in (1.1) vary.

PROPOSITION 4.1. *For any $k \ge 0$, the problem*

$$\begin{cases} -dU'' + kU' = aU - bU^2, & x > 0, \\ U(0) = 0 \end{cases} \tag{4.1}$$

admits a unique positive solution $U = U_k$. Moreover, $U_k'(x) > 0$ for $x \ge 0$, $U_{k_1}'(0) > U_{k_2}'(0)$, $U_{k_1}(x) > U_{k_2}(x)$ for $x > 0$ and $k_1 < k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu) > 0$ such that $\mu U_{k_0}'(0) = k_0$.

Before giving the proof of Proposition 4.1, we explain intuitively how problem (4.1) arises from (1.1). So we assume that (u, h) is the unique solution of (1.1) and $h(t) \to \infty$ as $t \to \infty$. Letting $v(t, x) = u(t, h(t) - x)$, we find that

$$\begin{cases} v_t - dv_{xx} + h'(t)v_x = v(a - bv), & t > 0,\ 0 < x < h(t), \\ v(t, 0) = 0, \quad h'(t) = \mu v_x(t, 0), \quad v_x(t, h(t)) = 0, & t > 0, \\ h(0) = h_0,\ v(0, x) = u_0(h_0 - x), & 0 \le x \le h_0. \end{cases}$$

Since $\lim_{t\to\infty} h(t) = \infty$, if $h'(t)$ approaches a constant k_0 and $v(t, x)$ approaches a positive function $U(x)$ as $t \to \infty$, then $U(x)$ must be a positive solution of (4.1) with $\mu U'(0) = k$.

Proof of Proposition 4.1. It is well known that for all large $l > 0$, the problem

$$-dU'' + kU' = aU - bU^2, \quad 0 < x < l, \quad U(0) = U(l) = 0$$

has a unique positive solution U^l. Define

$$V(x) = \begin{cases} U^l(x), & 0 \le x \le l, \\ 0, & x \ge l. \end{cases}$$

Then V is a lower solution of (4.1). Clearly any constant $C \ge \frac{a}{b}$ is an upper solution. It now follows from the standard upper and lower solutions argument over an unbounded domain that (4.1) has at least one solution $U(x)$ satisfying

$$V(x) \le U(x) \le \frac{a}{b} \text{ in } [0, +\infty).$$

By the strong maximum principle and Serrin's sweeping argument, we find that any nontrivial nonnegative solution of (4.1) satisfies

$$0 < U(x) < \frac{a}{b} \text{ in } (0, +\infty).$$

Next, we claim that $U(x)$ is increasing in x and $\lim_{x\to+\infty} U(x) = \frac{a}{b}$. Indeed, we may rewrite (4.1) as

$$-(de^{-\frac{k}{d}x}U')' = e^{-\frac{k}{d}x}(aU - bU^2). \tag{4.2}$$

Since $0 < U(x) < \frac{a}{b}$ in $(0, +\infty)$, we have $aU - bU^2 > 0$, and hence

$$-(de^{-\frac{k}{d}x}U')' > 0 \text{ in } (0, +\infty).$$

Hence $e^{-\frac{k}{d}x}U'(x)$ is a decreasing function. Since $U(x)$ is bounded in $(0, +\infty)$, we can find a sequence $x_n \to +\infty$ such that $U'(x_n) \to 0$ as $n \to +\infty$. It follows that

$$e^{-\frac{k}{d}x}U'(x) > \lim_{n\to+\infty} e^{-\frac{k}{d}x_n}U'(x_n) = 0 \text{ in } (0, +\infty).$$

We thus have $U'(x) > 0$ and $U(x)$ is increasing. Moreover, $\sigma = \lim_{x\to+\infty} U(x)$ exists. Using (4.2) we easily find that $\sigma = \frac{a}{b}$.

We now prove the uniqueness. Suppose U_1 and U_2 are both positive solutions of (4.1). Then for any $\varepsilon > 0$, it is easily checked that $w_i = (1 + \varepsilon)U_i$ satisfies

$$-(de^{-\frac{k}{d}x}w_i')' \geq e^{-\frac{k}{d}x}(aw_i - bw_i^2), \quad i = 1, 2.$$

Since $\lim_{t\to+\infty} w_i(x) = (1 + \varepsilon)\frac{a}{b}$, we can find $l_0 > 0$ large such that

$$w_1(l) > U_2(l), \quad w_2(l) > U_1(l) \ \ \forall l \geq l_0.$$

We may now apply Lemma 2.1 of [9] to conclude that

$$(1 + \varepsilon)U_1(x) \geq U_2(x), \ (1 + \varepsilon)U_2(x) \geq U_1(x) \text{ for } 0 < x < l \ \ \forall l \geq l_0.$$

It follows that $(1 + \varepsilon)U_1(x) \geq U_2(x)$ and $(1 + \varepsilon)U_2(x) \geq U_1(x)$ for all $x \geq 0$. Letting $\varepsilon \to 0$, we deduce that $U_1 = U_2$. This proves the uniqueness conclusion.

Finally, if $0 \leq k_1 < k_2$, then due to $U_{k_i}'(x) > 0$, we have

$$-dU_{k_1}'' + k_2U_{k_1}' > -dU_{k_1}'' + k_1U_{k_1}' = aU_{k_1} - bU_{k_1}^2, \quad x > 0.$$

It follows that for any $\varepsilon > 0$, $w := (1 + \varepsilon)U_{k_1}$ satisfies

$$-(de^{-\frac{k_2}{d}x}w')' \geq e^{-\frac{k_2}{d}x}(aw - bw^2).$$

As before we can apply Lemma 2.1 of [9] to conclude that $w \geq U_{k_2}$ in $[0, +\infty)$, i.e., $(1 + \varepsilon)U_{k_1} \geq U_{k_2}$ in $[0, +\infty)$. Letting $\varepsilon \to 0$, we deduce that $U_{k_1}(x) \geq U_{k_2}(x)$ in $[0, +\infty)$. By the strong maximum principle we deduce $U_{k_1}(x) > U_{k_2}(x)$ for $x > 0$. Since $U_{k_i}(0) = 0$, the above inequality implies that

$$U_{k_1}'(0) \geq U_{k_2}'(0) \text{ for } k_1 < k_2.$$

By the Hopf lemma, we have $U_k'(0) > 0$ and $U_{k_1}'(0) > U_{k_2}'(0)$. Thus for any fixed $\mu > 0$, the function $\sigma(k) = k - \mu U_k'(0)$ is a strictly increasing function. By a standard compactness argument, we can use the uniqueness of U_k to see that $k \to U_k$ is a continuous mapping from $[0, +\infty)$ to $C^1_{loc}[0, +\infty)$. Hence $\sigma(k)$ is a continuous function. Clearly,

$$\sigma(0) = -\mu U_0'(0) < 0 \quad \text{and} \quad \sigma(+\infty) = +\infty.$$

Therefore, there exists a unique $k_0 = k_0(\mu) > 0$ such that $\sigma(k_0) = 0$. ∎

THEOREM 4.2. *If $h_\infty = +\infty$, then $\lim_{t\to+\infty} \frac{h(t)}{t} = k_0$, where k_0 is uniquely determined in Proposition* 4.1.

Proof. It follows from the proof of Lemma 3.2 that $\limsup_{t\to+\infty} u(t,x) \le \lim_{t\to\infty} \overline{u}(t) = \frac{a}{b}$ uniformly for $x \ge 0$. Therefore, for any given $\varepsilon > 0$ small, there exists $T = T_\varepsilon > 0$ such that

$$u(t,x) \le \frac{a}{b}(1-\varepsilon)^{-1} \quad \forall t \ge T,\ x \ge 0.$$

Let $U_{k_0}(x)$ denote the unique positive solution of (4.1) with $k = k_0$. Since $U_{k_0}(x) \to \frac{a}{b}$ as $x \to +\infty$, there exists $X_0 > 0$ large such that

$$U_{k_0}(x) > \frac{a}{b}(1-\varepsilon) \text{ for } \ x \ge X_0.$$

We now define

$$\begin{aligned}
&\xi(t) = (1-\varepsilon)^{-2}k_0 t + X_0 + h(T), \quad t \ge 0,\\
&v(t,x) = (1-\varepsilon)^{-2}U_{k_0}(\xi(t) - x), \quad t \ge 0, \quad 0 \le x \le \xi(t).
\end{aligned}$$

Then

$$\xi'(t) = (1-\varepsilon)^{-2}k_0,$$

$$-\mu v_x(t,\xi(t)) = \mu(1-\varepsilon)^{-2}U'_{k_0}(0) = (1-\varepsilon)^{-2}k_0,$$

and so we have

$$\xi'(t) = -\mu v_x(t,\xi(t)).$$

Clearly,

$$v(t,\xi(t)) = 0 \quad \text{and} \quad v_x(t,0) = -(1-\varepsilon)^{-2}U'_{k_0}(\xi(t)) \le 0.$$

Moreover, for $0 \le x \le h(T)$,

$$v(0,x) = (1-\varepsilon)^{-2}U_{k_0}(\xi(0) - x) \ge (1-\varepsilon)^{-2}U_{k_0}(X_0) \ge \frac{a}{b}(1-\varepsilon)^{-1} \ge u(T,x)$$

and $v(0,x) > 0$ for $h(T) < x < \xi(0)$. Direct calculations show that

$$\begin{aligned}
v_t - dv_{xx} &= (1-\varepsilon)^{-2}(U'_{k_0}\xi' - dU''_{k_0})\\
&= (1-\varepsilon)^{-2}[(1-\varepsilon)^{-2}k_0U'_{k_0} - dU''_{k_0}]\\
&\ge (1-\varepsilon)^{-2}(k_0U'_{k_0} - dU''_{k_0}) \quad (\text{due to } U'_{k_0} \ge 0)\\
&= (1-\varepsilon)^{-2}(aU_{k_0} - bU^2_{k_0})\\
&= av - (1-\varepsilon)^2 bv^2\\
&\ge av - bv^2 \ \text{ for } t > 0, \quad 0 < x < \xi(t).
\end{aligned}$$

Hence we can use Lemma 3.5 to conclude that

$$u(t+T,x) \le v(t,x), \quad h(t+T) \le \xi(t) \ \text{ for } \ t \ge 0, \quad 0 \le x \le h(t+T).$$

It follows that

$$\limsup_{t\to+\infty} \frac{h(t)}{t} \le \lim_{t\to+\infty} \frac{\xi(t-T)}{t} = k_0(1-\varepsilon)^{-2}.$$

Since $\varepsilon > 0$ can be arbitrarily small, we deduce that

$$\limsup_{t\to+\infty} \frac{h(t)}{t} \le k_0.$$

Next, we show

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \ge k_0$$

by constructing a suitable lower solution. We consider the following problem:

$$\begin{cases} -dV'' + k_0 V' = aV - bV^2, \quad 0 < x < l, \\ V(0) = V(l) = 0. \end{cases} \tag{4.3}$$

As before we know that for all large l problem (4.3) admits a unique positive solution V_l and

$$V_l(x) < U_{k_0}(x) < \frac{a}{b} \text{ for } 0 < x \le l.$$

Moreover, as $l \to \infty$, $V_l(x)$ increases to a function $V_\infty(x)$ which solves (4.1) with $k = k_0$. By the uniqueness of the positive solution to (4.1), we deduce $V_\infty = U_{k_0}$. Moreover, a simple regularity and compactness consideration shows that

$$\lim_{l\to\infty} V_l = U_{k_0} \text{ in } C^1_{loc}[0,+\infty).$$

Therefore for any given small $\varepsilon > 0$, we can find $l_0 = l_0(\varepsilon) > 0$ large such that

$$V'_{l_0}(0) > \sqrt{1-\varepsilon}\, U'_{k_0}(0) = \sqrt{1-\varepsilon}\,\frac{k_0}{\mu}.$$

We now define

$$V_0(x) = \begin{cases} V_{l_0}(x), & 0 \le x \le \xi_0, \\ V_{l_0}(\xi_0), & x > \xi_0, \end{cases}$$

where $\xi_0 \in (0, l_0)$ is such that $V_{l_0}(\xi_0) = \max_{[0,l_0]} V_{l_0}$. From the equation for V_{l_0} we see that $e^{-(k_0/d)x} V'_{l_0}(x)$ is monotone decreasing in $(0, l_0)$. Thus $V'_{l_0}(x)$ changes sign exactly once in this interval. It follows that such ξ_0 is unique, $V'_{l_0}(x) > 0$ for $x \in (0, \xi_0)$, and $V'_{l_0}(x) < 0$ for $x \in (\xi_0, l_0)$. Thus we have

$$V_0(x) \le V_{l_0}(\xi_0) < U_{k_0}(\xi_0) < \frac{a}{b} \text{ for } x \ge 0,$$

$$V_0'(x) = 0 \ \text{ for } \ x \ge \xi_0, \ V_0'(x) > 0 \text{ for } 0 \le x < \xi_0.$$

Moreover, it is easily checked that V_0 satisfies (in the weak sense)

$$(4.4)\qquad \begin{cases} -dV_0'' + k_0 V_0' \le aV_0 - bV_0^2, & 0 \le x < +\infty, \\ V_0(0) = 0,\ V_0(x) < \frac{a}{b}, & x \ge 0. \end{cases}$$

Due to Lemma 3.2, we can choose $T = T_{\varepsilon,\xi_0} > 0$ large such that

$$h(T) > \xi_0 \text{ and } u(T,x) \ge \frac{a}{b}\sqrt{1-\varepsilon} \quad \forall x \in [0,\xi_0].$$

Then define

$$\begin{aligned} &\eta(t) = (1-\varepsilon)k_0 t + \xi_0, \quad t \ge 0, \\ &w(t,x) = \sqrt{1-\varepsilon}V_0(\eta(t) - x), \quad t \ge 0, \quad 0 \le x \le \eta(t). \end{aligned}$$

We have

$$\begin{aligned} &-\mu w_x(t,\eta(t)) = \sqrt{1-\varepsilon}\mu V_0'(0) = \sqrt{1-\varepsilon}\mu V_{l_0}'(0) > (1-\varepsilon)k_0, \\ &\eta'(t) = (1-\varepsilon)k_0 < -\mu w_x(t,\eta(t)), \\ &w(t,\eta(t)) = 0, \\ &w_x(t,0) = -\sqrt{1-\varepsilon}V_0'(\eta(t)) = 0 \quad (\text{since } \eta(t) \ge \xi_0), \\ &w(0,x) = \sqrt{1-\varepsilon}V_0(\xi_0 - x) < \sqrt{1-\varepsilon}\frac{a}{b} \le u(T,x) \quad \forall x \in [0,\xi_0]. \end{aligned}$$

Moreover,

$$\begin{aligned} w_t - dw_{xx} &= \sqrt{1-\varepsilon}[(1-\varepsilon)k_0 V_0' - dV_0''] \\ &\le \sqrt{1-\varepsilon}(k_0 V_0' - dV_0'') \\ &\le \sqrt{1-\varepsilon}(aV_0 - bV_0^2) \quad (\text{by } (4.4)) \\ &= aw - (\sqrt{1-\varepsilon})^{-1} bw^2 \\ &\le aw - bw^2 \end{aligned}$$

for $t > 0, 0 < x < \eta(t)$. Hence by Remark 3.6 we deduce

$$u(t+T,x) \ge w(t,x), \quad h(t+T) \ge \eta(t) \quad \text{for} \quad t \ge 0, \quad 0 \le x \le \eta(t).$$

It follows that

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \ge \lim_{t\to+\infty} \frac{\eta(t-T)}{t} = (1-\varepsilon)k_0.$$

Since $\varepsilon > 0$ can be arbitrarily small, this implies that

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \ge k_0.$$

The proof is now complete. ∎

Next, we analyze the dependence of k_0 on the parameters a, b, μ, and d. From the proof of Proposition 4.1 we know that k_0 is the unique solution of

$$k - \mu U_k'(0) = 0$$

or, equivalently, the unique value of k at which the increasing line $\sigma = \frac{1}{\mu}k$ and the decreasing curve $\sigma = U_k'(0)$ intersect in the $k-\sigma$ plane, as indicated in Figure 1.

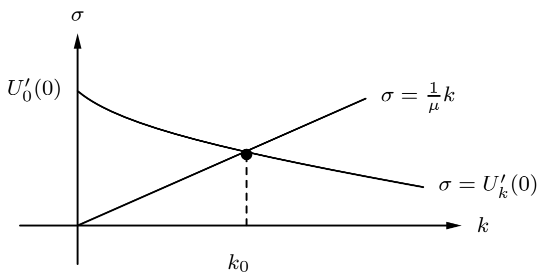

FIG. 1. *Graphical representation of* k_0.

Clearly, when all the other parameters are fixed, k_0 increases with μ, and $k_0 \to 0$ as $\mu \to 0$, and $k_0 \to +\infty$ as $\mu \to +\infty$. On the other hand, one easily sees by a comparison argument that for fixed k, $U_k(\cdot)$ increases with a and decreases with b, and it follows that $U_k'(0)$ increases with a and decreases with b. This implies that k_0 increases with a and decreases with b. Combining these, we find that for fixed d, if k_0 is viewed as a function of (μ, a, b), namely $k_0 = k_0(\mu, a, b)$, then

$$\mu_1 \geq \mu_2,\ a_1 \geq a_2, \text{ and } b_1 \leq b_2 \text{ imply } k_0(\mu_1, a_1, b_1) \geq k_0(\mu_2, a_2, b_2),$$

with strict inequality holding when $(\mu_1, a_1, b_1) \neq (\mu_2, a_2, b_2)$.

We next fix μ, a, b and examine the dependence of k_0 on d, and we write $k_0 = k_0(d)$ to emphasize this dependence.

PROPOSITION 4.3. *For fixed* $\mu, a,$ *and* b, *we have*

$$\lim_{d\to\infty} \sqrt{d}\, k_0(d) = \sigma_0 \in (0, \infty),$$

$$\lim_{d\to 0} \frac{k_0(d)}{\sqrt{d}\,|\ln d|} = 0 \quad \textit{and} \quad \lim_{d\to 0} \frac{k_0(d)}{\sqrt{d}} = +\infty.$$

Proof. Let U_{k_0} denote the unique positive solution of (4.1) with $k = k_0$, and define

$$V(x) = U_{k_0}(\sqrt{d}x).$$

Then

$$\begin{cases} -V'' + \frac{k_0}{\sqrt{d}}V' = -dU_{k_0}'' + k_0 U_{k_0}' = aV - bV^2, & x > 0, \\ V(0) = 0. \end{cases} \tag{4.5}$$

If for each $\lambda \geq 0$, we use V_λ to denote the unique positive solution of

$$-V'' + \lambda V' = aV - bV^2, \quad x > 0, \quad V(0) = 0.$$

It follows from $k_0 = \mu U_{k_0}'(0)$ that

$$k_0 = \frac{\mu}{\sqrt{d}} V'_{\frac{k_0}{\sqrt{d}}}(0)$$

or $k_0 = \sqrt{d}\lambda_0$, where $\lambda_0 = \lambda_0(d)$ is the unique solution of $\lambda = \frac{\mu}{d} V_\lambda'(0)$. In other words, λ_0 is the unique value of λ at which the graphs of

$$\eta = \frac{d}{\mu}\lambda \quad \text{and} \quad \eta = V_\lambda'(0)$$

intersect each other in the $\lambda - \eta$ plane, as shown in Figure 2.

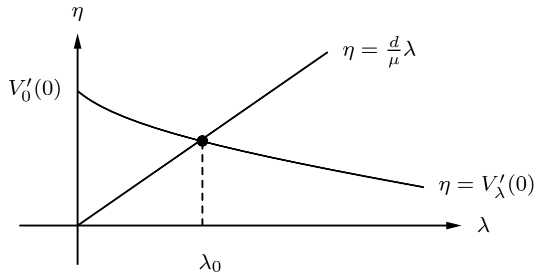

FIG. 2. *Graphical representation of* λ_0.

Clearly, as $d \to +\infty$, we have $\lambda_0 \to 0$ and $\frac{d}{\mu}\lambda_0 \to V_0'(0)$. Hence

$$\sqrt{d}k_0 = d\lambda_0 \to \mu V_0'(0) \quad \text{as} \quad d \to +\infty,$$

that is,

$$\lim_{d\to\infty} \sqrt{d}k_0 = \sigma_0 := \mu V_0'(0).$$

Consider next the case $d \to 0$. From Figure 2 it is clear that $\lambda_0 = \lambda_0(d) \to +\infty$ as $d \to 0$, i.e., $\lim_{d\to 0} \frac{k_0(d)}{\sqrt{d}} = \infty$. To obtain further estimate on λ_0, we rewrite the equation of V_λ as

$$-(e^{-\lambda x}V_\lambda')' = e^{-\lambda x}(aV_\lambda - bV_\lambda^2). \tag{4.6}$$

Since $V_\lambda < \frac{a}{b}$, from the equation of V_λ we deduce $(e^{-\lambda x}V_\lambda')' < 0$ and thus

$$e^{-\lambda x}V_\lambda'(x) < V_\lambda'(0) \text{ for } x > 0.$$

Hence $V_\lambda'(x) \le V_\lambda'(0)e^{\lambda x}$ and

$$V_\lambda(x) \le \frac{1}{\lambda}V_\lambda'(0)(e^{\lambda x} - 1) \le \frac{1}{\lambda}V_\lambda'(0)e^{\lambda x}.$$

On the other hand, integrating (4.6) over $(0, +\infty)$, we obtain

$$\begin{aligned}
V_\lambda'(0) &= \int_0^\infty e^{-\lambda x}(aV_\lambda - bV_\lambda^2)dx \quad (\text{using } V_\lambda'(+\infty) = 0)\\
&\le \int_0^M e^{-\lambda x}aV_\lambda dx + \int_M^{+\infty} e^{-\lambda x}\frac{a^2}{4b}dx \quad \left(\text{due to } aV_\lambda - bV_\lambda^2 \le \frac{a^2}{4b}\right)\\
&\le \frac{aM}{\lambda}V_\lambda'(0) + \frac{a^2}{4b\lambda}e^{-M\lambda} \text{ for any } M > 0.
\end{aligned}$$

Therefore,

$$V_\lambda'(0) \le \left(1 - \frac{aM}{\lambda}\right)^{-1}\frac{a^2}{4b\lambda}e^{-M\lambda} \le e^{-M\lambda}$$

for $\lambda \geq aM + \frac{a^2}{4b}$. It follows that $\frac{d}{\mu}\lambda_0 = V'_{\lambda_0}(0) \leq e^{-M\lambda_0}$ for all small d (since $\lambda_0 = \lambda_0(d)$ is large for such d). Thus for small d,

$$\begin{aligned}
&\ln d + \ln \lambda_0 - \ln \mu \leq -M\lambda_0, \\
&M\lambda_0 + \ln \lambda_0 \leq \ln \mu + \ln\left(\frac{1}{d}\right), \\
&(M-1)\lambda_0 \leq \ln\left(\frac{1}{d}\right), \\
&\lambda_0 \leq \frac{1}{M-1} \ln\left(\frac{1}{d}\right),
\end{aligned}$$

and hence

$$k_0 = \sqrt{d}\lambda_0 \leq \frac{1}{M-1}\sqrt{d}\ln\left(\frac{1}{d}\right), \quad \limsup_{d\to 0} \frac{k_0(d)}{\sqrt{d}\ln(1/d)} \leq \frac{1}{M-1}.$$

Since M can be arbitrarily large, it follows that $\lim_{d\to 0} \frac{d_0(d)}{\sqrt{d}\ln(1/d)} = 0$. $\square$

5. Double fronts spreading. In this section, we explain how the techniques developed for treating (1.1) can be easily modified to obtain similar results for (1.4).

We start with the existence uniqueness result. The local existence uniqueness result can be proved in the same way, except that we need to modify the transformation in the proof of Theorem 2.1 so that both boundaries are straightened. To do this, we define

$$x = y + \zeta(y)(h(t) - h_0) + \xi(y)(g(t) + h_0), \ -\infty < y < \infty,$$

with $\zeta(y)$ defined as before, and $\xi(y) = -\zeta(-y)$. The rest of the proof is the same.

The proof and the conclusion of Lemma 2.2 are still valid for (1.4), and the estimates

$$0 < u(t,x) \leq C_1, \quad 0 < -g'(t) \leq C_2 \text{ for } g(t) < x \leq 0, \quad t \in (0, T_0),$$

can be obtained analogously.

Finally, we can obtain the global existence and uniqueness result for (1.4) by exactly the same argument as in Theorem 2.3.

To summarize, we have the following result.

THEOREM 5.1. *Problem* (1.4) *has a unique solution* (u, g, h) *that is defined for all* $t \in (0, \infty)$. *Moreover,*

$$g'(t) < 0, \quad h'(t) > 0, \quad 0 < u(t,x) < \overline{u}(t) \textit{ for } t > 0 \quad \textit{and } g(t) < x < h(t),$$

where $\overline{u}(t)$ *is the unique solution of* (2.10).

We next consider the spreading-vanishing dichotomy. We define h_∞ as before and define g_∞ analogously.

LEMMA 5.2. *If* $h_\infty < \infty$ *or* $g_\infty > -\infty$, *then both* h_∞ *and* g_∞ *are finite and*

$$h_\infty - g_\infty \leq \pi\sqrt{\frac{d}{a}}, \quad \lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),\,h(t)])} = 0.$$

Proof. For definiteness, we assume that $h_\infty < \infty$. We first prove that $h_\infty - g_\infty \leq \pi\sqrt{\frac{d}{a}}$. Otherwise

$$h_\infty - g_\infty - \pi\sqrt{\frac{d}{a}} > 0,$$

and we can find $T > 0$ large and $\sigma > 0$ small such that

$$h(T) - g(T) > l := \sigma + \pi\sqrt{\frac{d}{a}}.$$

It follows that

$$g(t) \le g(T) < h_\infty - \sigma - \pi\sqrt{\frac{d}{a}} \text{ for } t \ge T.$$

We now choose $\epsilon > 0$ and define $v = v_\epsilon$ as in the proof of Lemma 3.1 and set

$$w(t,x) = v\left(\frac{x - x_0}{h(t) - x_0} l\right), \; x_0 = h_\infty - \frac{\pi}{2}\sqrt{\frac{d}{a}} - \frac{1}{2}\sigma.$$

We observe that

$$x \in [2x_0 - h(t), h(t)] \text{ implies } \frac{x - x_0}{h(t) - x_0} l \in [-l, l],$$

and

$$2x_0 - h(t) > 2x_0 - h_\infty = h_\infty - \pi\sqrt{\frac{d}{a}} - \sigma > g(t) \text{ for } t \ge T.$$

One can then follow the argument in the proof of Lemma 3.1 to find $T_0 > T$ and then $\delta > 0$ small such that $\underline{u}(t,x) := \delta w(t,x)$ is a lower solution to the equation satisfied by $u(t,x)$ over the region $\{(t,x) : t \ge T_0, 2x_0 - h(t) \le x \le h(t)\}$. Hence $\underline{u}(t,x) \le u(t,x)$ over this region and we can derive the same contradiction as in Lemma 3.1. This proves that g_∞ is finite and $h_\infty - g_\infty \le \pi\sqrt{\frac{d}{a}}$.

Let $\overline{u}(t,x)$ denote the unique solution of the problem

$$\begin{cases} \overline{u}_t - d\overline{u}_{xx} = a\overline{u} - b\overline{u}^2, & t > 0, \; g_\infty < x < h_\infty, \\ \overline{u}(t, g_\infty) = 0, \quad \overline{u}(t, h_\infty) = 0, & t > 0, \\ \overline{u}(0,x) = \tilde{u}_0(x), & g_\infty < x < h_\infty, \end{cases} \tag{5.1}$$

with

$$\tilde{u}_0(x) = \begin{cases} u_0(x), & g_0 \le x \le h_0, \\ 0 & \text{otherwise.} \end{cases}$$

The comparison principle gives $0 \le u(t,x) \le \overline{u}(t,x)$ for $t > 0$ and $x \in [g(t), h(t)]$. Since $h_\infty - g_\infty \le \pi\sqrt{\frac{d}{a}}$, it follows from a well-known conclusion on the logistic problem (5.1) that $\overline{u}(t,x) \to 0$ uniformly for $x \in [g_\infty, h_\infty]$ as $t \to \infty$. Thus $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([g(t),\,h(t)])} = 0$. ∎

LEMMA 5.3. *If $h_\infty = -g_\infty = +\infty$, then $\lim_{t\to\infty} u(t,x) = \frac{a}{b}$ uniformly on any compact subset of $\mathbb{R}^1$.*

Proof. This follows the proof of Lemma 3.2, except that now t_l is chosen such that $g(t_l) \le -l$ and $h(t_l) \ge l$, and in (3.3) instead of $0 < x < l$, we now require $-l < x < l$, and we replace the boundary condition $(\underline{u}_l)_x(t,0) = 0$ there by the boundary condition $\underline{u}_l(-l) = 0$. The change for $\tilde{u}_0$ is obvious. ∎

Combining Lemmas 5.2 and 5.3, we obtain the following spreading-vanishing dichotomy for (1.4).

THEOREM 5.4. *Let $(u(t,x), g(t), h(t))$ be the solution of the free boundary problem (1.4). Then the following alternative holds:*

Either

(i) *spreading:* $h_\infty = -g_\infty = +\infty$ *and* $\lim_{t\to+\infty} u(t,x) = \frac{a}{b}$ *uniformly for* x *in any bounded set of* $\mathbb{R}^1$*;*

or

(ii) *vanishing:* $h_\infty - g_\infty \le \pi\sqrt{\frac{d}{a}}$ *and* $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([g(t),h(t)])} = 0$.

Moreover, Lemma 5.2 implies the following result.

THEOREM 5.5. *If* $h_0 \ge \frac{\pi}{2}\sqrt{\frac{d}{a}}$*, then* $h_\infty = -g_\infty = +\infty$.

To find the sharp criteria governing the alternatives in the spreading-vanishing dichotomy for the case $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$, as in section 3, we need some comparison results.

LEMMA 5.6. *Suppose that* $T \in (0,\infty)$, $\overline{g}$, $\overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, \overline{g}(t) < x < \overline{h}(t)\}$*, and*

$$\begin{cases} \overline{u}_t - d\overline{u}_{xx} \ge \overline{u}(a - b\overline{u}), & t > 0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} \ge 0, & t > 0,\ x = \overline{g}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu\overline{u}_x, & t > 0,\ x = \overline{h}(t). \end{cases}$$

If

$$\overline{g}(t) \le g(t) \text{ in } [0,T], \quad h_0 \le \overline{h}(0), \quad \text{and } u_0(x) \le \overline{u}(0,x) \text{ in } [-h_0,h_0],$$

then the solution (u,g,h) *of the free boundary problem* (1.4) *satisfies*

$$h(t) \le \overline{h}(t) \text{ in } (0,T], \quad u(x,t) \le \overline{u}(x,t) \text{ for } t \in (0,T], \quad \text{and } g(t) < x < h(t).$$

The proof of Lemma 5.6 follows the same arguments as in Lemma 3.5. We also need a variant of Lemma 5.6, whose proof only requires some obvious modifications.

LEMMA 5.7. *Suppose that* $T \in (0,\infty)$, $\overline{g}, \overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, \overline{g}(t) < x < \overline{h}(t)\}$*, and*

$$\begin{cases} \overline{u}_t - d\overline{u}_{xx} \ge \overline{u}(a - b\overline{u}), & t > 0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} = 0, \quad \overline{g}'(t) \le -\mu\overline{u}_x, & t > 0,\ x = \overline{g}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu\overline{u}_x, & t > 0,\ x = \overline{h}(t). \end{cases}$$

If

$$[-h_0, h_0] \subseteq [\overline{g}(0), \overline{h}(0)] \quad \text{and} \quad u_0(x) \le \overline{u}(0,x) \text{ in } [-h_0,h_0],$$

then the solution (u,g,h) *of the free boundary problem* (1.4) *satisfies*

$$g(t) \ge \overline{g}(t),\ h(t) \le \overline{h}(t) \text{ in } (0,T],$$

$$u(x,t) \le \overline{u}(x,t) \text{ for } t \in (0,T] \quad \text{and} \quad x \in (g(t),h(t)).$$

Remark 5.8. There is a symmetric version of Lemma 5.6, where the conditions on the left and right boundaries are interchanged. We also have a corresponding comparison result for lower solutions in each case.

We can now prove the corresponding result of Lemma 3.7.

LEMMA 5.9. *Suppose* $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$*. If*

$$\mu \ge \mu_0 := \max\left\{1, \frac{b}{a}\|u_0\|_\infty\right\} d \left(\pi\sqrt{\frac{d}{a}} - 2h_0\right) \left(\int_{-h_0}^{h_0} u_0(x)dx\right)^{-1},$$

then $h_\infty = -g_\infty = +\infty$.

Proof. Only minor modifications in the proof of Lemma 3.7 are needed. Instead of starting from

$$\frac{d}{dt}\int_0^{h(t)} u(t,x)dx,$$

we now start with

$$\frac{d}{dt}\int_{g(t)}^{h(t)} u(t,x)dx.$$

The rest of the changes are obvious. ∎

It is easily seen that the lower solution constructed in the proof of Lemma 3.8 is also a lower solution for (1.4). Therefore, the following result holds.

LEMMA 5.10. *Suppose* $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$. *Then there exists* $\overline{\mu} > 0$ *depending on* u_0 *such that* $(h_\infty - g_\infty) \le \pi\sqrt{\frac{d}{a}}$ *when* $\mu \le \overline{\mu}$.

We can now use Lemmas 5.7, 5.9, and 5.10 to prove the following sharp criteria governing the spreading-vanishing dichotomy for the case $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$.

THEOREM 5.11. *Suppose* $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$. *Then there exists* $\mu^* > 0$ *depending on* u_0 *such that* $(h_\infty - g_\infty) \le \pi\sqrt{\frac{d}{a}}$ *when* $\mu \le \mu^*$, *and* $h_\infty = -g_\infty = \infty$ *when* $\mu > \mu^*$.

Proof. We define $\Sigma := \{\mu > 0 : h_\infty - g_\infty \le \pi\sqrt{\frac{d}{a}}\ \}$. For the rest of the proof we just follow the proof of Theorem 3.9. ∎

Finally, we consider the asymptotic spreading speed for (1.4).

THEOREM 5.12. *Let* (u, g, h) *be the unique solution of* (1.4) *with* $h_\infty = -g_\infty = \infty$. *Then*

$$\lim_{t\to\infty}\frac{g(t)}{t} = -k_0,\ \ \lim_{t\to\infty}\frac{h(t)}{t} = k_0,$$

where k_0 *is given by Proposition* 4.1.

Proof. We only prove the conclusion for $h(t)$, since the proof for $g(t)$ is parallel.

Examining the proof of Theorem 4.2, we find that the function $v(t, x)$ there is also an upper solution for (1.4), and hence

$$u(t+T,x) \le v(t,x), \quad h(t+T) \le \xi(t) \text{ for } t \ge 0, \quad \text{ and } g(t+T) \le x \le h(t+T).$$

We thus deduce

$$\limsup_{t\to\infty}\frac{h(t)}{t} \le k_0$$

in the same way.

The construction of the lower solution will be different from the proof of Theorem 4.2. For $l > 0$ and any small $\epsilon > 0$, we consider the following problem:

$$-dV'' + (1-\epsilon)k_0V' = aV - bV^2 \text{ in } (0,l), \quad V(0) = V(l) = 0.$$

As before, we know that for all large l this problem has a unique positive solution V_l and

$$V_l(x) < U_{(1-\epsilon)k_0}(x) < \frac{a}{b} \text{ for } 0 < x \le l.$$

Moreover, as $l \to \infty$, V_l converges to $U_{(1-\epsilon)k_0}$ in $C^1_{loc}([0,\infty))$. From Proposition 4.1 we know that $U'_{(1-\epsilon)k_0}(0) > U'_{k_0}(0)$. Hence we can find $l_0 > 0$ large enough such that

$$V'_{l_0}(0) > U'_{k_0}(0) = \frac{k_0}{\mu}.$$

Denote $V_0 = V_{l_0}$. We have $\max_{[0,l_0]} V_0 < a/b$, and hence, by Lemma 3.2, we can find $T = T_{l_0} > 0$ such that

$$h(T) > l_0 \text{ and } u(T,x) \geq \max_{[0,l_0]} V_0 \quad \forall x \in [0, l_0].$$

Define

$$\begin{aligned} &\eta(t) = (1-\varepsilon)k_0 t + l_0, \quad t \geq 0,\\ &w(t,x) = V_0(\eta(t) - x), \quad t \geq 0, \quad \eta(t) - l_0 \leq x \leq \eta(t). \end{aligned}$$

Clearly, $\eta(t) - l_0 \geq 0 > g(t)$ for $t \geq 0$, and

$$\begin{aligned} &-\mu w_x(t,\eta(t)) = \mu(1-\epsilon)V_0'(0) > (1-\epsilon)k_0,\\ &\eta'(t) = (1-\varepsilon)k_0 < -\mu w_x(t,\eta(t)),\\ &w(t,\eta(t)) = V_0(0) = 0,\\ &w(t,\eta(t)-l_0) = V_0(l_0) = 0,\\ &w(0,x) = V_0(l_0 - x) \leq u(T,x) \ \ \forall x \in [0,l_0]. \end{aligned}$$

Moreover,

$$\begin{aligned} w_t - dw_{xx} &= (1-\varepsilon)k_0 V_0' - dV_0''\\ &= aV_0 - bV_0^2\\ &= aw - bw^2 \end{aligned}$$

for $t > 0, \eta(t) - l_0 < x < \eta(t)$. Hence, by Remark 5.8, we deduce

$$u(t+T,x) \geq w(t,x), \quad h(t+T) \geq \eta(t) \ \text{ for } \ t \geq 0, \quad \eta(t) - l_0 \leq x \leq \eta(t).$$

It follows that

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \geq \lim_{t\to+\infty} \frac{\eta(t-T)}{t} = (1-\varepsilon)k_0.$$

Since $\varepsilon > 0$ can be arbitrarily small, this implies that

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \geq k_0.$$

The proof is now complete. ☐

6. Discussion. We have examined the dynamical behavior of the population $u(t,x)$ with spreading front $x = h(t)$ determined by (1.1), and also the dynamical behavior of the population $u(t,x)$ with double spreading fronts $x = g(t)$ and $x = h(t)$ modeled by (1.4).

We have proved that for both problems, a spreading-vanishing dichotomy holds (Theorems 3.3 and 5.4), and when spreading occurs the spreading fronts expand at a nearly constant speed for large time (Theorems 4.2 and 5.12). These phenomena are in agreement with numerous documented observations for the spreading of species in ecology (cf. [26, 19]), but differ from the mathematical conclusions obtained from (1.3), which predicts successful spreading for all initial data.

If we use "spreading radius" to mean the time-dependent distance between the fixed boundary $x = 0$ and the expanding front $x = h(t)$ for (1.1), or half the distance between the two expanding fronts for (1.4), then our spreading-vanishing dichotomy reveals a critical spreading radius, which may be called a "spreading barrier",

$$l^* = \frac{\pi}{2}\sqrt{\frac{d}{a}},$$

such that the population will spread to all the new environments and successfully establish itself if its spreading radius can break through this barrier l^* in some finite time, or the spreading never breaks through this barrier and the population vanishes in the long run. We note that once the spreading breaks through this barrier, the population will definitely establish and spread to the entire available space regardless of its size at the time the barrier is broken through, though according to our comparison results (Lemmas 3.5 and 5.7, and Remarks 3.6 and 5.8), when the initial spreading radius l_0 is below l^*, the initial population size u_0 has a significant positive influence on whether the spreading can break through the barrier at some later time. This feature of the dynamical behavior of our models seems to agree with the empirical evidence discussed in Chapter 4 of [19], where in particular a comprehensive experiment on an insect biocontrol agent in New Zealand (reported in Memmott et al. [20]) is reviewed. The observed data in this experiment during the six years after the introduction of 55 original populations show that the probability of establishment was significantly and positively related to the initial population size, but only during the first year in the field. Populations surviving after the initial year were not significantly related to the initial population size, as shown in Figure 4.3 of [19], which also reveals that the population size after year one of the introduction was mostly smaller than the introduction size.

Another fact revealed in our models which does not agree with (1.3) is about the spreading speed (also called spreading rate). The latter gives a spreading speed proportional to the square root of the dispersal rate d (namely $c^* = 2\sqrt{ad}$), while Proposition 4.3 shows that the spreading speed k_0 for (1.1) and (1.4) is not increasing in d (at least for large d). Moreover, in the course of the proof of Proposition 4.3, we have demonstrated that k_0 can be expressed in the form $k_0 = \lambda_0\sqrt{d}$, with $\lambda_0 = \lambda_0(d)$ determined as in Figure 2. Since $\lambda_0(d) \to 0$ as $d \to \infty$, and $\lambda_0(d) \to \infty$ as $d \to 0$, the expression $k_0 = \lambda_0\sqrt{d}$ suggests that, if k_0 is a good estimate for the real spreading rate, then the formula $c^* = 2\sqrt{ad}$ would underestimate the real spreading rate for small d, and overestimate this rate for large d. This turns out to be in agreement with the figures in Table 3.1 of [26], where the observed spreading rates are compared with the theoretically predicted rates based on the formula $c^* = 2\sqrt{ad}$, and the same trend is revealed there.

We feel it is reasonable to conclude that (1.1) and (1.4) are promising alternatives to (1.3) for the modeling of population spreading, and worth further investigation.

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