SPREADING SPEED FOR SOME COOPERATIVE SYSTEMS WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES, PART 2: SHARP ESTIMATE ON THE RATE OF SPREADING

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ABSTRACT. This is Part 2 of a two part series on a class of cooperative reaction-diffusion systems with free boundaries in one space dimension, where the diffusion terms are nonlocal, given by integral operators involving suitable kernel functions, and they are allowed not to appear in some of the equations in the system. Such a system covers various models arising from mathematical biology, including in particular a West Nile virus model [10] and an epidemic model [33], where a "spreading-vanishing" dichotomy is known to govern the long time dynamical behaviour, but the question on spreading speed was left open. In this two part series, we develop a systematic approach to determine the spreading profile of the system. In Part 1, we obtained threshold conditions on the kernel functions which decide exactly when the spreading has finite speed, or infinite speed (accelerated spreading), and when the spreading speed is finite, we showed that the speed is determined by a particular semi-wave. In Part 2 here, for some typical classes of kernel functions, we obtain more precise estimates on the spreading rate for both the finite speed case, and the infinite speed case. These extend the results for a single equation in [12] to a general system.

Key words: Free boundary, nonlocal diffusion, spreading rate.
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1. Introduction

This is Part 2 of a two part series aiming to determine the long-time behaviour of cooperative systems with nonlocal diffusion and free boundaries of the following form:

$$
\begin{cases}
\partial_t u_i = d_i \mathcal{L}_i[u_i](t,x) + f_i(u_1,u_2,\cdots,u_m), & t>0,\ x\in(g(t),h(t)),\ 1\le i\le m_0,\\
\partial_t u_i = f_i(u_1,u_2,\cdots,u_m), & t>0,\ x\in(g(t),h(t)),\ m_0< i\le m,\\
u_i(t,g(t)) = u_i(t,h(t)) = 0, & t>0,\ 1\le i\le m,\\
g'(t) = -\sum_{i=1}^{m_0}\mu_i\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
h'(t) = \sum_{i=1}^{m_0}\mu_i\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
u_i(0,x) = u_{i0}(x), & x\in[-h_0,h_0],\ 1\le i\le m,
\end{cases}
\tag{1.1}
$$

where $1 \le m_0 \le m$, and for $i \in \{1, ..., m_0\}$,

$$\mathcal{L}_i[v](t,x) := \int_{g(t)}^{h(t)} J_i(x-y)v(t,y)\mathrm{d}y - v(t,x),$$

$$d_i > 0 \text{ and } \mu_i \ge 0 \text{ are constants, with } \sum_{i=1}^{m_0}\mu_i > 0.$$

The initial functions satisfy

$$u_{i0} \in C([-h_0,h_0]),\quad u_{i0}(-h_0) = u_{i0}(h_0) = 0,\ u_{i0}(x) > 0 \ \text{ in } (-h_0,h_0),\ 1\le i\le m. \tag{1.2}$$

The kernel functions $J_i(x)$ $(i = 1, \cdots, m_0)$ satisfy

(**J**): $J_i \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is nonnegative, even, $J_i(0) > 0$, $\int_{\mathbb{R}} J_i(x)\mathrm{d}x = 1$ for $1 \le i \le m_0$.

As in Part 1 [11], we will write $F = (f_1, ..., f_m) \in [C^1(\mathbb{R}^m_+)]^m$ with

$$\mathbb{R}^m_+ := \{x = (x_1, ..., x_m) \in \mathbb{R}^m : x_i \ge 0 \text{ for } i = 1, ..., m\},$$

and use the following notations for vectors in $\mathbb{R}^m$:

(i) For $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, we simply write $(x_1, \cdots, x_m)$ as (x_i). For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$
\begin{aligned}
&x \succeq (\preceq)\ y && \text{means} \quad x_i \ge (\le)\ y_i \text{ for } 1 \le i \le m,\\
&x \succ (\prec)\ y && \text{means} \quad x \succeq (\preceq)\ y \text{ but } x \ne y,\\
&x \succ\!\succ (\prec\!\prec)\ y && \text{means} \quad x_i > (<)\ y_i \text{ for } 1 \le i \le m.
\end{aligned}
$$

(ii) If $x \preceq y$, then $[x, y] := \{z \in \mathbb{R}^m : x \preceq z \preceq y\}$.

(iii) Hadamard product: For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$x \circ y = (x_i y_i) \in \mathbb{R}^m.$$

(iv) Any $x \in \mathbb{R}^m$ is viewed as a row vector, namely a $1 \times m$ matrix, whose transpose is denoted by x^T.

Our basic assumptions on F are:

($\mathbf{f_1}$) (i) $F(u) = \mathbf{0}$ has only two roots in $\mathbb{R}^m_+$: $\mathbf{0} = (0, 0, \cdots, 0)$ and $\mathbf{u}^* = (u_1^*, u_2^*, \cdots, u_m^*) \succ\!\succ \mathbf{0}$.

(ii) $\partial_j f_i(u) \ge 0$ for $i \ne j$ and $u \in [\mathbf{0}, \hat{\mathbf{u}}]$, where either $\hat{\mathbf{u}} = \infty$ meaning $[\mathbf{0}, \hat{\mathbf{u}}] = \mathbb{R}^m_+$, or $\mathbf{u}^* \prec\!\prec \hat{\mathbf{u}} \in \mathbb{R}^m$; which implies that (1.1) is a cooperative system in $[\mathbf{0}, \hat{\mathbf{u}}]$.

(iii) The matrix $\nabla F(\mathbf{0})$ is irreducible with principal eigenvalue positive, where $\nabla F(\mathbf{0}) = (a_{ij})_{m\times m}$ with $a_{ij} = \partial_j f_i(\mathbf{0})$.

(iv) If $m_0 < m$ then $\partial_j f_i(u) > 0$ for $1 \le j \le m_0 < i \le m$ and $u \in [\mathbf{0}, \mathbf{u}^*]$.

($\mathbf{f_2}$) $F(ku) \ge kF(u)$ for any $0 \le k \le 1$ and $u \in [\mathbf{0}, \hat{\mathbf{u}}]$.

($\mathbf{f_3}$) The matrix $\nabla F(\mathbf{u}^*)$ is invertible, $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \preceq \mathbf{0}$ and for each $i \in \{1, ..., m\}$, either

(i) $\displaystyle\sum_{j=1}^m \partial_j f_i(\mathbf{u}^*) u_j^* < 0$, or

(ii) $\displaystyle\sum_{j=1}^m \partial_j f_i(\mathbf{u}^*) u_j^* = 0$ and $f_i(u)$ is linear in $[\mathbf{u}^* - \epsilon_0 \mathbf{1}, \mathbf{u}^*]$ for some small $\epsilon_0 > 0$, where $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^m$.

($\mathbf{f_4}$) The set $[\mathbf{0}, \hat{\mathbf{u}}]$ is invariant for

$$U_t = D \circ \int_{\mathbb{R}} \mathbf{J}(x-y) \circ U(t,y)\mathrm{d}y - D \circ U + F(U) \text{ for } t > 0,\ x \in \mathbb{R}, \tag{1.3}$$

and the equilibrium $\mathbf{u}^*$ attracts all the nontrivial solutions in $[\mathbf{0}, \hat{\mathbf{u}}]$; namely, $U(t,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ for all $t > 0, x \in \mathbb{R}$ if $U(0,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ for all $x \in \mathbb{R}$, and $\lim_{t\to\infty} U(t,\cdot) = \mathbf{u}^*$ in $L^\infty_{loc}(\mathbb{R})$ if additionally $U(0,x) \not\equiv \mathbf{0}$.

In (1.3) we have used the convention that $d_i = 0$ and $J_i \equiv 0$ for $m_0 < i \le m$, and

$$D = (d_i),\ \mathbf{J}(x) = (J_i(x)).$$

This convention will be used throughout the paper.

The above assumptions on F indicate that the system is cooperative in $[\mathbf{0}, \hat{\mathbf{u}}]$, and of monostable type, with $\mathbf{u}^*$ the unique stable equilibrium of (1.3), which is also the global attractor of all the nontrivial nonnegative solutions of (1.3) in $[\mathbf{0}, \hat{\mathbf{u}}]$.

Problems (1.1) and (1.3) arise frequently in population and epidemic models. For example, if $m_0 = m = 2$, (1.1) contains the West Nile virus model in [10] as a special case, and with $(m_0, m) = (1, 2)$, it covers the epidemic model in [33]. In these special cases, it is known that the long-time dynamical behaviour of the solution to (1.1) exhibits a spreading-vanishing dichotomy.

Similar to the special cases mentioned in the last paragraph, it can be shown that (1.1) with initial data satisfying (1.2) and $U(0,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ has a unique positive solution $(U(t,x), g(t), h(t))$ defined for all $t > 0$. We say spreading happens if, as $t \to \infty$,

$$(g(t), h(t)) \to (-\infty, \infty) \text{ and } U(t,\cdot) \to \mathbf{u}^* \text{ component-wise in } L^\infty_{loc}(\mathbb{R}),$$

and we say vanishing happens if

$$(g(t), h(t)) \to (g_\infty, h_\infty) \text{ is a finite interval, and } \max_{x\in[g(t),h(t)]} |U(t,x)| \to 0.$$

1.1. **Main results of Part 1.** Let us now recall the main results obtained in Part 1 [11]. When spreading happens for (1.1), we proved in Part 1 that the spreading speed is finite if and only if the following additional condition is satisfied by the kernel functions:

($\mathbf{J_1}$): $\displaystyle\int_0^\infty xJ_i(x)\mathrm{d}x < \infty$ for every $i \in \{1, ..., m_0\}$ such that $\mu_i > 0$.

If ($\mathbf{J_1}$) is not satisfied, then the spreading speed is infinite, namely accelerated spreading happens. Let us note that if for some $i \in \{1, ..., m_0\}$, $\mu_i = 0$, then no restriction on J_i is imposed by ($\mathbf{J_1}$).

The proof of these conclusions rely on a complete understanding of the associated semi-wave problem to (1.1), which consists of the following two equations (1.4) and (1.5) with unknowns $(c, \Phi(x))$:

$$
\begin{cases} D\circ \displaystyle\int_{-\infty}^{0} \mathbf{J}(x-y)\circ \Phi(y)\mathrm{d}y - D\circ\Phi + c\Phi'(x) + F(\Phi(x)) = 0 \text{ for } -\infty < x < 0, \\ \Phi(-\infty) = \mathbf{u}^*, \quad \Phi(0) = \mathbf{0}, \end{cases} \tag{1.4}
$$

and

$$
c = \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0}\int_{0}^{\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x, \tag{1.5}
$$

where $D = (d_i)$, $\mathbf{J} = (J_i)$, $\Phi = (\phi_i)$ and "$\circ$" is the Hadamard product.

If (c, Φ) solves (1.4), we say that Φ is a semi-wave solution to (1.3) with speed c. This is not to be confused with the semi-wave to (1.1), for which the extra equation (1.5) should be satisfied, yielding a semi-wave solution of (1.3) with a desired speed $c = c_0$, which determines the spreading speed of (1.1).

We are interested in semi-waves which are monotone and with positive speed. The following condition on the kernel functions will be used:

$(\mathbf{J_2})$: $\displaystyle\int_0^\infty e^{\lambda x} J_i(x)\mathrm{d}x < \infty$ for some $\lambda > 0$ and every $i \in \{1, ..., m_0\}$.

Theorem A. Suppose the kernel functions satisfy $(\mathbf{J})$ and F satisfies $(\mathbf{f_1}) - (\mathbf{f_4})$. Then there exists $C_* \in (0, +\infty]$ such that

(i) for $0 < c < C_*$, (1.4) has a unique monotone solution $\Phi^c = (\phi_i^c)$, and

$$
\lim_{c \nearrow C_*} \Phi^c(x) = \mathbf{0} \text{ locally uniformly in } (-\infty, 0];
$$

(ii) $C_* \neq \infty$ if and only if $(\mathbf{J_2})$ holds;

(iii) the system (1.4)-(1.5) has a solution pair (c, Φ) with $\Phi(x)$ monotone if and only if $(\mathbf{J_1})$ holds, and when $(\mathbf{J_1})$ holds, there exists a unique $c_0 \in (0, C_*)$ such that $(c, \Phi) = (c_0, \Phi^{c_0})$ solves (1.4) and (1.5).

The spreading speed of (1.1) is determined by the following result:

Theorem B. Suppose the conditions in Theorem A are satisfied, (U, g, h) is the solution of (1.1) with $U(0, x) \in [\mathbf{0}, \hat{\mathbf{u}}]$, and spreading happens. Then the following conclusions hold for the spreading speed:

(i) If $(\mathbf{J_1})$ is satisfied, then the spreading speed is finite, and is determined by

$$
-\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = c_0 \text{ with } c_0 \text{ given in Theorem A (iii).}
$$

(ii) If $(\mathbf{J_1})$ is not satisfied, then accelerated spreading happens, namely

$$
-\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = \infty.
$$

1.2. **Sharp estimates on the rate of spreading.** The main purpose of Part 2 here is to sharpen the conclusions in Theorem B for some typical kernel functions. The results here extend those for a single equation (namely (1.1) with $m = m_0 = 1$) in [12] to a general system.

For $\alpha > 0$, we introduce the condition

$(\mathbf{J}^\alpha)$: $\displaystyle\int_0^\infty x^\alpha J_i(x)dx < \infty$ for every $i \in \{1, ..., m_0\}$.

Let us note that $(\mathbf{J^1})$ implies $(\mathbf{J_1})$, but unless $\mu_i > 0$ for every $i \in \{1, ..., m_0\}$, $(\mathbf{J_1})$ does not imply $(\mathbf{J^1})$. On the other hand, if $(\mathbf{J_2})$ holds, then $(\mathbf{J}^\alpha)$ is satisfied for all $\alpha > 0$.

Theorem 1.1. *In Theorem B, suppose additionally* $(\mathbf{J}^\alpha)$ *holds for some* $\alpha \geq 2$, F *is* C^2 *and* $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \lll \mathbf{0}$. *Then there exist positive constants* θ, C *and* t_0 *such that, for all* $t > t_0$ *and* $x \in [g(t), h(t)]$,

$$|h(t) - c_0 t| + |g(t) + c_0 t| \leq C,$$

$$\begin{cases} U(t,x) \succeq [1-\epsilon(t)]\big[\Phi^{c_0}(x - c_0 t + C) + \Phi^{c_0}(-x - c_0 t + C) - \mathbf{u}^*\big], \\ U(t,x) \preceq [1+\epsilon(t)] \min\big\{\Phi^{c_0}(x - c_0 t - C),\ \Phi^{c_0}(-x - c_0 t - C)\big\}, \end{cases}$$

where $\epsilon(t) := (t+\theta)^{-\alpha}$, *and* (c_0, Φ^{c_0}) *is the unique pair solving* (1.4) *and* (1.5) *obtained in Theorem A (iii), with* $\Phi^{c_0}(x)$ *extended by* $\mathbf{0}$ *for* $x > 0$.

Further estimates on $g(t)$ and $h(t)$ can be obtained if we narrow down more on the class of kernel functions $\{J_i : i = 1, ..., m_0\}$. We will write

$$\eta(t) \approx \xi(t) \quad \text{if} \quad C_1\xi(t) \leq \eta(t) \leq C_2\xi(t)$$

for some positive constants $C_1 \leq C_2$ and all t in the concerned range.

Our next two theorems are about kernel functions satisfying, for some $\gamma > 0$,

$(\hat{\mathbf{J}}^\gamma)$: $\quad J_i(x) \approx |x|^{-\gamma}$ for $|x| \gg 1$ and all $i \in \{1, ..., m_0\}$.

Note that for kernel functions satisfying $(\hat{\mathbf{J}}^\gamma)$, condition $(\mathbf{J})$ is satisfied only if $\gamma > 1$, and $(\mathbf{J_1})$ is satisfied only if $\gamma > 2$. The next result determines the orders of accelerated spreading when $\gamma \in (1, 2]$.

Theorem 1.2. *In Theorem B, if additionally the kernel functions satisfy* $(\hat{\mathbf{J}}^\gamma)$ *for some* $\gamma \in (1, 2]$, *then for* $t \gg 1$,

$$\begin{array}{lcll} -g(t),\ h(t) & \approx & t \ln t & \text{if } \gamma = 2, \\ -g(t),\ h(t) & \approx & t^{1/(\gamma-1)} & \text{if } \gamma \in (1,2). \end{array}$$

For kernel functions satisfying $(\hat{\mathbf{J}}^\gamma)$, clearly $(\mathbf{J}^\alpha)$ holds if and only if $\gamma > 1 + \alpha$. Therefore the case $\gamma > 3$ is already covered by Theorem 1.1. The following theorem is concerned with the remaining case $\gamma \in (2, 3]$, which indicates that the result in Theorem 1.1 is sharp.

Theorem 1.3. *In Theorem B, suppose additionally the kernel functions satisfy* $(\hat{\mathbf{J}}^\gamma)$ *for some* $\gamma \in (2, 3]$, F *is* C^2 *and*

$$F(v) - v[\nabla F(v)]^T \ggg \mathbf{0} \quad \text{for} \quad \mathbf{0} \lll v \preceq \mathbf{u}^*. \tag{1.6}$$

Then for $t \gg 1$,

$$\begin{array}{lcll} c_0 t + g(t),\ c_0 t - h(t) & \approx & \ln t & \text{if } \gamma = 3, \\ c_0 t + g(t),\ c_0 t - h(t) & \approx & t^{3-\gamma} & \text{if } \gamma \in (2,3). \end{array}$$

Note that $(\mathbf{f_2})$ implies

$$F(v) - v[\nabla F(v)]^T \succeq \mathbf{0} \quad \text{for } v \in [\mathbf{0}, \mathbf{u}^*].$$

Therefore (1.6) is a strengthened version of $(\mathbf{f_2})$. If we take $v = \mathbf{u}^*$ in (1.6), then it yields $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \lll \mathbf{0}$. When $m = 1$, (1.6) reduces to $F(v) > F'(v)v$ for $0 < v \leq \hat{u}$, which is satisfied, for example, by $F(v) = av - bv^p$ with $a, b > 0$ and $p > 1$.

The proofs of Theorems 1.1 and 1.3 rely on some of the following estimates on the semi-wave solutions of (1.3), which are of independent interests.

Theorem 1.4. *Suppose that* F *satisfies* $(\mathbf{f_1}) - (\mathbf{f_4})$ *and the kernel functions satisfy* $(\mathbf{J})$, *and* $\Phi(x) = (\phi_i(x))$ *is a monotone solution of* (1.4) *for some* $c > 0$. *Then the following conclusions hold:*

(i) *If* $(\mathbf{J}^\alpha)$ *holds for some* $\alpha > 0$, *then for every* $i \in \{1, ..., m\}$,

$$\int_{-\infty}^{-1} \big[u_i^* - \phi_i(x)\big]|x|^{\alpha-1}dx < \infty,$$

which implies, by the monotonicity of $\phi_i(x)$,

$$0 < u_i^* - \phi_i(x) \le C|x|^{-\alpha} \text{ for some } C > 0 \text{ and all } x < -1.$$

(ii) *If* $(\mathbf{J}^\alpha)$ *does not hold for some* $\alpha > 0$, *then*

$$\sum_{i=1}^{m} \int_{-\infty}^{-1} \big[u_i^* - \phi_i(x)\big]|x|^{\alpha-1}dx = \infty.$$

(iii) *If* $(\mathbf{J_2})$ *holds, then there exist positive constants* C *and* β *such that*

$$0 < u_i^* - \phi_i(x) \le Ce^{\beta x} \text{ for all } x < 0,\ i \in \{1, ..., m\}.$$

1.3. **Applications to epidemic models.** Let us now apply the results above to the models in [10] and [33].

The West Nile virus model in [10] is given by

$$(1.7)\quad \begin{cases} H_t = d_1\mathcal{L}_1[H](t,x) + a_1(e_1 - H)V - b_1H, & x \in (g(t), h(t)),\ t > 0, \\ V_t = d_2\mathcal{L}_2[V](t,x) + a_1(e_2 - V)H - b_2V, & x \in (g(t), h(t)),\ t > 0, \\ H(t,x) = V(t,x) = 0, & t > 0,\ x \in \{g(t), h(t)\}, \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x-y)V(t,x)\mathrm{d}y\mathrm{d}x, & t > 0, \\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(x-y)V(t,x)\mathrm{d}y\mathrm{d}x, & t > 0, \\ -g(0) = h(0) = h_0,\ H(0,x) = u_1^0(x),\ V(0,x) = u_2^0(x), & x \in [-h_0, h_0]. \end{cases}$$

where a_i, e_i and b_i $(i = 1, 2)$ are positive constants satisfying $a_1a_2e_1e_2 > b_1b_2$ (which is necessary for spreading to happen). We thus have

$$F(u) = F_1(u) := \Big(a_1(e_1 - u_1)u_2 - b_1u_1, a_2(e_2 - u_2)u_1 - b_2u_2\Big),$$

$$\mathbf{u}^* = \left(\frac{a_1a_2 - e_1e_2 - b_1b_2}{a_1a_2e_2 + a_2b_1}, \frac{a_1a_2 - e_1e_2 - b_1b_2}{a_1a_2e_1 + a_1b_2}\right).$$

It is straightforward to check that conditions $(\mathbf{f_1}) - (\mathbf{f_3})$ are satisfied by F_1 with $\hat{\mathbf{u}} = (e_1, e_2)$. Condition $(\mathbf{f_4})$ was shown to hold in [10]. It is also easy to see that F_1 is C^2 and

$$F_1(u) - u[\nabla F_1(u)]^T = (a_1u_1u_2, a_2u_1u_2).$$

Therefore (1.6) holds as well. Thus all our results apply to (1.7).

The epidemic model in [33] is given by

$$(1.8)\quad \begin{cases} u_t = d\mathcal{L}_1[u] - au + cv, & t > 0,\ x \in (g(t), h(t)), \\ v_t = -bv + G(u), & t > 0,\ x \in (g(t), h(t)), \\ u(t,x) = v(t,x) = 0, & t > 0,\ x = g(t) \text{ or } x = h(t), \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x-y)u(t,x)dydx, & t > 0, \\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J_1(x-y)u(t,x)dydx, & t > 0, \\ -g(0) = h(0) = h_0,\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in [-h_0, h_0], \end{cases}$$

where $a,\ b,\ c,\ d,\ \mu$ and h_0 are positive constants, and the function G is assumed to satisfy

(i) $G \in C^1([0,\infty)),\ G(0)=0,\ G'(z)>0$ for $z \geq 0$;
(ii) $\left[\frac{G(z)}{z}\right]' < 0$ for $z > 0$ and $\lim_{z\to+\infty} \frac{G(z)}{z} < \frac{ab}{c}$;
(iii) $G'(0) > \frac{ab}{c}$ (necessary for spreading to happen).

In this example,

$$F(u) = F_2(u) := (-au_1 + cu_2, G(u_1) - bu_2), \quad \mathbf{u}^* = (K_1, K_2)$$

where $(K_1, K_2) \succ\succ \mathbf{0}$ are uniquely determined by

$$\frac{G(K_1)}{K_1} = \frac{ab}{c}, \quad K_2 = \frac{G(K_1)}{b}.$$

One easily checks that F_2 satisfies $(\mathbf{f_1}) - (\mathbf{f_3})$ with $\hat{\mathbf{u}} = \infty$. In [33], it was proved that $(\mathbf{f_4})$ also holds. Clearly F_2 is C^2. However, $\mathbf{u}^*[\nabla F_2(\mathbf{u}^*)]^T \prec\prec \mathbf{0}$ does not hold. Therefore all our results apply to (1.8) except Theorems 1.1 and 1.3.

1.4. **Organisation of the paper.** The rest of the paper is organised as follows. In Section 2, we prove Theorems 1.1 and 1.4. The proof of the former is built on the proof and conclusions of the latter, where subtle analysis is used to find out the relationship between the behaviour of the semi-wave solution and that of the kernel functions.

Sections 3 and 4 are devoted to the proof of Theorems 1.2 and 1.3 for kernel functions behaving like $|x|^{-\gamma}$ near infinity. In Section 3, we completely determine the growth orders of $c_0 t - h(t)$ for γ in the range $(2, 3]$, while in Section 4, we completely determine the accelerated spreading orders of $h(t)$ when γ falls into the range $(1, 2]$. Note that when $\gamma > 3$, the spreading behaviour is already covered by the more general results in Section 2.

2. Sharper estimates for the semi-wave and spreading rate

2.1. **Asymptotic behaviour of semi-wave solutions to (1.3).** The purpose of this subsection is to prove the following three theorems, which imply Theorem 1.4.

Theorem 2.1. *Suppose that F satisfies $(\mathbf{f_1}) - (\mathbf{f_4})$ and the kernel functions satisfy $(\mathbf{J})$ and $(\mathbf{J}^\alpha)$ for some $\alpha > 0$. If $\Phi(x) = (\phi_i(x))$ is a monotone solution of (1.4) for some $c > 0$, then for every $i \in \{1, ..., m\}$,*

$$\int_{-\infty}^{-1} [u_i^* - \phi_i(x)]|x|^{\alpha-1} dx < \infty,$$

which implies, by the monotonicity of $\Phi(x)$,

$$0 < |x|^\alpha [u_i^* - \phi_i(x)] \leq C \text{ for some } C > 0 \text{ and all } x < 0, \ i \in \{1, ..., m\}.$$

Under the condition $(\mathbf{J})$, if the kernel functions satisfy $(\mathbf{J}^\alpha)$ for some $\alpha = \alpha_0 > 0$, then it is easily seen that $(\mathbf{J}^\alpha)$ is satisfied for all $\alpha \in [0, \alpha_0]$. Therefore if $(\mathbf{J}^\alpha)$ is satisfied for some but not for all $\alpha > 0$, then there exists $\alpha^* \in (0, \infty)$ such that the kernel functions satisfy $(\mathbf{J}^\alpha)$ if and only if $\alpha \in I^{\alpha^*} = [0, \alpha^*)$ or $[0, \alpha^*]$ (depending on whether or not $\mathbf{J}^{\alpha^*}$ is satisfied), namely

$$\begin{cases} \displaystyle\sum_{i=1}^{m_0} \int_0^\infty x^\alpha J_i(x) dx < \infty & \text{for } \alpha \in I^{\alpha^*}, \\ \displaystyle\sum_{i=1}^{m_0} \int_0^\infty x^\alpha J_i(x) dx = \infty & \text{for } \alpha \in (0, \infty) \setminus I^{\alpha^*}. \end{cases}$$

Therefore, by Theorem 2.1 we have

$$\sum_{i=1}^{m} \int_{-\infty}^{-1} [u_i^* - \phi_i(x)]|x|^{\alpha-1} dx < \infty \text{ for every } \alpha \in I^{\alpha^*}. \tag{2.1}$$

The next result shows that this estimate is sharp.

Theorem 2.2. *Suppose that F satisfies $(\mathbf{f_1})-(\mathbf{f_4})$ and the kernel functions satisfy $(\mathbf{J})$. If $(\mathbf{J}^\alpha)$ is not satisfied for some $\alpha>0$, and $\Phi(x)=(\phi_i(x))$ is a monotone solution of (1.4) for some $c>0$, then*

$$\sum_{i=1}^m \int_{-\infty}^{-1}[u_i^*-\phi_i(x)]|x|^{\alpha-1}dx=\infty. \tag{2.2}$$

Theorem 2.3. *Suppose that F satisfies $(\mathbf{f_1})-(\mathbf{f_4})$ and the kernel functions satisfy $(\mathbf{J})$. If $(\mathbf{J_2})$ holds, and $\Phi(x)=(\phi_i(x))$ is a monotone solution of (1.4) for some $c>0$, then there exist positive constants β and C such that*

$$0<u_i^*-\phi_i(x)\le Ce^{\beta x} \text{ for all } x<0,\ i\in\{1,...,m\}. \tag{2.3}$$

The following three lemmas play a crucial role in the proof of Theorem 2.1.

Lemma 2.4. *Suppose that $J(x)$ has the properties described in $(\mathbf{J})$ and satisfies $(\mathbf{J}^\alpha)$ for some $\alpha\ge 1$. If $\psi\in L^1((-\infty,0])$ is nonnegative, continuous and nondecreasing in $(-\infty,0]$, and*

$$\int_{-\infty}^0 |x|^\beta\psi(x)dx<\infty \text{ for some } \beta\ge 0, \tag{2.4}$$

then for any $\sigma\in(0,\min\{\beta+1,\alpha\}]$, there exists $C>0$ such that

$$I=I_M:=\int_{-M}^0|x|^\sigma\left[\int_{-\infty}^0 J(x-y)\psi(y)dy-\psi(x)\right]dx\in[-C,C] \text{ for all } M>0.$$

Proof. For fixed $M>0$ we have

$$\begin{aligned}
&\int_{-M}^0\int_{-\infty}^0|x|^\sigma J(x-y)\psi(y)dydx\\
&=\int_0^M\int_{-\infty}^x x^\sigma J(y)\psi(y-x)dydx\\
&=\int_0^M\int_{-\infty}^0 x^\sigma J(y)\psi(y-x)dydx+\int_0^M\int_0^x x^\sigma J(y)\psi(y-x)dydx\\
&=\int_{-\infty}^0\int_0^M x^\sigma J(y)\psi(y-x)dxdy+\int_0^M\int_y^M x^\sigma J(y)\psi(y-x)dxdy\\
&=\int_{-\infty}^0\int_{-y}^{M-y}(x+y)^\sigma J(y)\psi(-x)dxdy+\int_0^M\int_0^{M-y}(x+y)^\sigma J(y)\psi(-x)dxdy,
\end{aligned}$$

and

$$\int_{-M}^0|x|^\sigma\psi(x)dx=\int_{\mathbb{R}}\int_0^M x^\sigma J(y)\psi(-x)dxdy.$$

Therefore we can write

$$I=\sum_{j=1}^3 I_j$$

with

$$\begin{aligned}
I_1:=&\int_{-\infty}^0\int_{-y}^{M-y}\left[(x+y)^\sigma-x^\sigma\right]J(y)\psi(-x)dxdy\\
&+\int_0^M\int_0^{M-y}\left[(x+y)^\sigma-x^\sigma\right]J(y)\psi(-x)dxdy,\\
I_2:=&\int_{-\infty}^0\int_{-y}^{M-y}x^\sigma J(y)\psi(-x)dxdy-\int_{-\infty}^0\int_0^M x^\sigma J(y)\psi(-x)dxdy
\end{aligned}$$

$$= \int_{-\infty}^{0} \int_{M}^{M-y} x^\sigma J(y)\psi(-x)dxdy - \int_{-\infty}^{0} \int_{0}^{-y} x^\sigma J(y)\psi(-x)dxdy,$$

$$I_3 := -\int_{0}^{M} \int_{M-y}^{M} x^\sigma J(y)\psi(-x)dxdy - \int_{M}^{\infty} \int_{0}^{M} x^\sigma J(y)\psi(-x)dxdy.$$

To estimate I_1 we will make use of some elementary inequalities. If $s, t > 0$ and $\sigma \in (0, 1]$, then it is easily checked that

$$(s+t)^\sigma - s^\sigma \le t^\sigma. \tag{2.5}$$

If $\sigma = n + \theta$ with $n \ge 1$ an integer, and $\theta \in (0, 1]$, then by the mean value theorem

$$\begin{aligned}
(s+t)^\sigma - s^\sigma &= \sigma(s+\zeta t)^{\sigma-1}t \le \sigma t(s+t)^{\sigma-1} = \sigma t s^{\sigma-1} + \sigma t\left[(s+t)^{\sigma-1} - s^{\sigma-1}\right] \\
&\le \sum_{k=1}^{n}\left[\Pi_{j=0}^{k-1}(\sigma-j)t^k s^{\sigma-k}\right] + \Pi_{j=0}^{n-1}(\sigma-j)t^n\left[(s^\theta + t^\theta) - s^\theta\right] \\
&\le \sum_{k=1}^{n}\left[\Pi_{j=0}^{k-1}(\sigma-j)t^k s^{\sigma-k}\right] + \Pi_{j=0}^{n-1}(\sigma-j)t^{n+\theta} \\
&= \sum_{k=1}^{n} c_k t^k s^{\sigma-k} + c_{n+1}t^\sigma
\end{aligned}$$

where $\zeta \in [0, 1]$, and $c_k = c_k(\sigma) > 0$ for $k \in \{1, ..., n+1\}$.

Applying this inequality to $(x+y)^\sigma - x^\sigma$ with $x + y > 0$ and $x > 0$, we obtain, for the case $\sigma > 1$,

$$|(x+y)^\sigma - x^\sigma| \le \sum_{k=1}^{n} c_k |y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma$$

with $\sigma - n = \theta \in (0, 1]$ and $n \ge 1$ an integer, $c_k = c_k(\sigma) > 0$ for $k \in \{1, ..., n+1\}$.

Therefore, in the case $\sigma > 1$,

$$\begin{aligned}
|I_1| \le\ & \int_{-\infty}^{0} \int_{-y}^{M-y} \left[\sum_{k=1}^{n} c_k |y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma\right] J(y)\psi(-x)dxdy \\
&+ \int_{0}^{M} \int_{0}^{M-y} \left[\sum_{k=1}^{n} c_k |y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma\right] J(y)\psi(-x)dxdy \\
\le\ & 2\sum_{k=1}^{n} c_k \int_{0}^{\infty} x^{\sigma-k}\psi(-x)dx \int_{0}^{\infty} y^k J(y)dy + 2c_{n+1}\int_{0}^{\infty}\psi(-x)dx \int_{0}^{\infty} y^\sigma J(y)dy \\
:=\ & C_1.
\end{aligned}$$

Since $1 \le k \le n < \sigma \le \min\{\beta + 1, \alpha\}$, by the assumptions on J and ψ we see that C_1 is a finite number.

If $\sigma \in (0, 1]$, then

$$\begin{aligned}
|I_1| &\le \int_{-\infty}^{0} \int_{-y}^{M-y} |y|^\sigma J(y)\psi(-x)dxdy + \int_{0}^{M} \int_{0}^{M-y} |y|^\sigma J(y)\psi(-x)dxdy \\
&\le 2\int_{0}^{\infty}\psi(-x)dx \int_{0}^{\infty} y^\sigma J(y)dy := \tilde{C}_1 < \infty.
\end{aligned}$$

Since $\psi(x)$ is nondecreasing, from (2.4) we easily deduce

$$\psi(-x) \le \frac{M_1}{x^\sigma} \text{ for some } M_1 > 0 \text{ and all } x > 1.$$

Similarly, using $(\mathbf{J}^\alpha)$ we obtain

$$M\int_M^\infty J(y)dy \le M^{1-\alpha}\int_M^\infty y^\alpha J(y)dy \le \int_1^\infty y^\alpha J(y)dy := M_2 \text{ for } M\ge 1,$$

and hence

$$M\int_M^\infty J(y)dy \le \min\left\{\int_0^\infty J(y)dy, M_2\right\} := M_3 < \infty \text{ for all } M>0.$$

Therefore

$$\begin{aligned}|I_2| &\le \int_{-\infty}^0\int_M^{M-y} M_1 J(y)dxdy + \int_{-\infty}^0\int_0^{-y} M_1 J(y)dxdy\\ &= 2M_1\int_0^\infty yJ(y)dy := C_2<\infty,\end{aligned}$$

and

$$\begin{aligned}|I_3| &\le \int_0^M M_1 yJ(y)dy + \int_M^\infty M_1 M J(y)dy\\ &\le M_1\int_0^\infty yJ(y)dy + M_1M_3 := C_3 < \infty.\end{aligned}$$

We thus have

$$|I| \le C_1 + \tilde C_1 + C_2 + C_3 := C < \infty \text{ for all } M>0.$$

The proof is complete. □

Lemma 2.5. *Suppose that $J(x)$ has the properties described in $(\mathbf{J})$ and satisfies $(\mathbf{J}^\alpha)$ for some $\alpha\in(0,1)$. Let ψ be nonnegative, continuous and nondecreasing in $(-\infty,0]$. Then there exists $C>0$ such that*

$$S = S_M := \int_{-M}^0 |x|^{\alpha-1}\left[\int_{-\infty}^0 J(x-y)\psi(y)dy - \psi(x)\right]dx \le C \textit{ for all } M>0.$$

Proof. As in the proof of Lemma 2.4, we deduce for fixed $M>0$ and $\sigma>-1$,

$$\begin{aligned}&\int_{-M}^0\int_{-\infty}^0 |x|^\sigma J(x-y)\psi(y)dydx\\ &= \int_{-\infty}^0\int_{-y}^{M-y}(x+y)^\sigma J(y)\psi(-x)dxdy + \int_0^M\int_0^{M-y}(x+y)^\sigma J(y)\psi(-x)dxdy.\end{aligned}$$

and

$$\int_{-M}^0 |x|^\sigma\psi(x)dx = \int_{\mathbb{R}}\int_0^M |x|^\sigma J(y)\psi(-x)dxdy.$$

Hence

$$S = \sum_{i=1}^3 \tilde I_i$$

with

$$\begin{aligned}\tilde I_1 &:= \int_{-\infty}^0\int_{-y}^{M-y}\left[(x+y)^\sigma - x^\sigma\right]J(y)\psi(-x)dxdy\\ &\quad + \int_0^M\int_0^{M-y}\left[(x+y)^\sigma - x^\sigma\right]J(y)\psi(-x)dxdy,\\ \tilde I_2 &:= \int_{-\infty}^0\int_M^{M-y} x^\sigma J(y)\psi(-x)dxdy - \int_{-\infty}^0\int_0^{-y} x^\sigma J(y)\psi(-x)dxdy,\end{aligned}$$

$$\tilde{I}_3 := -\int_0^M \int_{M-y}^M x^\sigma J(y)\psi(-x)dxdy - \int_M^\infty \int_0^M x^\sigma J(y)\psi(-x)dxdy.$$

Take $\sigma = \alpha - 1$. It is clear that $\tilde{I}_3 \le 0$. For $\tilde{I}_1$, since $\sigma < 0$,

$$(x+y)^\sigma - x^\sigma < 0 \text{ when } x > 0 \text{ and } y > 0,$$

and hence, by ($\mathbf{J}^\alpha$) and $\sigma + 1 = \alpha \in (0,1)$,

$$\begin{aligned}\tilde{I}_1 \le & \int_{-\infty}^0 \int_{-y}^{M-y} [(x+y)^\sigma - x^\sigma] J(y)\psi(-x)dxdy \\ \le & \psi(0) \int_{-\infty}^0 \int_{-y}^{M-y} [(x+y)^\sigma - x^\sigma] J(y)dxdy \\ = & \frac{\psi(0)}{\sigma+1} \int_{-\infty}^0 [M^{\sigma+1} - (M-y)^{\sigma+1} + (-y)^{\sigma+1}] J(y)\mathrm{d}y \\ \le & \frac{\psi(0)}{\sigma+1} \int_{-\infty}^0 (-y)^{\sigma+1} J(y)\mathrm{d}y = \frac{\psi(0)}{\sigma+1} \int_0^\infty y^{\sigma+1} J(y)\mathrm{d}y := C_1 < \infty.\end{aligned}$$

Moreover, by ($\mathbf{J}^\alpha$), $\sigma + 1 = \alpha \in (0,1)$ and (2.5),

$$\begin{aligned}\tilde{I}_2 \le & \int_{-\infty}^0 \int_M^{M-y} x^\sigma J(y)\psi(-x)\mathrm{d}x\mathrm{d}y \le \psi(0) \int_{-\infty}^0 \int_M^{M-y} x^\sigma J(y)\mathrm{d}x\mathrm{d}y \\ = & \frac{\psi(0)}{\sigma+1} \int_{-\infty}^0 [(M-y)^{\sigma+1} - M^{\sigma+1}] J(y)\mathrm{d}y \\ \le & \frac{\psi(0)}{\sigma+1} \int_0^\infty y^{\sigma+1} J(y)\mathrm{d}y := C_2 < \infty.\end{aligned}$$

Therefore,

$$S \le C_1 + C_2 := C < \infty \text{ for all } M > 0.$$

The proof is complete. □

Denote

$$\Psi(x) = (\psi_i(x)) := \mathbf{u}^* - \Phi(x) \text{ and } G(u) = (g_i(u)) := -F(\mathbf{u}^* - u).$$

Then Ψ satisfies

$$\text{(2.6)} \qquad \begin{cases} \mathbf{0} = D \circ \int_{-\infty}^0 \mathbf{J}(x-y) \circ \Psi(y)\mathrm{d}y - D \circ \Psi + D \circ \mathbf{u}^* \circ \int_0^\infty \mathbf{J}(x-y)\mathrm{d}y \\ \qquad + c\Psi'(x) + G(\Psi(x)) \text{ for } -\infty < x < 0, \\ \Psi(-\infty) = \mathbf{0}, \quad \Psi(0) = \mathbf{u}^*. \end{cases}$$

Since $\mathbf{u}^*$ is stable and $\nabla F(\mathbf{u}^*) = \nabla G(\mathbf{0})$ is invertible, the eigenvalues of $\nabla F(\mathbf{u}^*)$ are all negative. Therefore we can use the same reasoning as in the proof of Lemma **??** to find two vectors $\widetilde{A} = (\tilde{a}_i) \gg \mathbf{0}$ and $\widetilde{B} = (\tilde{b}_i) \ll \mathbf{0}$ such that, for $U = (u_i) \in [\mathbf{0}, \epsilon\mathbf{1}]$ with $\epsilon > 0$ sufficiently small,

$$\sum_{i=1}^m \tilde{a}_i g_i(U) \le \sum_{i=1}^m \tilde{b}_i u_i \le -\hat{b} \sum_{j=1}^m \tilde{a}_i u_i,$$

for some $\hat{b} > 0$.

Since $\Psi(-\infty) = \mathbf{0}$ and $\Psi(x) = (\psi_i(x)) \gg \mathbf{0}$ for $x < 0$, we have $0 < \psi_i(x) < \epsilon$ for $x \ll -1$, and so

$$\text{(2.7)} \qquad \sum_{i=1}^m \tilde{a}_i g_i(\Psi(x)) \le -\hat{b}\widetilde{\psi}(x) \quad \text{for } x \ll -1, \text{ with}$$

$$\widetilde{\psi}(x) := \sum_{j=1}^{m} \tilde{a}_j \psi_j(x). \tag{2.8}$$

Lemma 2.6. *Suppose* $(\mathbf{J})$ *and* $(\mathbf{f_1}) - (\mathbf{f_4})$ *are satisfied. If* $(\mathbf{J}^\alpha)$ *holds for some* $\alpha \geq 1$*, then*

$$\int_{-\infty}^{0} \widetilde{\psi}(x)\mathrm{d}x < \infty.$$

Proof. A simple calculation gives

$$\begin{aligned} &D \circ \int_{-\infty}^{0} \mathbf{J}(x-y) \circ \Psi(y)\mathrm{d}y - D \circ \Psi + D \circ \mathbf{u}^* \circ \int_{0}^{\infty} \mathbf{J}(x-y)\mathrm{d}y \\ =& - D \circ \int_{-\infty}^{0} \mathbf{J}(x-y) \circ \Phi(y)\mathrm{d}y + D \circ \Phi. \end{aligned}$$

Integrating the equation satisfied by $\widetilde{\psi}$ over the interval (x, y) with $x < y \ll -1$, and making use of (2.7), we obtain

$$\begin{aligned} &c(\widetilde{\psi}(y) - \widetilde{\psi}(x)) + \sum_{i=1}^{m} \int_{x}^{y} \tilde{a}_i d_i \left[\int_{-\infty}^{0} J_i(z-w)\psi_i(w)\mathrm{d}w - \psi_i(z) \right] \mathrm{d}z \\ &+ \sum_{i=1}^{m} \int_{x}^{y} \tilde{a}_i d_i u_i^* \int_{0}^{\infty} J_i(z-w)\mathrm{d}w\mathrm{d}z \\ =&c(\widetilde{\psi}(y) - \widetilde{\psi}(x)) - \sum_{i=1}^{m} \int_{x}^{y} \tilde{a}_i d_i \left[\int_{-\infty}^{0} J_i(z-w)\phi_i(w)\mathrm{d}w - \phi_i(z) \right] \mathrm{d}z \\ =& - \int_{x}^{y} \sum_{i=1}^{m} \tilde{a}_i g_i(\Psi(z))\mathrm{d}z \geq \hat{b} \int_{x}^{y} \widetilde{\psi}(z)\mathrm{d}z. \end{aligned}$$

We extend Φ to $\mathbb{R}$ by define $\phi_i(x) = 0$ for $x > 0$. Then the new function Φ is differentiable on $\mathbb{R}$ except at $x = 0$. Due to $(\mathbf{J}^\alpha)$, we have, for $i \in \{1, ..., m_0\}$,

$$\begin{aligned} &\left| \int_{x}^{y} \left(\int_{-\infty}^{0} J_i(z-w)\phi_i(w)\mathrm{d}w - \phi_i(z) \right) \mathrm{d}z \right| = \left| \int_{x}^{y} \left(\int_{\mathbb{R}} J_i(z-w)\phi_i(w)\mathrm{d}w - \phi_i(z) \right) \mathrm{d}z \right| \\ =& \left| \int_{x}^{y} \int_{\mathbb{R}} J_i(w)(\phi_i(z+w) - \phi(z))\mathrm{d}w\mathrm{d}z \right| = \left| \int_{x}^{y} \int_{\mathbb{R}} J_i(w) \int_{0}^{1} w\phi_i'(z+sw)\mathrm{d}s\mathrm{d}w\mathrm{d}z \right| \\ =& \left| \int_{\mathbb{R}} wJ_i(w) \int_{0}^{1} [\phi_i(y+sw) - \phi_i(x+sw)]\mathrm{d}s\mathrm{d}w \right| \\ \leq& \ a_i^* \int_{\mathbb{R}} |y|J_i(y)\mathrm{d}y =: M_i < \infty. \end{aligned}$$

Thus, for $x < y \ll -1$,

$$\hat{b} \int_{x}^{y} \widetilde{\psi}(z)\mathrm{d}z \leq c(\widetilde{\psi}(y) - \widetilde{\psi}(x)) + \sum_{i=1}^{m} \tilde{a}_i d_i M_i \leq \sum_{i=1}^{m} \tilde{a}_i (c u_i^* + d_i M_i),$$

which implies $\int_{-\infty}^{0} \widetilde{\psi}(z)\mathrm{d}z < \infty$. □

Proof of Theorem 2.1: **Case 1**. $\alpha \geq 1$.

With $\widetilde{\psi} = \sum_{i=1}^{m} \tilde{a}_i \psi_i$ given by (2.8), it suffices to show

$$\int_{-\infty}^{0} \widetilde{\psi}(x)|x|^{\alpha-1}\mathrm{d}x < \infty.$$

By Lemma 2.6 we have

$$\int_{-\infty}^{0} \tilde{\psi}(x)dx < \infty \text{ and hence } \int_{-\infty}^{0} \psi_i(x)dx < \infty \text{ for } i \in \{1, ..., m\}.$$

So there is nothing to prove if $\alpha = 1$, and we only need to consider the case $\alpha > 1$.

Suppose $\alpha > 1$ and

$$\int_{-\infty}^{0} |x|^{\gamma}\tilde{\psi}(x)dx < \infty \text{ for some } \gamma \geq 0. \tag{2.9}$$

Then by Lemma 2.4, for any β satisfying $0 < \beta \leq \min\{\gamma+1, \alpha\}$, and $i \in \{1, ..., m_0\}$,

$$\int_{-M}^{0} \left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x)\right] |x|^{\beta}\mathrm{d}x \leq C \text{ for some } C > 0 \text{ and all } M > 0. \tag{2.10}$$

Moreover, if we fix $M_0 > 1$ so that (2.7) holds for $x \leq -M_0$, then for $M > M_0$ and β as above, we have

$$\begin{aligned}
&\hat{b}\int_{-M}^{-M_0} \tilde{\psi}(x)|x|^{\beta}dx \\
&\leq -\sum_{i=1}^{m}\int_{-M}^{-M_0} \tilde{a}_i g_i(\Psi(x))|x|^{\beta}dx \\
&= c\int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^{\beta}dx + \sum_{i=1}^{m_0}\tilde{a}_i d_i\int_{-M}^{-M_0}\left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x)\right]|x|^{\beta}\mathrm{d}x \\
&\quad + \sum_{i=1}^{m_0}\tilde{a}_i d_i u_i^*\int_{-M}^{-M_0}\int_{0}^{\infty}|x|^{\beta}J_i(x-y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

By (2.10),

$$\begin{aligned}
&\sum_{i=1}^{m_0}\tilde{a}_i d_i\int_{-M}^{-M_0}\left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x)\right]|x|^{\beta}\mathrm{d}x \\
&\leq C\sum_{i=1}^{m_0}\tilde{a}_i d_i - \sum_{i=1}^{m_0}\tilde{a}_i d_i\int_{-M_0}^{0}\left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x)\right]|x|^{\beta}\mathrm{d}x \\
&:= C_1 < \infty \text{ for all } M > M_0.
\end{aligned}$$

Moreover, if we assume additionally that $\beta \leq \alpha - 1$, then we have, for $i \in \{1, ..., m_0\}$,

$$\begin{aligned}
&\int_{-M}^{-M_0}\int_{0}^{\infty}|x|^{\beta}J_i(x-y)\mathrm{d}y\mathrm{d}x \\
&\leq \int_{0}^{M}\int_{0}^{\infty}x^{\beta}J_i(x+y)dydx = \int_{0}^{M}\int_{x}^{\infty}x^{\beta}J_i(y)dydx \\
&\leq \int_{0}^{\infty}\int_{x}^{\infty}x^{\beta}J_i(y)dydx = \frac{1}{\beta+1}\int_{0}^{\infty}y^{\beta+1}J_i(y)dy := C_2 < \infty.
\end{aligned}$$

Therefore, for $\beta \in (0, \min\{\gamma+1, \alpha-1\}]$ and $M > M_0$,

$$\begin{aligned}
&\hat{b}\int_{-M}^{-M_0}\tilde{\psi}(x)|x|^{\beta}dx \leq \ c\int_{-M}^{-M_0}\tilde{\psi}'(x)|x|^{\beta}dx + C_1 + \sum_{i=1}^{m}\tilde{a}_i d_i u_i^* C_2 \\
&\leq c\int_{1}^{M}x^{\beta}\tilde{\psi}'(-x)dx + C_3 \leq c\int_{1}^{M}x^{\gamma+1}\tilde{\psi}'(-x)dx + C_3 \\
&\leq c\tilde{\psi}(-1) + c\int_{1}^{M}(\gamma+1)x^{\gamma}\tilde{\psi}(-x)dx + C_3 := C_4 < \infty \text{ by (2.9)}.
\end{aligned}$$

It follows that

$$\int_{-\infty}^{0} \tilde{\psi}(x)|x|^{\beta}dx < \infty. \tag{2.11}$$

Thus we have proved that (2.9) implies (2.11) for any $\beta \in (0, \min\{\gamma+1, \alpha-1\}]$.

If we write $\alpha - 1 = n + \theta$ with $n \geq 0$ an integer and $\theta \in (0, 1]$. Then by the above conclusion and an induction argument we see that (2.11) holds with $\beta = n$. Thus (2.9) holds for $\gamma = n$. So applying the above conclusion once more we see that (2.11) holds for every $\beta \in (0, \min\{n + 1, \alpha - 1\}] = (0, \alpha - 1]$, as desired.

Case 2. $\alpha \in (0, 1)$.

Let $\beta = \alpha - 1$. As in Case 1, for $M > M_0$,

$$\begin{aligned}
&\hat{b}\int_{-M}^{-M_0} \tilde{\psi}(x)|x|^{\beta}dx \\
&\leq c\int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^{\beta}dx + \sum_{i=1}^{m_0} \tilde{a}_i d_i \int_{-M}^{-M_0} \left[\int_{-\infty}^{0} J_i(x-y)\psi_i(y)\mathrm{d}y - \psi_i(x)\right]|x|^{\beta}\mathrm{d}x \\
&\quad + \sum_{i=1}^{m_0} \tilde{a}_i d_i u_i^* \int_{-M}^{-M_0}\int_{0}^{\infty} |x|^{\beta} J_i(x-y)\mathrm{d}y\mathrm{d}x \\
&\leq c\int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^{\beta}dx + \widetilde{C}_1 + \sum_{i=1}^{m_0} \tilde{a}_i d_i u_i^* \int_{-M}^{-M_0}\int_{0}^{\infty} |x|^{\beta} J_i(x-y)\mathrm{d}y\mathrm{d}x,
\end{aligned}$$

where $\widetilde{C}_1 > 0$ is obtained by making use of Lemma 2.5. By $(\mathbf{J}^{\alpha})$ and $\beta + 1 = \alpha$,

$$\begin{aligned}
&\int_{-M}^{-M_0}\int_{0}^{\infty} |x|^{\beta} J_i(x-y)\mathrm{d}y\mathrm{d}x \leq \int_{0}^{\infty}\int_{x}^{\infty} x^{\beta} J_i(y)\mathrm{d}y\mathrm{d}x \\
=&\frac{1}{\alpha}\int_{0}^{\infty} y^{\alpha} J_i(y)\mathrm{d}y := \widetilde{C}_2 < \infty.
\end{aligned}$$

Due to $\beta < 0$, we have

$$\begin{aligned}
&\int_{-M}^{-M_0} \tilde{\psi}'(x)|x|^{\beta}dx = \int_{M_0}^{M} \tilde{\psi}'(-x)x^{\beta}dx \\
=&\tilde{\psi}(-M_0)M_0^{\beta} - \tilde{\psi}(-M)M^{\beta} + \beta\int_{M_0}^{M} \tilde{\psi}(-x)x^{\beta-1}dx \\
\leq&\tilde{\psi}(-M_0)M_0^{\beta} := \widetilde{C}_3 < \infty.
\end{aligned}$$

Hence

$$\hat{b}\int_{-M}^{-M_0} \tilde{\psi}(x)|x|^{\beta}dx \leq \widetilde{C}_1 + \widetilde{C}_2\sum_{i=1}^{m_0} \tilde{a}_i d_i u_i^* + c\widetilde{C}_3 < \infty$$

for all $M > M_0$, which implies

$$\int_{-\infty}^{-1} \tilde{\psi}(x)|x|^{\alpha-1}dx < \infty.$$

The proof is completed.

□

Proof of Theorem 2.2: We have

$$|g_i(\Psi(x))| \leq L\sum_{j=1}^{m} \psi_j(x) := L\hat{\psi}(x) \text{ for some } L > 0 \text{ and all } x < 0,\ i \in \{1, ..., m\}.$$

Now for $M>1$ and $\beta=\alpha-1$,

$$
\begin{aligned}
L\int_{-M}^{-1}\hat\psi(x)|x|^{\beta}dx &\geq -\sum_{i=1}^{m}\int_{-M}^{-1} g_i(\Psi(x))|x|^{\beta}dx\\
&= c\int_{-M}^{-1}\hat\psi'(x)|x|^{\beta}dx+\sum_{i=1}^{m_0}d_i\int_{-M}^{-1}\left[\int_{-\infty}^{0}J_i(x-y)\psi_i(y)\mathrm{d}y-\psi_i(x)\right]|x|^{\beta}\mathrm{d}x\\
&\quad+\sum_{i=1}^{m_0}d_iu_i^*\int_{-M}^{-1}\int_0^{\infty}|x|^{\beta}J_i(x-y)\mathrm{d}y\mathrm{d}x\\
&\geq -\sum_{i=1}^{m_0}d_i\int_{-M}^{-1}\psi_i(x)|x|^{\beta}\mathrm{d}x+\sum_{i=1}^{m_0}d_iu_i^*\int_{-M}^{-1}\int_0^{\infty}|x|^{\beta}J_i(x-y)\mathrm{d}y\mathrm{d}x
\end{aligned}
$$

Therefore, with $\widetilde L:=L+\sum_{i=1}^{m_0}d_i$, we have

$$
\begin{aligned}
\widetilde L\int_{-M}^{-1}\hat\psi(x)|x|^{\beta}dx &\geq \sum_{i=1}^{m_0}d_iu_i^*\int_{-M}^{-1}\int_0^{\infty}|x|^{\beta}J_i(x-y)\mathrm{d}y\mathrm{d}x\\
&=\sum_{i=1}^{m_0}d_iu_i^*\int_1^{M}\int_x^{\infty}x^{\beta}J_i(y)dydx\\
&=\sum_{i=1}^{m_0}d_iu_i^*\Big[\int_1^{M}\int_1^{\infty}-\int_1^{M}\int_1^{x}\Big]x^{\beta}J_i(y)dydx\\
&=\sum_{i=1}^{m_0}\frac{d_iu_i^*}{\beta+1}\Big[\int_1^{\infty}(M^{\beta+1}-1)J_i(y)dy+\int_1^{M}(y^{\beta+1}-M^{\beta+1})J_i(y)dy\Big]\\
&\geq\sum_{i=1}^{m_0}\frac{d_iu_i^*}{\beta+1}\Big[\int_1^{M}y^{\beta+1}J_i(y)dy-\int_1^{\infty}J_i(y)dy\Big]\to\infty \text{ as } M\to\infty,
\end{aligned}
$$

since $\beta+1=\alpha$. Therefore (2.2) holds, as we wanted. □

To prove Theorem 2.3, we need the following lemma.

Lemma 2.7. *Let the assumptions in Theorem 2.3 be satisfied and $\Psi(x)=(\psi_i(x))=:\mathbf{u}^*-\Phi(x)$. Then for every small $\epsilon>0$, there exist $\beta=\beta(\epsilon)\in(0,\lambda]$ and $C=C(\epsilon)>0$ such that for all $M>0$ and $i\in\{1,...,m\}$,*

$$
Q^{(i)}=Q^{(i)}_M:=\int_{-M}^{0}e^{-\beta x}\int_{-\infty}^{0}J_i(x-y)\psi_i(y)\mathrm{d}y\mathrm{d}x\leq(1+\epsilon)\int_{-M}^{0}e^{-\beta x}\psi_i(x)\mathrm{d}x+C. \tag{2.12}
$$

Proof. By a change of variables, we deduce

$$
\begin{aligned}
Q^{(i)}&=\int_{-M}^{0}e^{-\beta x}\int_{-\infty}^{-x}J_i(y)\psi_i(x+y)\mathrm{d}y\mathrm{d}x=\int_0^{M}\int_{-\infty}^{x}e^{\beta x}J_i(y)\psi_i(y-x)\mathrm{d}y\mathrm{d}x\\
&=\int_0^{M}\left(\int_{-\infty}^{0}+\int_0^{x}\right)e^{\beta x}J_i(y)\psi_i(y-x)\mathrm{d}y\mathrm{d}x\\
&=\int_{-\infty}^{0}\int_0^{M}e^{\beta x}J_i(y)\psi_i(y-x)\mathrm{d}x\mathrm{d}y+\int_0^{M}\int_y^{M}e^{\beta x}J_i(y)\psi_i(y-x)\mathrm{d}x\mathrm{d}y\\
&=\int_{-\infty}^{0}e^{\beta y}J_i(y)\int_{-y}^{M-y}e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y+\int_0^{M}e^{\beta y}J_i(y)\int_0^{M-y}e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
&:=I+II.
\end{aligned}
$$

We have

$$\begin{aligned}
I=&\int_{-M}^{0} e^{\beta y}J_i(y)\int_{-y}^{M-y} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y+\int_{-\infty}^{-M} e^{\beta y}J_i(y)\int_{-y}^{M-y} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
=&\int_{-M}^{0} e^{\beta y}J_i(y)\left(\int_{-y}^{M}+\int_{M}^{M-y}\right) e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y+\int_{-\infty}^{-M} e^{\beta y}J_i(y)\int_{-y}^{M-y} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
=&\int_{-M}^{0} e^{\beta y}J_i(y)\int_{-y}^{M} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y+\int_{-M}^{0} e^{\beta y}J_i(y)\int_{M}^{M-y} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
&+\int_{-\infty}^{-M} e^{\beta y}J_i(y)\int_{-y}^{M-y} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
:=&B_1^{(i)}+A_1^{(i)}+A_2^{(i)},
\end{aligned}$$

and

$$\begin{aligned}
II=&\int_{0}^{M} e^{\beta y}J_i(y)\int_{0}^{M} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y-\int_{0}^{M} e^{\beta y}J_i(y)\int_{M-y}^{M} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
:=&B_2^{(i)}+A_3^{(i)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
Q^{(i)}=&I+II=(B_1^{(i)}+B_2^{(i)})+(A_1^{(i)}+A_2^{(i)}+A_3^{(i)})\\
\le&\int_{-M}^{0} e^{\beta y}J_i(y)\int_{0}^{M} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y+\int_{0}^{M} e^{\beta y}J_i(y)\int_{0}^{M} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
&+(A_1^{(i)}+A_2^{(i)}+A_3^{(i)})\\
=&\int_{-M}^{M} e^{\beta y}J_i(y)\mathrm{d}y\int_{0}^{M} e^{\beta x}\psi_i(-x)\mathrm{d}x+(A_1^{(i)}+A_2^{(i)}+A_3^{(i)}).
\end{aligned}$$

Set

$$P(\gamma):=\int_{\mathbb{R}} e^{\gamma y}J_i(y)\mathrm{d}y=\int_0^\infty [e^{\gamma y}+e^{-\gamma y}]J_i(y)dy.$$

Clearly $P(\gamma)$ is increasing and continuous in $\gamma\in[0,\alpha]$, with $P(0)=1$. Hence there exists small $\beta_*=\beta_*(\epsilon)\in(0,\lambda]$ such that for all $0<\beta\le\beta_*(\epsilon)$,

$$P(\beta)=\int_{\mathbb{R}} e^{\beta y}J_i(y)\mathrm{d}y\le 1+\epsilon.$$

Thus, for such β,

$$Q^{(i)}\le(1+\epsilon)\int_0^M e^{\beta x}\psi_i(-x)\mathrm{d}x+(A_1^{(i)}+A_2^{(i)}+A_3^{(i)}).$$

It remains to verify that $A_1^{(i)}+A_2^{(i)}+A_3^{(i)}$ has an upper bound which is independent of $M\in(0,\infty)$. Using the monotonicity of ψ_i, we deduce

$$\begin{aligned}
A_1^{(i)}+A_3^{(i)}=&\int_{-M}^{0} e^{\beta y}J_i(y)\int_{M}^{M-y} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y-\int_{0}^{M} e^{\beta y}J_i(y)\int_{M-y}^{M} e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\\
\le&\psi_i(-M)\int_{-M}^{0} e^{\beta y}J_i(y)\int_{M}^{M-y} e^{\beta x}\mathrm{d}x\mathrm{d}y-\psi_i(-M)\int_{0}^{M} e^{\beta y}J_i(y)\int_{M-y}^{M} e^{\beta x}\mathrm{d}x\mathrm{d}y\\
=&\frac{\psi_i(-M)}{\beta}\int_{-M}^{0} e^{\beta y}J_i(y)[e^{\beta(M-y)}-e^{\beta M}]\mathrm{d}y-\frac{\psi_i(-M)}{\beta}\int_{0}^{M} e^{\beta y}J_i(y)[e^{\beta M}-e^{\beta(M-y)}]\mathrm{d}y\\
=&\frac{\psi_i(-M)e^{\beta M}}{\beta}\int_{-M}^{0} J_i(y)[1-e^{\beta y}]\mathrm{d}y-\frac{\psi_i(-M)e^{\beta M}}{\beta}\int_{0}^{M} J_i(y)[e^{\beta y}-1]\mathrm{d}y
\end{aligned}$$

$$=\frac{\psi_i(-M)e^{\beta M}}{\beta}\int_0^M J_i(y)[2-e^{-\beta y}-e^{\beta y}]\mathrm{d}y\leq 0,$$

and

$$\begin{aligned}A_2^{(i)}=&\int_{-\infty}^{-M}e^{\beta y}J_i(y)\int_{-y}^{M-y}e^{\beta x}\psi_i(-x)\mathrm{d}x\mathrm{d}y\leq\psi_i(0)\int_{-\infty}^{-M}e^{\beta y}J_i(y)\int_{-y}^{M-y}e^{\beta x}\mathrm{d}x\mathrm{d}y\\ =&\frac{u_i^*}{\beta}\int_{-\infty}^{-M}e^{\beta y}J_i(y)[e^{\beta(M-y)}-e^{-\beta y}]\mathrm{d}y=\frac{u_i^*(e^{\beta M}-1)}{\beta}\int_M^\infty J_i(y)\mathrm{d}y\\ \leq&\frac{u_i^*(e^{\beta M}-1)}{\beta}e^{-\beta M}\int_M^\infty e^{\beta y}J_i(y)\mathrm{d}y\leq\frac{u_i^*}{\beta}\int_0^\infty e^{\beta y}J_i(y)\mathrm{d}y:=C<\infty,\end{aligned}$$

since $\beta\leq\lambda$. Hence (2.12) holds. □

<u>Proof of Theorem 2.3</u>. With $\widetilde{\psi}=\sum_{i=1}^m\tilde{a}_i\psi_i$ given by (2.8), it suffices to show that there exists $\beta\in(0,\lambda]$ such that

$$\widetilde{\psi}(x)=O(e^{\beta x})\text{ for large negative }x.$$

By Lemma 2.7, there exist $\epsilon>0$ and $\beta\in(0,\lambda]$ small such that (2.12) holds and $\hat{b}\geq\sum_{i=1}^m\tilde{a}_id_i\epsilon+c\beta$. Multiplying $e^{-\beta x}$ on both sides of the equation satisfied by $\widetilde{\psi}$ and then integrating the resulting equation over the interval $[-M,0]$ with an arbitrary $M>0$, we obtain

$$\begin{aligned}&-\sum_{i=1}^m\int_{-M}^0\tilde{a}_ig_i(\Psi(x))e^{-\beta x}\mathrm{d}x-\int_{-M}^0c\widetilde{\psi}'(x)(-x)^\beta\mathrm{d}x\\ =&\sum_{i=1}^m\tilde{a}_id_i\int_{-M}^0\left[\int_{-\infty}^0J_i(x-y)\psi_i(y)\mathrm{d}y-\psi_i(x)\right]e^{-\beta x}\mathrm{d}x\\ &+\sum_{i=1}^m\tilde{a}_id_iu_i^*\int_{-M}^0e^{-\beta x}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x=:S_1(M)+S_2(M).\end{aligned}\tag{2.13}$$

In view of $(\mathbf{J_2})$ and $\beta\in(0,\lambda]$, we have

$$\begin{aligned}S_2(M)=&\sum_{i=1}^m\tilde{a}_id_iu_i^*\int_{-M}^0e^{-\beta x}\int_{-x}^\infty J_i(y)\mathrm{d}y\mathrm{d}x\leq\sum_{i=1}^m\tilde{a}_id_iu_i^*\int_{-\infty}^0e^{-\beta x}\int_{-x}^\infty J_i(y)\mathrm{d}y\mathrm{d}x\\ =&\sum_{i=1}^m\tilde{a}_id_iu_i^*\int_0^\infty\int_{-y}^0e^{-\beta x}J_i(y)\mathrm{d}x\mathrm{d}y=\sum_{i=1}^m\frac{\tilde{a}_id_iu_i^*}{\beta}\int_0^\infty[e^{\beta y}-1]J_i(y)\mathrm{d}y<\infty.\end{aligned}$$

This together with (2.12) implies

$$S_1(M)+S_2(M)\leq\sum_{i=1}^m\tilde{a}_id_i\epsilon\int_{-M}^0e^{-\beta x}\psi_i(x)\mathrm{d}x+C_1\tag{2.14}$$

for some $C_1>0$ independent of M.

On the other hand, by (2.7) and $\hat{b}\geq\sum_{i=1}^m\tilde{a}_id_i\epsilon+c\beta$ we obtain, for $M>M_0\gg1$,

$$\begin{aligned}&-\sum_{i=1}^m\int_{-M}^0\tilde{a}_ig_i(\Psi(x))e^{-\beta x}\mathrm{d}x-\int_{-M}^0c\widetilde{\psi}'(x)e^{-\beta x}\mathrm{d}x\\ \geq&\hat{b}\int_{-M}^{-M_0}\widetilde{\psi}(x)e^{-\beta x}\mathrm{d}x-\int_{-M}^0c\widetilde{\psi}'(x)e^{-\beta x}\mathrm{d}x\\ &-\sum_{i=1}^m\int_{-M_0}^0\tilde{a}_ig_i(\Psi(x))e^{-\beta x}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&=\hat{b}\int_{-M}^{0}\widetilde{\psi}(x)e^{-\beta x}\mathrm{d}x-\int_{-M}^{0}c\widetilde{\psi}'(x)e^{-\beta x}\mathrm{d}x+C_2\\
&\geq\sum_{i=1}^{m}\tilde{a}_i d_i\epsilon\int_{-M}^{0}\widetilde{\psi}(x)e^{-\beta x}\mathrm{d}x+c\beta\int_{-M}^{0}\widetilde{\psi}(x)e^{-\beta x}\mathrm{d}x-\int_{-M}^{0}c\widetilde{\psi}'(x)e^{-\beta x}\mathrm{d}x+C_2\\
&=\sum_{i=1}^{m}\tilde{a}_i d_i\epsilon\int_{-M}^{0}\widetilde{\psi}(x)e^{-\beta x}\mathrm{d}x-c\int_{-M}^{0}[\widetilde{\psi}(x)e^{-\beta x}]'\mathrm{d}x+C_2\\
&=\sum_{i=1}^{m}\tilde{a}_i d_i\epsilon\int_{-M}^{0}\widetilde{\psi}(x)e^{-\beta x}\mathrm{d}x-c\widetilde{\psi}(0)+c\widetilde{\psi}(-M)e^{\beta M}+C_2,
\end{aligned}
$$

where

$$
C_2:=-\sum_{i=1}^{m}\int_{-M_0}^{0}\tilde{a}_i g_i(\Psi(x))e^{-\beta x}\mathrm{d}x-\int_{-M_0}^{0}\widetilde{\psi}(x)e^{-\beta x}\mathrm{d}x.
$$

Therefore, by (2.13) and (2.14),

$$
c\widetilde{\psi}(-M)e^{\beta M}\leq c\widetilde{\psi}(0)+C_1-C_2 \text{ for all } M>M_0,
$$

which implies $\widetilde{\psi}(x)=O(e^{\beta x})$ for $x\ll -1$. The proof is completed. □

2.2. **Bounds for $c_0t-h(t)$, $c_0t+g(t)$ and $U(t,x)$ for kernels of type $(\mathbf{J}^{\alpha})$.** Let us first observe that it suffices to estimate $h(t)-c_0t$, since that for $g(t)+c_0t$ follows by considering (1.1) with initial function $u_0(-x)$.

Theorem 1.1 will follow easily from Lemmas 2.8, 2.10 below and their proofs, where more general and stronger conclusions are proved.

Lemma 2.8. *In Theorem B, if additionally $(\mathbf{J}^{\alpha})$ holds for some $\alpha\geq 1$, F is C^2 and $\mathbf{u}^*\nabla F(\mathbf{u}^*)\prec\!\!\prec\mathbf{0}$, then there exists $C>0$ such that for $t\geq 0$,*

$$
h(t)-c_0t\geq -C\left[1+\int_0^t(1+x)^{-\alpha}dx+\int_0^{\frac{c_0}{2}t}x^2\hat{J}(x)dx+t\int_{\frac{c_0}{2}t}^{\infty}x\hat{J}(x)dx\right],
$$

where $c_0>0$ is given in Theorem A and $\hat{J}(x):=\sum_{i=1}^{m_0}\mu_iJ_i(x)$.

To prove Lemma 2.8, we will need the following result.

Lemma 2.9. *Suppose that $F=(f_i)\in C^2(\mathbb{R}^m,\mathbb{R}^m)$, $\mathbf{u}^*\succ\!\!\succ 0$ and*

$$
F(\mathbf{u}^*)=\mathbf{0},\ \ \mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T\prec\!\!\prec\mathbf{0}.
$$

Then there exists $\delta_0>0$ small such that for $0<\epsilon\ll 1$ and $u,v\in[(1-\delta_0)\mathbf{u}^*,\mathbf{u}^*]$ satisfying

$$
(u_i^*-u_i)(u_j^*-v_j)\leq C\delta_0\epsilon \text{ for some } C>0 \text{ and all } i,j\in\{1,...,m\},
$$

we have

$$
(1-\epsilon)[F(u)+F(v)]-F((1-\epsilon)(u+v-\mathbf{u}^*))\preceq\frac{\epsilon}{2}\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T.
$$

Proof. Define

$$
G(u,v)=(g_i(u,v)):=(1-\epsilon)[F(u)+F(v)]-F((1-\epsilon)(u+v-\mathbf{u}^*)),\quad u,v\in\mathbb{R}^m.
$$

For $u,\ v\in[(1-\delta_0)\mathbf{u}^*,\mathbf{u}^*]$ and each $i\in\{1,...,m\}$, we may apply the mean value theorem to the function

$$
\xi_i(t):=g_i(\mathbf{u}^*+t(u-\mathbf{u}^*),\mathbf{u}^*+t(v-\mathbf{u}^*)
$$

to obtain

$$
\xi_i(1)=\xi_i(0)+\xi_i'(\zeta_i) \text{ for some } \zeta_i\in[0,1].
$$

Denote

$$
\tilde{u}=\tilde{u}^i:=\mathbf{u}^*+\zeta_i(u-\mathbf{u}^*),\quad \tilde{v}=\tilde{v}^i:=\mathbf{u}^*+\zeta_i(v-\mathbf{u}^*).
$$

Then the above identity is equivalent to

$$\begin{aligned} g_i(u,v) =& g_i(\mathbf{u}^*,\mathbf{u}^*) + \nabla_u\, g_i(\tilde u,\tilde v)\cdot(u-\mathbf{u}^*) + \nabla_v\, g_i(\tilde u,\tilde v)\cdot(v-\mathbf{u}^*)\\ =& -f_i((1-\epsilon)\mathbf{u}^*) + (1-\epsilon)\nabla f_i(\tilde u)\cdot(u-\mathbf{u}^*) + (1-\epsilon)\nabla f_i(\tilde v)\cdot(v-\mathbf{u}^*)\\ & -(1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde u+\tilde v-\mathbf{u}^*)\big)\cdot(u-\mathbf{u}^*)\\ & -(1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde u+\tilde v-\mathbf{u}^*)\big)\cdot(v-\mathbf{u}^*). \end{aligned}$$

Let us note that $\tilde u\in[u,\mathbf{u}^*]$ and $\tilde v\in[v,\mathbf{u}^*]$. Since $F\in C^2$, there is C_1 such that

$$|\partial_{jk} f_i(u)|\le C_1 \ \text{ for } \ u\in[\mathbf{0},\mathbf{u}^*],\ i,j,k\in\{1,...,m\}.$$

A simple calculation gives

$$\begin{aligned} &(1-\epsilon)\nabla f_i(\tilde u)(u-\mathbf{u}^*) - (1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde u+\tilde v-\mathbf{u}^*)\big)\cdot(u-\mathbf{u}^*)\\ =&(1-\epsilon)\Big[\nabla f_i(\tilde u) - \nabla f_i\big((1-\epsilon)(\tilde u+\tilde v-\mathbf{u}^*)\big)\Big]\cdot(u-\mathbf{u}^*)\\ \le&(1-\epsilon)b_1\sum_{j=1}^m(u_j^*-u_j), \end{aligned}$$

where

$$\begin{aligned} b_1 &:= C_1|\tilde u-(1-\epsilon)(\tilde u+\tilde v-\mathbf{u}^*)|\\ &= C_1|\epsilon\tilde u-(1-\epsilon)(\tilde v-\mathbf{u}^*)| \le C_1\sum_{j=1}^m[\epsilon\tilde u_j+(1-\epsilon)(u_j^*-\tilde v_j)]\\ &\le C_2\epsilon + C_1\sum_{j=1}^m(u_j^*-v_j) \text{ with } C_2 := C_1\textstyle\sum_{j=1}^m u_j^*. \end{aligned}$$

Similarly,

$$\begin{aligned} &(1-\epsilon)\nabla f_i(\tilde u)\cdot(v-\mathbf{u}^*) - (1-\epsilon)\nabla f_i\big((1-\epsilon)(\tilde u+\tilde v-\mathbf{u}^*)\big)\cdot(v-\mathbf{u}^*)\\ \le&\ (1-\epsilon)b_2\sum_{j=1}^m(u_j^*-v_j), \end{aligned}$$

where

$$b_2 := C_1|\epsilon\tilde u-(1-\epsilon)(\tilde u-\mathbf{u}^*)| \le\ C_2\epsilon + C_1\sum_{j=1}^m(u_j^*-u_j).$$

Thus

$$\begin{aligned} g_i(u,v) \le& -f_i((1-\epsilon)\mathbf{u}^*) + (1-\epsilon)b_1\sum_{j=1}^m(u_j^*-v_j) + (1-\epsilon)b_2\sum_{j=1}^m(u_j^*-u_j)\\ \le& -f_i((1-\epsilon)\mathbf{u}^*) + \left[C_2\epsilon + C_1\sum_{j=1}^m(u_j^*-v_j)\right]\sum_{k=1}^m(u_k^*-u_k)\\ &+\left[C_2\epsilon + C_1\sum_{j=1}^m(u_j^*-u_j)\right]\sum_{k=1}^m(u_k^*-v_k)\\ =& -f_i((1-\epsilon)\mathbf{u}^*) + C_2\epsilon\sum_{k=1}^m\Big[(u_k^*-u_k)+(u_k^*-v_k)\Big] \end{aligned}$$

$$
\begin{aligned}
&+ C_1 \sum_{j,k=1}^{m} (u_j^* - v_j)(u_k^* - u_k) + C_1 \sum_{j,k=1}^{m} (u_j^* - u_j)(u_k^* - v_k) \\
&= \epsilon \nabla f_i(\mathbf{u}^*) \cdot \mathbf{u}^* + o(\epsilon) + C_2 \epsilon \sum_{k=1}^{m} \Big[(u_k^* - u_k) + (u_k^* - v_k) \Big] \\
&+ 2C_1 \sum_{j,k=1}^{m} (u_j^* - v_j)(u_k^* - u_k),
\end{aligned}
$$

where $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$.

If $u, v \in [(1-\delta_0)\mathbf{u}^*, \mathbf{u}^*]$, then

$$
P = (p_i) := \mathbf{u}^* - u, \quad Q = (q_i) := \mathbf{u}^* - v \in [\mathbf{0}, \delta_0 \mathbf{u}^*], \tag{2.15}
$$

and hence

$$
\begin{aligned}
g_i(u, v) &= g_i(\mathbf{u}^* - P, \mathbf{u}^* - Q) \\
&\le \epsilon \nabla f_i(\mathbf{u}^*) \cdot \mathbf{u}^* + o(\epsilon) + C_2 \epsilon \sum_{k=1}^{m} (p_k + q_k) + 2C_1 \sum_{j,k=1}^{m} p_j q_k \\
&\le \epsilon \Big[\mathbf{u}^* \cdot \nabla f_i(\mathbf{u}^*) + o(1) + 2(C_2 + C_1)\delta_0 \Big] \\
&\le \frac{\epsilon}{2} \mathbf{u}^* \cdot \nabla f_i(\mathbf{u}^*) \quad \text{for} \ \ i \in \{1, ..., m\}, \ 0 < \epsilon \ll 1
\end{aligned}
$$

provided that $\delta_0 > 0$ is sufficiently small. □

Proof of Lemma 2.8. Let (c_0, Φ^{c_0}) be the unique solution pair of (1.4)-(1.5) in Theorem A. To simplify notations we write $\Phi^{c_0}(x) = \Phi(x) = (\phi_i(x))$. By Theorem 2.1 there is $C > 0$ such that

$$
\sum_{i=1}^{m_0} \int_0^\infty J_i(y)|y|^\alpha \mathrm{d}y \le C, \quad 0 < u_i^* - \phi_i(x) \le \frac{C}{x^\alpha} \quad \text{for} \ \ x < -1, \ i \in \{1, ..., m\}. \tag{2.16}
$$

Define

$$
\begin{cases}
\underline{h}(t) := c_0 t + \delta(t), \quad t \ge 0, \\
\underline{U}(t, x) := (1 - \epsilon(t))[\Phi(x - \underline{h}(t)) + \Phi(-x - \underline{h}(t)) - \mathbf{u}^*], \quad t \ge 0, x \in [-\underline{h}(t), \underline{h}(t)],
\end{cases}
$$

where $\epsilon(t) := (t + \theta)^{-\alpha}$ and

$$
\delta(t) := K_1 - K_2 \int_0^t \epsilon(\tau) \mathrm{d}\tau - 2 \sum_{i=1}^{m_0} \mu_i u_i^* \int_0^t \int_{-\infty}^{-\frac{c_0}{2}(\tau + \theta)} \int_0^\infty J_i(x - y) \mathrm{d}y \mathrm{d}x \mathrm{d}\tau,
$$

with θ, K_1 and K_2 large positive constants to be determined.

For any $M > 0$ and $i \in \{1, ..., m_0\}$,

$$
\begin{aligned}
&\int_{-\infty}^{-M} \int_0^\infty J_i(x - y) \mathrm{d}y \mathrm{d}x = \int_M^\infty \int_x^\infty J_i(y) \mathrm{d}y \mathrm{d}x \\
&= \int_M^\infty \int_M^y J_i(y) \mathrm{d}x \mathrm{d}y = \int_M^\infty (y - k) J_i(y) \mathrm{d}y \le \int_M^\infty y J_i(y) \mathrm{d}y.
\end{aligned}
$$

Hence, due to $\int_0^\infty y J_i(y) \mathrm{d}y < \infty$, we have

$$
2 \sum_{i=1}^{m_0} \mu_i u_i^* \int_0^t \int_{-\infty}^{-\frac{c_0}{2}(\tau + \theta)} \int_0^\infty J_i(x - y) \mathrm{d}y \mathrm{d}x \mathrm{d}\tau
$$

$$\leq 2\sum_{i=1}^{m_0}\mu_i u_i^*\int_0^t\int_{-\infty}^{-\frac{c_0}{2}\theta}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau$$

$$\leq \left[2\sum_{i=1}^{m_0}\mu_i u_i^*\int_{\frac{c_0}{2}\theta}^\infty yJ_i(y)\mathrm{d}y\right] t\leq \frac{c_0}{4}t$$

provided that $\theta>0$ is large enough, say $\theta\geq\theta_0$.

For any given small $\epsilon_0>0$, due to $\Phi(-\infty)=\mathbf{u}^*$ there is $K_0=K_0(\epsilon_0)>0$ such that

$$(1-\epsilon_0)\mathbf{u}^*\preceq\Phi(-K_0),$$

which implies that

$$\Phi(x-\underline{h}(t)),\Phi(-x-\underline{h}(t))\in[(1-\epsilon_0)\mathbf{u}^*,\mathbf{u}^*]\ \text{ for }\ x\in[-\underline{h}(t)+K_0,\underline{h}(t)-K_0], \tag{2.17}$$

where we have assumed $\underline{h}(0)=K_1>K_0$.

Clearly

$$K_2\int_0^t(\tau+\theta)^{-\alpha}\mathrm{d}\tau\leq K_2\theta^{-\alpha}t\leq\frac{c_0}{4}t$$

provided $\theta\geq(4K_2/c_0)^{1/\alpha}$. Therefore

$$\underline{h}(t)\geq\frac{c_0}{2}t+K_1\geq\frac{c_0}{2}(t+\theta)>K_0 \text{ for all } t\geq 0 \text{ provided that} \tag{2.18}$$

$$K_1\geq\frac{c_0}{2}\theta \text{ and } \theta\geq\max\left\{(4K_2/c_0)^{1/\alpha},\theta_0,2K_0/c_0\right\}. \tag{2.19}$$

Define

$$\epsilon_1:=\inf_{1\leq i\leq m}\inf_{x\in[-K_0,0]}|\phi_i'(x)|>0.$$

Then

$$\begin{cases}\Phi'(x-\underline{h}(t))<-\epsilon_1\mathbf{1} & \text{for } x\in[\underline{h}(t)-K_0,\underline{h}(t)],\\ \Phi'(-x-\underline{h}(t))<-\epsilon_1\mathbf{1} & \text{for } x\in[-\underline{h}(t),-\underline{h}(t)+K_0].\end{cases} \tag{2.20}$$

Claim 1: With $\underline{U}=(\underline{u}_i)$, and suitably chosen $\theta,\ K_1,\ K_2$, we have

$$\underline{h}'(t)\leq\sum_{i=1}^m\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^\infty J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y,\qquad t>0 \tag{2.21}$$

and

$$-\underline{h}'(t)\geq-\sum_{i=1}^m\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{-\infty}^{-\underline{h}(t)}J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y,\qquad t>0.$$

Due to $\underline{U}(t,x)=\underline{U}(t,-x)$ and $\mathbf{J}(x)=\mathbf{J}(-x)$, we just need to verify (2.21). We calculate

$$\sum_{i=1}^{m_0}\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^\infty J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x$$

$$=(1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-2\underline{h}(t)}^0\int_0^\infty J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x$$

$$+(1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-2\underline{h}(t)}^0\int_0^\infty J_i(x-y)[\phi_i(-x-2\underline{h}(t))-u_i^*]\mathrm{d}y\mathrm{d}x$$

$$=(1-\epsilon)c_0-(1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{-2\underline{h}(t)}\int_0^\infty J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x$$

$$
- (1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-2\underline{h}(t)}^{0}\int_0^\infty J_i(x-y)[u_i^*-\phi_i(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x.
$$

From (2.18), for $t\geq 0$,

$$
\begin{aligned}
&(1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{-2\underline{h}(t)}\int_0^\infty J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x\\
&\quad+(1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-2\underline{h}(t)}^{-\underline{h}(t)}\int_0^\infty J_i(x-y)[u_i^*-\phi_i(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x\\
&\leq 2\sum_{i=1}^{m_0}\mu_i u_i^*\int_{-\infty}^{-\underline{h}(t)}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\\
&\leq 2\sum_{i=1}^{m_0}\mu_i u_i^*\int_{-\infty}^{-\frac{c_0}{2}(t+\theta)}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

And by (2.16), we have, for $t>0$,

$$
\begin{aligned}
&(1-\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\underline{h}(t)}^{0}\int_0^\infty J_i(x-y)[u_i^*-\phi_i(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x\\
\leq&\sum_{i=1}^{m_0}\mu_i[u_i^*-\phi_i(-\underline{h}(t))]\int_{-\underline{h}(t)}^{0}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\\
\leq&\sum_{i=1}^{m_0}\mu_i\frac{C}{h(t)^\alpha}\int_{-\infty}^{0}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\\
=&\sum_{i=1}^{m_0}\mu_i\frac{C}{h(t)^\alpha}\int_0^\infty yJ_i(y)\mathrm{d}y\leq\sum_{i=1}^{m_0}\mu_i\frac{C^2}{(c_0/2)^\alpha(t+\theta)^\alpha}\leq\frac{K_2-c_0}{(t+\theta)^\alpha}
\end{aligned}
$$

if

$$
K_2\geq c_0+\frac{C^2}{(c_0/2)^\alpha}\sum_{i=1}^{m}\mu_i. \tag{2.22}
$$

Hence, when θ, K_1 and K_2 are chosen such that (2.19) and (2.22) hold, then

$$
\begin{aligned}
&\sum_{i=1}^{m}\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^\infty J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x\\
&\geq(1-\epsilon)c_0-2\sum_{i=1}^{m}\mu_i u_i^*\int_{-\infty}^{-\frac{c_0}{2}(t+\theta)}\int_0^\infty J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x-\frac{K_2-c_0}{(t+\theta)^\alpha}\\
&=c_0-K_2\epsilon(t)-2\sum_{i=1}^{m}\mu_i u_i^*\int_{-\infty}^{-\frac{c_0}{2}(t+\theta)}\int_0^\infty J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x\\
&=h'(t)\quad\text{for all } t>0,
\end{aligned}
$$

which finishes the proof of (2.21).

Claim 2: With $\theta,\ K_1,\ K_2$ chosen such that (2.19) and (2.22) hold, and K_2 suitably further enlarged (see (2.23) below), $\theta_0\gg 1$ and $0<\epsilon_0\ll 1$, we have, for all $t>0$ and $x\in(-\underline{h}(t),\underline{h}(t))$,

$$
\underline{U}_t(t,x)\preceq D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-D\circ\underline{U}(t,x)+F(\underline{U}(t,x)).
$$

A simple calculation gives

$$
\underline{U}_t=-\epsilon'(t)[\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*]
$$

$$
\begin{aligned}
&- (1-\epsilon)h'(t)[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]\\
=&\alpha(t+\theta)^{-\alpha-1}[\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*]\\
&- (1-\epsilon)[c_0+\delta'(t)][\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))],
\end{aligned}
$$

and using the equation satisfied by Φ we deduce

$$
\begin{aligned}
&- (1-\epsilon)c_0[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]\\
=&(1-\epsilon)\Big[D\circ\int_{-\infty}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\Phi(y-\underline{h}(t))\mathrm{d}y-D\circ\Phi(x-\underline{h}(t))\\
&\qquad+D\circ\int_{-\underline{h}(t)}^{\infty}\mathbf{J}(-x-y)\circ\Phi(-y-\underline{h}(t))\mathrm{d}y-D\circ\Phi(-x-\underline{h}(t))\Big]\\
&+(1-\epsilon)\Big[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))\Big]\\
=&D\circ\left[\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-\underline{U}(t,x)\right]\\
&+(1-\epsilon)\Big[D\circ\int_{-\infty}^{-\underline{h}(t)}\mathbf{J}(x-y)\circ[\Phi(y-\underline{h}(t))-\mathbf{u}^*]\mathrm{d}y\\
&\qquad+D\circ\int_{\underline{h}(t)}^{\infty}\mathbf{J}(-x-y)\circ[\Phi(-y-\underline{h}(t))\mathrm{d}y-\mathbf{u}^*]\mathrm{d}y\Big]\\
&+(1-\epsilon)\Big[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))\Big]\\
\preceq&D\circ\left[\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-\underline{U}(t,x)\right]\\
&+(1-\epsilon)\Big[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))\Big].
\end{aligned}
$$

Hence

$$
\underline{U}_t\preceq D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-\underline{U}(t,x)+F(\underline{U}(t,x))+A_1(t,x)+A_2(t,x),
$$

where

$$
\begin{aligned}
A_1(t,x):=&\alpha(t+\theta)^{-\alpha-1}[\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*],\\
A_2(t,x):=&-(1-\epsilon)\delta'(t)[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]\\
&+(1-\epsilon)[F(\Phi(x-\underline{h}(t)))+F(\Phi(-x-\underline{h}(t)))]-F(\underline{U}(t,x)).
\end{aligned}
$$

To finish the proof of Claim 2, it remains to check that

$$
A_1(t,x)+A_2(t,x)\preceq\mathbf{0}\ \text{ for }\ t>0,\ x\in(-\underline{h}(t),\underline{h}(t)).
$$

We next prove this inequality for x in the following three intervals, separately:

$$
I_1(t):=[\underline{h}(t)-K_0,\underline{h}(t)],\ I_2(t):=[-\underline{h}(t),-\underline{h}(t)+K_0],\ I_3(t):=[-\underline{h}(t)+K_0,\underline{h}(t)-K_0].
$$

For $x\in I_1(t)$, by (2.16),

$$
\mathbf{0}\succ\Phi(-x-\underline{h}(t))-\mathbf{u}^*\succeq\Phi(K_0-2\underline{h}(t))-\mathbf{u}^*\succeq\Phi(-\underline{h}(t))-\mathbf{u}^*\succeq\frac{-C}{h(t)^\alpha}\mathbf{1}
$$

Then by $(\mathbf{f}_2)$, there exists $L>0$ such that

$$
F(\Phi(-x-\underline{h}(t)))=F(\Phi(-x-\underline{h}(t)))-F(\mathbf{u}^*)\preceq L\frac{C}{h(t)^\alpha}\mathbf{1}
$$

and

$$F(\underline{U}(t,x)) \succeq (1-\epsilon)F\Big(\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*\Big)$$
$$\succeq (1-\epsilon)\big[F(\Phi(x-\underline{h}(t)))-L\frac{C}{h(t)^\alpha}\mathbf{1}\big].$$

Thus from the definition of $\delta(t)$, (2.18) and (2.20), we deduce

$$\begin{aligned}A_2(t,x) \preceq&(1-\epsilon)\Big[\delta'(t)[\Phi'(x-\underline{h}(t))+\Phi'(-x-\underline{h}(t))]+F(\Phi(x-\underline{h}(t)))\\&+F(\Phi(-x-\underline{h}(t)))-F\Big(\Phi(x-\underline{h}(t))+\Phi(-x-\underline{h}(t))-\mathbf{u}^*\Big)\Big]\\\preceq&(1-\epsilon)\left[-\delta'(t)\epsilon_1+2L\frac{C}{h(t)^\alpha}\right]\mathbf{1}\preceq(1-\epsilon)\left[-K_2(t+\theta)^{-\alpha}\epsilon_1+\frac{2LC}{h(t)^\alpha}\right]\mathbf{1}\\\preceq&(1-\epsilon)(t+\theta)^{-\alpha}\big[-K_2\epsilon_1+2LC(2/c_0)^\alpha\big]\mathbf{1}.\end{aligned}$$

Moreover,

$$A_1(t,x)\preceq\alpha(t+\theta)^{-\alpha-1}\mathbf{u}^*\leq 2|\mathbf{u}^*|(1-\epsilon)\alpha(t+\theta)^{-\alpha-1}\mathbf{1},$$

where $|\mathbf{u}^*|:=\max_{1\leq i\leq m}u_i^*$ and by enlarging θ_0 we have assumed that $\epsilon(t)\leq\theta_0^{-\alpha}<1/2$. Hence

$$A_1(t,x)+A_2(t,x)\preceq(1-\epsilon)(t+\theta)^{-\alpha}\Big[-K_2\epsilon_1+2LC(2/c_0)^\alpha+2|\mathbf{u}^*|\alpha\theta_0^{-1}\Big]\mathbf{1}\preceq\mathbf{0}$$

if additionally

$$K_2\geq\epsilon_1^{-1}\Big[2LC(2/c_0)^\alpha+2|\mathbf{u}^*|\alpha\theta_0^{-1}\Big]. \tag{2.23}$$

This proves the desired inequality for $x\in I_1(t)$.

Since $A_1(t,x)+A_2(t,x)$ is even in x, the desired inequality is also valid for $x\in I_2(t)=-I_1(t)$. It remains to prove the desired inequality for $x\in I_3(t)$.

We apply Lemma 2.9 with $u=\Phi(x-\underline{h}(t))$ and $v=\Phi(-x-\underline{h}(t))$. Let

$$P(t,x)=(p_i(t,x)):=\mathbf{u}^*-\Phi(x-\underline{h}(t)),\quad Q(t,x)=(q_i(t,x)):=\mathbf{u}^*-\Phi(-x-\underline{h}(t)).$$

Then by (2.17) we have

$$P(t,x),\ Q(t,x)\in[\mathbf{0},\epsilon_0\mathbf{u}^*]\text{ for }x\in I_3(t),t>0. \tag{2.24}$$

Moreover, since $\min\{x-\underline{h}(t),-x-\underline{h}(t)\}\leq-\underline{h}(t)$ always holds, by (2.16) and (2.18), if we denote $C_3:=C(c_0/2)^{-\alpha}$, then

$$p_j(t,x)q_k(t,x)\leq\frac{C\epsilon_0}{\underline{h}(t)^\alpha}\leq C_3\epsilon_0\epsilon(t)\ \text{ for }x\in I_3(t),\ t>0,\quad j,k\in\{1,...,m\}. \tag{2.25}$$

Let A_2^i denote the i-th component of A_2. Now due to $\delta'(t)<0$ and $\Phi'\lll\mathbf{0}$, we have, by (2.24), (2.25) and Lemma 2.9, assuming $\epsilon_0>0$ is sufficiently small,

$$A_2^i(t,x)\leq g_i(\mathbf{u}^*-P,\mathbf{u}^*-Q)\leq\frac{\epsilon}{2}\mathbf{u}^*\cdot\nabla f_i(\mathbf{u}^*)\ \text{ for }x\in I_3(t),\ t>0,\ i\in\{1,...,m\}\text{ and all }\theta_0\gg1.$$

Since

$$A_1^i(t,x)\leq\alpha(t+\theta)^{-\alpha-1}u_i^*\leq\alpha|u_i^*|\theta_0^{-1}\epsilon(t),$$

we thus obtain

$$A_1^i+A_2^i\leq\epsilon\Big(\mathbf{u}^*\cdot\nabla f_i(\mathbf{u}^*)/2+\alpha u_i^*\theta_0^{-1}\Big)<0\qquad\text{for }x\in I_3(t),\ t>0,\ i\in\{1,...,m\},\ \theta_0\gg1,$$

provided that ϵ_0 is sufficiently small. The proof of Claim 2 is now complete.

Claim 3: There exists $t_0>0$ such that

$$\begin{cases}g(t+t_0)\leq-\underline{h}(t),\ h(t+t_0)\geq\underline{h}(t)\text{ for }t\geq0,\\U(t+t_0,x)\succeq\underline{U}(t,x)\text{ for }t\geq0,\ x\in[-\underline{h}(t),\underline{h}(t)].\end{cases} \tag{2.26}$$

It is clear that

$$\underline{U}(t, \pm\underline{h}(t)) = (1-\epsilon(t))[\Phi(-2\underline{h}(t)) - \mathbf{u}^*] \prec \mathbf{0} \text{ for } \; t \geq 0.$$

Since spreading happens for (U, g, h), there exists a large constant $t_0 > 0$ such that

$$\begin{aligned} & g(t_0) < -K_1 = -\underline{h}(0) \text{ and } \underline{h}(0) = K_1 < h(t_0), \\ & U(t_0, x) \succeq (1-\theta^{-\alpha})\mathbf{u}^* \succeq \underline{U}(0,x) \quad \text{for} \quad x \in [-\underline{h}(0), \underline{h}(0)]. \end{aligned}$$

which together with the inequalities proved in Claims 1 and 2 allows us to apply the comparison principle to conclude that (2.26) is valid.

Claim 4: There exists $C > 0$ such that

$$\delta(t) \geq -C\left[1 + \int_0^t (1+x)^{-\alpha}dx + \int_0^{\frac{c_0}{2}t} x^2\hat{J}(x)dx + t\int_{\frac{c_0}{2}t}^{\infty} x\hat{J}(x)dx\right].$$

Clearly

$$\int_0^t \epsilon(\tau)\mathrm{d}\tau = \int_0^t (x+\theta)^{-\alpha}\mathrm{d}x < \int_0^t (x+1)^{-\alpha}\mathrm{d}x.$$

By changing order of integrations we have

$$\begin{aligned} & \int_0^t \int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)} \int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau \leq \int_0^t \int_{-\infty}^{-\frac{c_0}{2}\tau} \int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau \\ = & \int_0^t \int_{\frac{c_0}{2}\tau}^{\infty} \left[y - \frac{c_0}{2}\tau\right] J_i(y)\mathrm{d}y\mathrm{d}\tau \leq \int_0^t \int_{\frac{c_0}{2}\tau}^{\infty} yJ_i(y)\mathrm{d}y\mathrm{d}\tau \\ = & \frac{c_0}{2}\int_0^{\frac{c_0}{2}t} y^2 J_i(y)dy + t\int_{\frac{c_0}{2}t}^{\infty} yJ_i(y)dy. \end{aligned}$$

The desired inequality now follows directly from the definition of $\delta(t)$. □

Next we prove an upper bound for $h(t) - c_0t$. Let us note that we do not need the condition $(\mathbf{J}^\alpha)$ in the following result.

Lemma 2.10. *Under the assumptions of Theorem B (i), if* $(\mathbf{J_1})$ *holds, and additionally* F *is* C^2 *and* $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \prec\!\!\prec \mathbf{0}$, *then there exits* $C > 0$ *such that*

$$h(t) - c_0 t \leq C \quad \textit{for all} \;\; t > 0. \tag{2.27}$$

Proof. As in the proof of Lemma 2.8, (c_0, Φ^{c_0}) denotes the unique solution pair of (1.4)-(1.5) in Theorem A, and to simplify notations we write $\Phi^{c_0}(x) = \Phi(x) = (\phi_i(x))$.

For fixed $\beta > 1$, and some large constants $\theta > 0$ and $K_1 > 0$ to be determined, define

$$\begin{cases} \bar{h}(t) := c_0 t + \delta(t), \quad t \geq 0, \\ \overline{U}(t,x) := (1+\epsilon(t))\Phi(x - \bar{h}(t)), \quad t \geq 0, \; x \leq \bar{h}(t), \end{cases}$$

where $\epsilon(t) := (t+\theta)^{-\beta}$ and

$$\delta(t) := K_1 + \frac{c_0}{1-\beta}[(t+\theta)^{1-\beta} - \theta^{1-\beta}].$$

Clearly, there is a large constant $t_0 > 0$ such that

$$U(t+t_0, x) \preceq (1 + \frac{1}{2}\epsilon(0))\mathbf{u}^* \;\; \text{for} \quad t \geq 0, \; x \in [g(t), h(t)].$$

Due to $\Phi(-\infty) = \mathbf{u}^*$, we may choose sufficient large $K_1 > 0$ such that $\underline{h}(0) = K_1 > 2h(t_0)$, $-\underline{h}(0) = -K_1 < 2g(t_0)$, and also

$$\overline{U}(0,x) = (1+\epsilon(0))\Phi(-K_1/2) \succ (1+\frac{1}{2}\epsilon(0))\mathbf{u}^* \succeq U(t_0,x) \;\; \text{for} \;\; x \in [g(t_0), h(t_0)]. \tag{2.28}$$

Claim 1: We have, with $\overline{U} = (\bar{u}_i)$,

$$\bar{h}'(t) \geq \sum_{i=1}^m \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y \quad \text{for} \qquad t>0.$$

A direct calculation shows

$$\begin{aligned}
&\sum_{i=1}^m \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y \\
\leq &\sum_{i=1}^m \mu_i \int_{-\infty}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y \\
=&(1+\epsilon)\sum_{i=1}^m \mu_i \int_{-\infty}^{0} \int_{0}^{+\infty} J_i(x-y)\phi_i(x)\mathrm{d}y \\
=&(1+\epsilon)c_0 = \bar{h}'(t),
\end{aligned}$$

as desired.

Claim 2: If $\theta > 0$ is sufficiently large, then for $t > 0$ and $x \in (g(t+t_0), \underline{h}(t))$, we have

$$\overline{U}_t(t,x) \succeq D \circ \int_{g(t+t_0)}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \overline{U}(t,y)\mathrm{d}y - D \circ \overline{U}(t,x) + F(\overline{U}(t,x)). \tag{2.29}$$

By (1.4), we have

$$\begin{aligned}
\overline{U}_t(t,x) =& -(1+\epsilon)[c_0 + \delta'(t)]\Phi'(x-\bar{h}(t)) + \epsilon'(t)\Phi(x-\underline{h}(t)) \\
=& -(1+\epsilon)c_0\Phi'(x-\bar{h}(t)) - (1+\epsilon)\delta'(t)\Phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t)) \\
\succeq& D \circ \int_{g(t_0+t)}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \overline{U}(t,y)\mathrm{d}y - D \circ \overline{U}(t,x) + F(\overline{U}(t,x)) + A(t,x)
\end{aligned}$$

with

$$\begin{aligned}
A(t,x) :=&(1+\epsilon)F(\Phi(x-\bar{h}(t))) - F((1+\epsilon)\Phi(x-\bar{h}(t))) \\
&-(1+\epsilon)\delta'(t)\Phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t)).
\end{aligned}$$

To prove the claim, we need to show

$$A(t,x) \succeq \mathbf{0} \quad \text{for } x \in [g(t_0+t), \bar{h}(t)] \text{ and } t>0.$$

Let ϵ_0, ϵ_1 and K_0 be given as in the proof of Lemma 2.8. For $x \in [\bar{h}(t)-K_0, \bar{h}(t)]$ and $t > 0$, by (2.20), we have

$$\begin{aligned}
A(t,x) \succeq& -(1+\epsilon)\delta'(t)\Phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t)) \\
=& -(1+\epsilon)c_0(t+\theta)^{-\beta}\Phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\Phi(x-\underline{h}(t)) \\
\succeq& c_0(t+\theta)^{-\beta}\epsilon_1\mathbf{1} - \beta(t+\theta)^{-\beta-1}\mathbf{u}^* \\
\succeq& (t+\theta)^{-\beta-1}\big[c_0\theta\epsilon_1\mathbf{1} - \beta\mathbf{u}^*\big] \succeq \mathbf{0},
\end{aligned}$$

provided θ is large enough.

We next estimate $A(t,x)$ for $x \in [g(t+t_0), \underline{h}(t)-K_0]$. Define

$$G(u) = (g_i(u)) := (1+\epsilon)F(u) - F((1+\epsilon)u), \quad u,v \in \mathbb{R}^m.$$

Then for $u, v \in [\mathbf{0}, \mathbf{u}^*]$ and $i \in \{1, ..., m\}$,

$$\begin{aligned}
g_i(u) =& g_i(\mathbf{u}^*) + \nabla g_i(\tilde{u}) \cdot (u-\mathbf{u}^*) \\
=& -f_i((1+\epsilon)\mathbf{u}^*) + (1+\epsilon)\nabla f_i(\tilde{u}) \cdot (u-\mathbf{u}^*) - (1+\epsilon)\nabla f_i((1+\epsilon)\tilde{u}) \cdot (u-\mathbf{u}^*)
\end{aligned}$$

$$= -f_i((1+\epsilon)\mathbf{u}^*) + (1+\epsilon)\Big[\nabla f_i(\tilde{u}) - \nabla f_i((1+\epsilon)\tilde{u})\Big]\cdot(u-\mathbf{u}^*)$$

for some $\tilde{u} = \tilde{u}^i \in [u, \mathbf{u}^*]$. Since $F \in C^2$, there exists $C_1 > 0$ such that

$$|\partial_{jk} f_i(u)| \le C_1 \ \text{ for } \ u \in [0, \hat{\mathbf{u}}],\ i,j,k \in \{1,...,m\}.$$

Therefore

$$g_i(u) \ge -f_i((1+\epsilon)\mathbf{u}^*) - (1+\epsilon)b_1 \sum_{j=1}^m (u_j^* - u_j)$$

with

$$b_1 := C_1|\epsilon\tilde{u}| \le C_1\epsilon|\mathbf{u}^*| := C_2\epsilon.$$

Thus

$$g_i(u) \ge -\epsilon\nabla f_i(\mathbf{u}^*)\cdot\mathbf{u}^* + o(\epsilon) - 2C_2\epsilon\sum_{j=1}^m (u_j^* - u_j).$$

By (2.17) we have

$$-\epsilon_0\mathbf{u}^* \preceq \Phi(x-\bar{h}(t)) - \mathbf{u}^* \prec\!\!\prec \mathbf{0} \ \text{ for } \ x \in [g(t_0+t), \underline{h}(t) - K_0],\ t>0. \tag{2.30}$$

Using (2.17), $\delta' > 0$, $\Phi' \preceq \mathbf{0}$ and $\epsilon = (t+\theta)^{-\beta} \le \theta^{-\beta}$, we obtain

$$\begin{aligned}
A^i(t,x) \ge& (1+\epsilon)f_i(\Phi(x-\bar{h}(t))) - f_i((1+\epsilon)\Phi(x-\bar{h}(t))) - \beta(t+\theta)^{-\beta-1}\phi_i(x-\underline{h}(t))\\
=&\ g_i(\Phi(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\phi_i(x-\underline{h}(t))\\
\ge&\ \epsilon\Big[-\mathbf{u}^*\cdot\nabla f_i(\mathbf{u}^*) + o(1) - 2\epsilon_0 C_2\sum_{j=1}^m u_j^* - \beta\theta^{-\beta-1}u_i^*\Big]\\
>&\ 0 \quad \text{for} \quad x \in [g(t_0+t), \underline{h}(t) - K_0],\ t>0,\ i \in \{1,...,m\},
\end{aligned}$$

provided θ is large enough and $\epsilon_0 > 0$ is small enough, since $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \prec\!\!\prec \mathbf{0}$. We have now proved (2.29).

Due to the inequalities proved in Claims 1 and 2, (2.28) and

$$\overline{U}(t, g(t+t_0)) > 0, \quad \overline{U}(t, \bar{h}(t)) = (1+\epsilon)\Phi(\bar{h}(t) - \bar{h}(t)) = 0 \ \text{ for } \ t \ge 0,$$

we are now able to apply Lemma **??** to conclude that

$$\begin{aligned}
&h(t+t_0) \le \bar{h}(t), && t \ge 0,\\
&U(t+t_0, x) \preceq \overline{U}(t,x), && t \ge 0,\ x \in [g(t+t_0), \underline{h}(t)].
\end{aligned}$$

The desired inequality (2.27) follows directly from $\delta(t) \le K_1 + \frac{c_0}{\beta-1}\theta^{1-\beta}$ and $h(t+t_0) \le \bar{h}(t)$. The proof is complete. □

Proof of Theorem 1.1. Since $\alpha \ge 2$, from the proof of Lemmas 2.8 and 2.10, it is easily seen that

$$C_0 := \sup_{t>0}\big[|\bar{h}(t) - c_0 t| + |\underline{h}(t) - c_0 t|\big] < \infty.$$

Hence for large fixed $\theta > 0$ and all large t, say $t \ge t_0$,

$$[g(t), h(t)] \supset [-\underline{h}(t-t_0), \underline{h}(t-t_0)] \supset [-c_0 t + C, c_0 t - C] \text{ with } C := C_0 + c_0 t_0,$$

and

$$U(t,x) \succeq \underline{U}(t,x) \succeq (1-\epsilon(t))\big[\Phi^{c_0}(x - c_0 t + C) + \Phi^{c_0}(-x - c_0 t + C) - \mathbf{u}^*\big]$$

for $x \in [-c_0 t + C, c_0 t - C]$, where $\epsilon(t) = (t+\theta)^{-\alpha}$. This inequality for $U(t,x)$ also holds for $x \in [g(t), h(t)]$ if we assume that $\Phi^{c_0}(x) = 0$ for $x > 0$, since when x lies outside of $[-c_0 t + C, c_0 t - C]$ the right side is $\prec \mathbf{0}$.

By considering (1.1) with initial function $u_0(-x)$, from the proof of Lemma 2.10 we see that the following analogous inequalities hold:

$$g(t) \geq -\bar{h}(t-t_0),\ U(t,x) \preceq (1+\epsilon(t))\Phi^{c_0}(-x-\bar{h}(t-t_0))$$

for $t > t_0$ and $x \in [g(t), h(t)]$. We thus have

$$[g(t), h(t)] \subset [-\bar{h}(t-t_0), \bar{h}(t-t_0)] \subset [-c_0 t - C, c_0 t + C],$$

and

$$U(t,x) \preceq \overline{U}(t,x) \preceq (1-\epsilon(t)) \min\Big\{\Phi^{c_0}(x-c_0 t - C), \Phi^{c_0}(-x-c_0 t - C)\Big\}$$

for $t > t_0$ and $x \in [g(t), h(t)]$. The proof is complete. □

3. Growth rate of $c_0 t - h(t)$ and $c_0 t + g(t)$ for kernels of type $(\hat{\mathbf{J}}^\gamma)$

Recall that $(U(t,x), g(t), h(t))$ is the unique positive solution of (1.1), and we assume that spreading happens. Under the assumptions of Theorem B (i), we have

$$-\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = c_0 > 0.$$

In this section we determine the growth order of $c_0 t - h(t)$ and $c_0 t + g(t)$ when the kernel functions satisfy, for some $\gamma \in (2,3]$, $\omega \in (\gamma-1, \gamma]$, $C > 0$ and all $|x| \geq 1$,

$$\begin{cases} J_i(x) \approx |x|^{-\gamma} & \text{if } i \in \{1,...,m_0\} \text{ and } \mu_i \neq 0, \\ J_i(x) \leq C|x|^{-\omega} & \text{if } i \in \{1,...,m_0\} \text{ and } \mu_i = 0. \end{cases} \tag{3.1}$$

Clearly, $(\hat{\mathbf{J}}^\gamma)$ implies (3.1).

The main result of this section is the following theorem.

Theorem 3.1. *In Theorem B, if additionally* $(\mathbf{J^1})$, (3.1) *and* (1.6) *hold, then for* $t \gg 1$,

$$\begin{cases} c_0 t + g(t),\ c_0 t - h(t) \approx\ t^{3-\gamma} & \text{if } \gamma \in (2,3], \\ c_0 t + g(t),\ c_0 t - h(t) \approx\ \ln t & \text{if } \gamma = 3. \end{cases}$$

It is clear that the conclusion of Theorem 1.3 follows directly from Theorem 3.1. Note that if $\omega > 2$ in (3.1), then $(\mathbf{J^1})$ automatically holds.

By $(\mathbf{f_1})$ and the Perron-Frobenius theorem, we know that the matrix $\nabla F(0) - \widetilde{D}$ with $\widetilde{D} = \text{diag}(d_1, ..., d_m)$ has a principal eigenvalue $\tilde{\lambda}_1$ with a corresponding eigenvector $V^* = (v_1^*, \cdots, v_m^*) \gg \mathbf{0}$, namely

$$V^*\Big([\nabla F(0)]^T - \widetilde{D}\Big) = \tilde{\lambda}_1 V^*. \tag{3.2}$$

To prove Theorem 3.1, the difficult part is to find the lower bound for $c_0 t - h(t)$, which will be established according to the following two cases: (i) $\tilde{\lambda}_1 < 0$, (ii) $\tilde{\lambda}_1 \geq 0$.

As before, we will only estimate $c_0 t - h(t)$, since the estimate for $c_0 t + g(t)$ follows by making the variable change $x \to -x$ in the initial functions.

3.1. The case $\tilde{\lambda}_1 < 0$.

Lemma 3.2. *Suppose that the assumptions in Theorem 3.1 are satisfied. If* $\tilde{\lambda}_1 < 0$, *then there exists* $\sigma = \sigma(\gamma) > 0$ *such that for all large* $t > 0$,

$$\begin{cases} c_0 t - h(t) \geq \sigma\, t^{3-\gamma} & \text{if } \gamma \in (2,3), \\ c_0 t - h(t) \geq \sigma \ln t & \text{if } \gamma = 3. \end{cases} \tag{3.3}$$

Proof. Let $\beta := \gamma - 2 \in (0,1]$, and (c_0, Φ) be the solution of (1.4)-(1.5). Define

$$\epsilon(t) := K_1(t+\theta)^{-\beta}, \quad \delta(t) := K_2 - K_3 \int_0^t \epsilon(\tau)\mathrm{d}\tau$$

and

$$\begin{cases} \bar{h}(t) := c_0 t + \delta(t), & t \geq 0, \\ \overline{U}(t,x) := (1+\epsilon(t))\Phi(x - \bar{h}(t)) + \rho(t,x), & t \geq 0,\ x \leq \bar{h}(t), \end{cases}$$

where

$$\rho(t,x) := K_4 \xi(x - \overline{h}(t))\epsilon(t) V^*,$$

with $\xi \in C^2(\mathbb{R})$ satisfying

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\epsilon},\ \xi(x) = 0 \text{ for } |x| > 2\tilde{\epsilon}, \tag{3.4}$$

and the positive constants θ, K_1, K_2, K_3, K_4, $\tilde{\epsilon}$ are to be determined.

We are going to show that, it is possible to choose these constants and some $t_0 > 0$ such that

$$\overline{U}_t(t,x) \succeq D \circ \int_{g(t+t_0)}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \overline{U}(t,y)\mathrm{d}y - \overline{U}(t,x) + F(\overline{U}(t,x)) \tag{3.5}$$

$$\text{for } t > 0,\ x \in (g(t+t_0), \overline{h}(t)),$$

$$\bar{h}'(t) \geq \sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y \qquad \text{for } t > 0, \tag{3.6}$$

$$\overline{U}(t, g(t+t_0)) \succeq \mathbf{0}, \quad \overline{U}(t, \bar{h}(t)) \succeq \mathbf{0} \qquad \text{for } t \geq 0, \tag{3.7}$$

$$\overline{U}(0,x) \succeq U(t_0, x),\ \overline{h}(0) \geq h(t_0) \qquad \text{for } x \in [g(t_0), h(t_0)]. \tag{3.8}$$

If these inequalities are proved, then by the comparison principle, we obtain

$$\overline{h}(t) \geq h(t+t_0),\ \overline{U}(t,x) \succeq U(t+t_0, x) \text{ for } t > 0,\ x \in [g(t+t_0), h(t+t_0)],$$

and the desired inequality for $c_0 t - h(t)$ follows easily from the definition of $\overline{h}(t)$.

Therefore, to complete the proof, it suffices to prove the above inequalities. We divide the arguments below into several steps.

Firstly, by Theorem B, there is $C_1 > 1$ such that

$$-g(t), h(t) \leq (c_0+1)t + C_1 \quad \text{for} \quad t \geq 0. \tag{3.9}$$

Let us also note that (3.7) holds trivially.

Step 1. Choose $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (3.8) holds.

For later analysis, we need to find $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (3.8) holds and at the same time they have less than linear growth in θ.

Let $W^* \succ\!\!\succ \mathbf{0}$ be an eigenvector corresponding to the maximal eigenvalue $\tilde{\lambda}$ of $\nabla F(\mathbf{u}^*)$. By our assumptions on F, we have $\tilde{\lambda} < 0$. Hence there exists small $\epsilon_* > 0$ such that for any $k \in (0, \epsilon_*]$,

$$F(\mathbf{u}^* + kW^*) = kW^*\Big([\nabla F(\mathbf{u}^*)]^T + o(1)\mathbf{I}_m\Big) \preceq \frac{k}{2}\tilde{\lambda} W^* \prec\!\!\prec \mathbf{0},$$

$$F(\mathbf{u}^* - kW^*) = -kW^*\Big([\nabla F(\mathbf{u}^*)]^T + o(1)\mathbf{I}_m\Big) \succeq -\frac{k}{2}\tilde{\lambda} W^* \succ\!\!\succ \mathbf{0}.$$

It follows that, for $\tilde{\sigma} = \tilde{\lambda}/2$,

$$\overline{W}(t) = \mathbf{u}^* + \epsilon_* e^{\tilde{\sigma}t} W^*, \quad \underline{W}(t) = \mathbf{u}^* - \epsilon_* e^{\tilde{\sigma}t} W^*$$

are a pair of upper and lower solution of the ODE system $W' = F(W)$ with initial data $W(0) \in [\mathbf{u}^* - \epsilon_* W^*, \mathbf{u}^* + \epsilon_* W^*]$.

By $(\mathbf{f_4})$, the unique solution of the ODE system

$$W' = F(W),\ W(0) = (\|u_{10}\|_\infty, \cdots, \|u_{m0}\|_\infty)$$

satisfies $\lim_{t\to\infty} W(t) = \mathbf{u}^*$. Hence there exists $t_* > 0$ such that

$$W(t_*) \in [\mathbf{u}^* - \epsilon_* W^*, \mathbf{u}^* + \epsilon_* W^*].$$

Using the above defined upper solution $\overline{W}(t)$ we obtain

$$W(t+t_*) \preceq \mathbf{u}^* + \epsilon_* e^{\tilde{\sigma} t} W^* \preceq (1+\tilde{\epsilon}_* e^{\tilde{\sigma} t})\mathbf{u}^* \text{ for } t \geq 0,$$

where $\tilde{\epsilon}_* > 0$ is chosen such that $\epsilon_* W^* \leq \tilde{\epsilon}_* \mathbf{u}^*$. By the comparison principle we deduce

$$U(t+t_*, x) \preceq W(t+t_*) \preceq (1+\tilde{\epsilon}_* e^{\tilde{\sigma} t})\mathbf{u}^* \text{ for } t \geq 0,\ x \in [g(t+t_*), h(t+t_*)].$$

Hence

$$U(t_0, x) \preceq (1 + \frac{\epsilon(0)}{2})\mathbf{u}^* \text{ for } x \in [g(t_0), h(t_0)]$$

provided that

$$t_0 = t_0(\theta) := \frac{\beta}{|\tilde{\sigma}|}\ln\theta + \frac{\ln(2\tilde{\epsilon}_*/K_1)}{|\tilde{\sigma}|} + t_*.$$

By (3.1), for any fixed $\omega_* \in (\beta, \omega - 1)$, we have

$$\int_{\mathbb{R}} J(x)|x|^{\omega_*}\mathrm{d}x < \infty.$$

Then by Theorem 1.4, there is C_2 such that

$$\mathbf{u}^* - \Phi(x) \leq \frac{C_2}{|x|^{\omega_*}}\mathbf{u}^* \text{ for } \ x \leq -1.$$

Hence, for $K > 1$ we have

$$\begin{aligned}
&(1+\epsilon(0))\Phi(-K) - (1+\epsilon(0)/2)\mathbf{u}^* \\
\succeq&(1+\epsilon(0))\big[1 - C_2 K^{-\omega_*}\big]\mathbf{u}^* - (1+\epsilon(0)/2)\mathbf{u}^* \\
=&\big[K_1\theta^{-\beta}/2 - C_2 K^{-\omega_*}(1+K_1\theta^{-\beta})\big]\mathbf{u}^* \\
\succeq&\mathbf{0}
\end{aligned}$$

provided that

$$K^{\omega_*} \geq 2C_2 + \frac{2C_2}{K_1}\theta^{\beta}.$$

Therefore, for all $K_1 \in (0,1]$, $\theta \geq 1$ and $K \geq (4C_2/K_1)^{1/\omega_*}\theta^{\beta/\omega_*}$, we have

$$(1+\epsilon(0))\Phi(-K) - (1+\epsilon(0)/2)\mathbf{u}^* \succeq \mathbf{0}.$$

Now define

$$K_2(\theta) := 2\max\left\{(4C_2/K_1)^{1/\omega_*}\theta^{\beta/\omega_*}, (c_0+1)t_0(\theta) + C_1\right\}. \tag{3.10}$$

Then for $K_2 = K_2(\theta)$ we have

$$\overline{h}(0) = K_2 > K_2/2 \geq (c_0+1)t_0 + C_1 \geq h(t_0),$$

and for $x \in [g(t_0), h(t_0)]$,

$$\overline{U}(0,x) = (1+\epsilon(0))\Phi(x - K_2) \succeq (1+\epsilon(0))\Phi(-K_2/2) \succeq (1+\epsilon(0)/2)\mathbf{u}^*.$$

Thus (3.8) holds if t_0 and K_2 are chosen as above, for any $\theta \geq 1$, $K_1 \in (0,1]$.

Step 2. We verify that (3.6) holds if θ, K_1, K_3 and K_4 are chosen suitably.

Denote

$$C_3 := \sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_{0}^{+\infty} J_i(x-y)\mathrm{d}y\mathrm{d}x = \sum_{i=1}^{m_0}\mu_i\int_{0}^{+\infty} J_i(y)y\mathrm{d}y. \tag{3.11}$$

With $\rho=(\rho_i)$, a direct calculation shows

$$
\begin{aligned}
&\sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\\
=&\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x-\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\\
=&\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_{0}^{+\infty}J_i(x-y)[(1+\epsilon)\phi_i(x)+\rho_i(t,x+\overline{h}(t))]\mathrm{d}y\mathrm{d}x\\
&-\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty}J_i(x-y)[(1+\epsilon)\phi_i(x)+\rho_i(t,x+\overline{h}(t))]\mathrm{d}y\mathrm{d}x\\
\leq&(1+\epsilon)c_0+C_3K_4\epsilon|V^*|-\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty}J_i(x-y)(1+\epsilon)\phi_i(x)\mathrm{d}y\mathrm{d}x\\
\leq&(1+\epsilon)c_0+C_3K_4\epsilon|V^*|-\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty}J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x,
\end{aligned}
$$

where

$$|V^*|:=\max_{1\leq i\leq m}v_i^*.$$

By elementary calculus, for any $k>1$,

$$
\begin{aligned}
&\int_{-\infty}^{-k}\int_0^{\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x=\int_{-\infty}^{-k}\int_{-x}^{\infty}\frac{1}{y^{2+\beta}}\mathrm{d}y\mathrm{d}x=\int_k^{\infty}\int_x^{\infty}\frac{1}{y^{2+\beta}}\mathrm{d}y\mathrm{d}x\\
&=\int_k^{\infty}\int_k^{y}\frac{1}{y^{2+\beta}}\mathrm{d}x\mathrm{d}y=\int_k^{\infty}\frac{y-k}{y^{2+\beta}}\mathrm{d}y=\beta^{-1}(1+\beta)^{-1}k^{-\beta}.
\end{aligned}
\tag{3.12}
$$

From (3.1) and (3.9), there exists $C_4>0$ such that

$$
\begin{aligned}
&\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty}J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x\\
\geq&C_4\left[\min_{1\leq i\leq m}\phi_i(g(t+t_0)-\bar h(t))\right]\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x\\
\geq&\phi_*C_4\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_{0}^{+\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x=\frac{\phi_*C_4}{\beta(1+\beta)}(|g(t+t_0)|+\bar h(t))^{-\beta}\\
\geq&\frac{\phi_*C_4}{\beta(1+\beta)}[(c_0+1)(t+t_0)+C_1+c_0t+K_2]^{-\beta}\\
=&\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^{\beta}}\left[t+\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)}\right]^{-\beta},
\end{aligned}
\tag{3.13}
$$

where $\phi_*=\min_{1\leq i\leq m}\phi_i(-1)\leq\min_{1\leq i\leq m}\phi_i(-K_2)\leq\min_{1\leq i\leq m}\phi_i(g(t+t_0)-\bar h(t))$. Therefore, for all large $\theta>0$ so that

$$\theta>\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)},\tag{3.14}$$

which is possible since $t_0(\theta)$ and $K_2(\theta)$ grow slower than linearly in θ, we have

$$\sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&\le (1+\epsilon(t))c_0 + C_4K_4\epsilon(t)|V^*| - \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}(t+\theta)^{-\beta}\\
&= c_0 + \epsilon(t)\left[c_0 + C_4K_4|V^*| - \frac{\phi_* C_4}{K_1\beta(1+\beta)(2c_0+1)^\beta}\right]\\
&\le c_0 - K_3\epsilon(t) = h'(t)
\end{aligned}
$$

provided that K_1, K_3 and K_4 are small enough so that

$$
K_1(c_0 + C_4K_4|V^*| + K_3) \le \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}. \tag{3.15}
$$

Therefore (3.6) holds if we first fix K_1, K_3, K_4 small so that (3.15) holds, and then choose θ large such that (3.14) is satisfied.

Step 3. We show that (3.5) holds when K_3 and K_4 are chosen suitably small and θ is large. From (1.4), we deduce

$$
\overline{U}_t(t,x) = -(1+\epsilon)[c_0 + \delta'(t)]\Phi'(x - \bar h(t)) + \epsilon'(t)\Phi(x - \underline{h}(t)) + \rho_t(t,x),
$$

and

$$
\begin{aligned}
&-(1+\epsilon)c_0\Phi'(x-\bar h(t))\\
&=(1+\epsilon)\left[D\circ\int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\circ\Phi(y-\bar h(t))\mathrm{d}y - D\circ\Phi(x-\bar h(t)) + F(\Phi(x-\bar h(t)))\right]\\
&=D\circ\int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\circ[\overline{U}(t,y)-\rho(t,y)]\mathrm{d}y - D\circ[\overline{U}(t,x)-\rho(t,x)] + (1+\epsilon)F(\Phi(x-\bar h(t)))\\
&=D\circ\int_{g(t+t_0)}^{\bar h(t)} \mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y - D\circ\overline{U}(t,x) + F(\overline{U}(t,x))\\
&\quad + D\circ\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\overline{U}_t(t,x) =& D\circ\int_{g(t+t_0)}^{\bar h(t)} \mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y - D\circ\overline{U}(t,x) + F(\overline{U}(t,x))\\
&+ A(t,x)
\end{aligned}
$$

with

$$
\begin{aligned}
A(t,x) :=& D\circ\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x))\\
&-(1+\epsilon)\delta'(t)\Phi'(x-\bar h(t)) + \epsilon'(t)\Phi(x-\underline{h}(t)) + \rho_t(t,x).
\end{aligned}
$$

Therefore to complete this step, it suffices to show that we can choose K_3, K_4 and θ such that $A(t,x) \succeq \mathbf{0}$. We will do that for $x \in [\bar h(t) - \tilde\epsilon, \bar h(t)]$ and for $x \in [g(t_0+t), \bar h(t) - \tilde\epsilon]$ separately.

Claim 1. If $\tilde\epsilon > 0$ in (3.4) is sufficiently small and θ is sufficiently large, then

$$
\begin{aligned}
&D\circ\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x))\\
&\succeq \frac{|\tilde\lambda_1|}{4}\rho(t,x) \succ \mathbf{0} \quad \text{for} \quad x \in [\bar h(t) - \tilde\epsilon, \bar h(t)].
\end{aligned} \tag{3.16}
$$

Since $\tilde{\lambda}_1 < 0$ and $D \circ V^* = V^* \widetilde{D}$, using (3.2) we deduce, for $x \in [\bar{h}(t) - \tilde{\epsilon}, \bar{h}(t)]$,

$$
\begin{aligned}
& D \circ \left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \rho(t,y)\mathrm{d}y\right] \\
=& K_4\epsilon(t)\left[D \circ V^* - D \circ \int_{-\infty}^{0} \mathbf{J}(x - \bar{h}(t) - y) \circ \xi(y) V^* \mathrm{d}y\right] \\
\succeq& K_4\epsilon(t)\left[D \circ V^* - D \circ \int_{-2\tilde{\epsilon}}^{0} \mathbf{J}(x - \bar{h}(t) - y) \circ V^* \mathrm{d}y\right] \\
=& K_4\epsilon(t)\left[V^*\nabla F(0) - \tilde{\lambda}_1 V^* - D \circ \int_{\bar{h}(t)-x-2\tilde{\epsilon}}^{\bar{h}(t)-x} \mathbf{J}(y) \circ V^* \mathrm{d}y\right] \\
\succeq& K_4\epsilon(t)\left[V^*\nabla F(0) - \tilde{\lambda}_1 V^* - D \circ \int_{-2\tilde{\epsilon}}^{\tilde{\epsilon}} \mathbf{J}(y) \circ V^* \mathrm{d}y\right] \\
\succeq& K_4\epsilon(t)\left[V^*\nabla F(0) - \frac{\tilde{\lambda}_1}{2} V^*\right] = \rho(t,x)\nabla F(0) - \frac{\tilde{\lambda}_1}{2}\rho(t,x),
\end{aligned}
$$

provided $\tilde{\epsilon} \in (0, \epsilon_1]$ for some small $\epsilon_1 > 0$.

On the other hand, for $x \in [\bar{h}(t) - \tilde{\epsilon}, \bar{h}(t)]$, by $(\mathbf{f_2})$ we obtain

$$
\begin{aligned}
& (1+\epsilon)F(\Phi(x - \bar{h}(t))) - F(\overline{U}(t,x)) \\
\succeq& F((1+\epsilon)\Phi(x - \bar{h}(t))) - F(\overline{U}(t,x)) \\
=& F(\overline{U}(t,x) - \rho(t,x)) - F(\overline{U}(t,x)),
\end{aligned}
$$

and

$$
\mathbf{0} \preceq \overline{U}(t,x) \preceq (1+\epsilon)\Phi(\tilde{\epsilon}) + K_4\epsilon V^* \preceq 2\Phi(\tilde{\epsilon}) + \theta^{-\beta} V^*,
$$

So the components of $\overline{U}(t,x)$ and $\rho(t,x)$ are small for small $\tilde{\epsilon}$ and large θ. It follows that

$$
\begin{aligned}
& F(\overline{U}(t,x) - \rho(t,x)) - F(\overline{U}(t,x)) = -\rho(t,x)[\nabla F(\overline{U}(t,x)) + o(1)\mathbf{I}_m] \\
&= -\rho(t,x)[\nabla F(0) + o(1)\mathbf{I}_m] \succeq -\rho(t,x)\nabla F(0) + \frac{\tilde{\lambda}_1}{4}\rho(t,x)
\end{aligned}
$$

for $x \in [\bar{h}(t) - \tilde{\epsilon}, \bar{h}(t)]$, provided that $\tilde{\epsilon}$ is small and θ is large. Hence, (3.16) holds.

Denote

$$
M := \max_{1 \le i \le m} \sup_{x \le 0} |\phi_i'(x)|.
$$

For $x \in [\bar{h} - \tilde{\epsilon}, \bar{h}]$, by (3.16) we have

$$
\begin{aligned}
A(t,x) \succeq& \frac{|\tilde{\lambda}_1|}{4}\rho(t,x) - (1+\epsilon)\delta'(t)\Phi'(x - \bar{h}(t)) + \epsilon'(t)\Phi(x - \underline{h}(t)) + \rho_t(t,x) \\
\succeq& \epsilon(t)\left[\frac{|\tilde{\lambda}_1|}{4}K_4 V^* - 2K_3 M\mathbf{1} - \beta(t+\theta)^{-1}\mathbf{u}^* - K_4\beta(t+\theta)^{-1}V^*\right] \\
\succeq& \epsilon(t)\left[\frac{|\tilde{\lambda}_1|}{4}K_4 V^* - 2K_3 M\mathbf{1} - \theta^{-1}\beta\Big(\mathbf{u}^* + K_4 V^*\Big)\right] \\
\succeq& \mathbf{0}
\end{aligned}
$$

provided that we first fix K_3 and K_4 so that (3.15) holds and at the same time

$$
\frac{|\tilde{\lambda}_1|}{4}K_4 V^* - 2K_3 M\mathbf{1} \gg \mathbf{0}, \tag{3.17}
$$

and then choose θ sufficiently large.

Next, for fixed small $\tilde{\epsilon} > 0$, we estimate $A(t,x)$ for $x \in [g(t+t_0), \bar{h}(t) - \tilde{\epsilon}]$.

Claim 2. For any given $1 \gg \eta > 0$, there is $c_1 = c_1(\eta)$ such that

$$(1+\epsilon)F(v) - F((1+\epsilon)v) \succeq c_1\epsilon\mathbf{1} \quad \text{for} \quad v \in [\eta\mathbf{1}, \mathbf{u}^*] \text{ and } 0 < \epsilon \ll 1. \tag{3.18}$$

Indeed, by (1.6) there exists $c_1 > 0$ depending on η such that

$$F(v) - v[\nabla F(v)]^T \succeq 2c_1\mathbf{1} \quad \text{for} \quad v \in [\eta\mathbf{1}, \mathbf{u}^*].$$

Since

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{(1+\epsilon)F(v) - F((1+\epsilon)v)}{\epsilon} =& \lim_{\epsilon\to 0} \frac{\epsilon F(v) - [F(v+\epsilon v) - F(v)]}{\epsilon} \\ =& F(v) - v[\nabla F(v)]^T \succeq 2c_1\mathbf{1}\end{aligned}$$

uniformly for $v \in [\eta\mathbf{1}, \mathbf{u}^*]$, there exists $\epsilon_0 > 0$ small so that

$$\frac{(1+\epsilon)F(v) - F((1+\epsilon)v)}{\epsilon} \succeq c_1\mathbf{1}$$

for $v \in [\eta\mathbf{1}, \mathbf{u}^*]$ and $\epsilon \in (0, \epsilon_0]$. This proves Claim 2.

By Claim 2 and the Lipschitz continuity of F, there exist positive constants C_l and C_f such that, for $v = \Phi(x - \bar{h}(t)) \in [\Phi(-\tilde{\epsilon}), \mathbf{u}^*]$,

$$\begin{aligned}&(1+\epsilon)F(v) - F((1+\epsilon)v + \rho) \\ =&(1+\epsilon)F(v) - F((1+\epsilon)v) + F((1+\epsilon)v) - F((1+\epsilon)v + \rho) \\ \succeq& C_l\epsilon\mathbf{1} - C_f K_4 \epsilon\mathbf{1}\end{aligned}$$

when $\epsilon = \epsilon(t)$ is small.

We also have

$$\begin{aligned}&D \circ \left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \rho(t,x)\mathrm{d}y\right] \succeq -D \circ \int_{-\infty}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \rho(t,x)\mathrm{d}y \\ \succeq& -K_4\epsilon(t) D \circ V^* \succeq -C_d K_4 \epsilon(t)\mathbf{1}\end{aligned}$$

for some $C_d > 0$, and

$$\begin{aligned}\rho_t(t,x) =& -\xi'\bar{h}'K_4\epsilon(t)V^* + \xi K_4\epsilon'(t)V^* \\ \succeq& -\xi_* K_4\epsilon(t)V^* - K_4\beta(t+\theta)^{-1}\epsilon(t)V^* \\ \succeq& -(\xi_* + \beta\theta^{-1})K_4\epsilon(t)V^*,\end{aligned}$$

with $\xi_* := c_0 \max_{x\in\mathbb{R}} |\xi'(x)|$.

Using these we obtain, for $x \in [g(t_0+t), \bar{h}(t) - \tilde{\epsilon}]$,

$$\begin{aligned}A(t,x) \succeq& -C_d K_4 \epsilon(t)\mathbf{1} + (1+\epsilon)F(\Phi(x - \bar{h}(t))) - F(\bar{U}(t,x)) + 2M\delta'(t)\mathbf{1} + \epsilon'(t)\mathbf{u}^* + \rho_t(t,x) \\ \succeq& C_l\epsilon(t)\mathbf{1} - (C_f + C_d)K_4\epsilon(t)\mathbf{1} - 2MK_3\epsilon(t)\mathbf{1} - \beta(t+\theta)^{-1}\epsilon(t)\mathbf{u}^* - (\xi_* + \beta\theta^{-1})K_4\epsilon(t)V^* \\ =& \epsilon(t)\left[C_l\mathbf{1} - K_4(C_f + C_d)\mathbf{1} - 2MK_3\mathbf{1} - \beta(t+\theta)^{-1}\mathbf{u}^* - (\xi_* + \beta\theta^{-1})K_4V^*\right] \\ \succeq& \epsilon(t)\left[C_l\mathbf{1} - K_4(C_f + C_d)\mathbf{1} - 2MK_3\mathbf{1} - \xi_* K_4V^* - \beta\theta^{-1}\big(\mathbf{u}^* + K_4V^*\big)\right] \\ \succeq& \mathbf{0}\end{aligned}$$

provided that we first choose K_3 and K_4 small such that

$$C_l\mathbf{1} - K_4(C_f + C_d)\mathbf{1} - 2MK_3\mathbf{1} - \xi_* K_4 V^* \succ\succ \mathbf{0}$$

while keeping both (3.15) and (3.17) hold, and then choose $\theta > 0$ sufficiently large.

Therefore, (3.5) holds when K_3, K_4 and θ are chosen as above. The proof of the lemma is now complete. $\square$

3.2. The case $\tilde{\lambda}_1 \geq 0$.

Lemma 3.3. *Suppose that the assumptions in Theorem 3.1 are satisfied. If $\tilde{\lambda}_1 \geq 0$, then* (3.3) *still holds.*

Proof. This is a modification of the proof of Lemma 3.2. We will use similar notations. Let $\beta = \gamma - 2 \in (0, 1]$, and (c_0, Φ) be the solution of (1.4)-(1.5). For fixed $\tilde{\epsilon} > 0$, let $\xi \in C^2(\mathbb{R})$ satisfy

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\epsilon},\ \xi(x) = 0 \text{ for } |x| > 2\tilde{\epsilon}.$$

Define

$$\begin{cases} \bar{h}(t) := c_0 t + \delta(t), & t \geq 0, \\ \overline{U}(t,x) := (1+\epsilon(t))\Phi\big(x - \bar{h}(t) - \lambda(t)\big) - \rho(t,x), & t \geq 0,\ x \leq \bar{h}(t), \end{cases}$$

where

$$\epsilon(t) := K_1(t+\theta)^{-\beta},\ \delta(t) := K_2 - K_3 \int_0^t \epsilon(\tau)\mathrm{d}\tau,$$
$$\rho(t,x) := K_4 \xi(x - \bar{h}(t))\epsilon(t) V^*,\ \lambda(t) := K_5 \epsilon(t),$$

and the positive constants θ and K_1, K_2, K_3, K_4, K_5 are to be determined.

Let

$$C_{\tilde{\epsilon}} := \min_{1 \leq i \leq m} \min_{x \in [-2\tilde{\epsilon}, 0]} |\phi_i'(x)|.$$

Then for $x \in [\bar{h}(t) - 2\tilde{\epsilon}, \bar{h}(t)]$ and $i \in \{1, ..., m\}$, with $\rho(t,x) = (\rho_i(t,x))$,

$$\begin{aligned} \bar{u}_i(t,x) \geq& \phi_i\big(-\lambda(t)\big) - \rho_i(t,x) \geq C_{\tilde{\epsilon}}\lambda(t) - K_4 \epsilon(t) v_i^* \\ \geq& \epsilon(t)(C_{\tilde{\epsilon}} K_5 - K_4 v_i^*) > 0 \end{aligned}$$

if

$$K_4 = C_{\tilde{\epsilon}} K_5 / (2 \max_{1 \leq i \leq m} v_i^*), \tag{3.19}$$

which combined with $\xi(x) = 0$ for $|x| \geq 2\tilde{\epsilon}$ implies

$$\overline{U}(t,x) \succeq \mathbf{0} \text{ for } \ t \geq 0,\ x \leq \bar{h}(t). \tag{3.20}$$

Let $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ be given by Step 1 in the proof of Lemma 3.2. Then $[g(t_0), h(t_0)] \subset (-\infty, K_2/2)$, and due to $\rho(0,x) = 0$ for $x \leq h(t_0) < K_2/2 < K_2 = \bar{h}(0)$, we have

$$\begin{aligned} \overline{U}(0,x) =& (1+\epsilon(0))\Phi(x - K_2 - \lambda) \succeq (1+\epsilon(0))\Phi(-K_2/2) \\ \succeq& (1+\epsilon(0)/2)\mathbf{u}^* \succeq U(t_0, x) \ \text{ for } \ x \in [g(t_0), h(t_0)]. \end{aligned} \tag{3.21}$$

Step 1. We verify that by choosing K_1, K_3 and K_5 suitably small,

$$\bar{h}'(t) \geq \sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y\mathrm{d}x \ \text{ for all } \ t > 0. \tag{3.22}$$

By direct calculations we have

$$\begin{aligned} &\sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)\bar{u}_i(t,x)\mathrm{d}y\mathrm{d}x \\ \leq& \sum_{i=1}^{m_0} \mu_i \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J_i(x-y)(1+\epsilon)\phi_i(x - \bar{h}(t) - \lambda(t))\mathrm{d}y\mathrm{d}x \\ =& (1+\epsilon) \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{+\infty} J_i(x-y)\phi_i(x - \lambda(t))\mathrm{d}y\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&-(1+\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}J_i(x-y)\phi_i(x-\lambda(t))\mathrm{d}y\mathrm{d}x\\
\le&(1+\epsilon)c_0+(1+\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_0^{+\infty}J_i(x-y)[\phi_i(x-\lambda)-\phi_i(x)]\mathrm{d}y\mathrm{d}x\\
&-(1+\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x
\end{aligned}
$$

Let $M_1:=\max_{1\le i\le m}\sup_{x\le 0}|\phi_i'(x)|$ and C_3 be given by (3.11). Then

$$
(1+\epsilon)\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_0^{+\infty}J_i(x-y)[\phi_i(x-\lambda(t))-\phi_i(x)]\mathrm{d}y\mathrm{d}x\le 2C_3M_1\lambda(t).
$$

By (3.13),

$$
\begin{aligned}
&\sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x\\
\ge&\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}\left[t+\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)}\right]^{-\beta}.
\end{aligned}
$$

Therefore, as in the proof of Lemma 3.2, for sufficiently large θ so that

$$
\theta>\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)} \tag{3.23}
$$

holds, we have

$$
\begin{aligned}
&\sum_{i=1}^{m_0}\mu_i\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\\
\le&(1+\epsilon)c_0+2C_3M_1\lambda(t)-\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}(t+\theta)^{-\beta}\\
=&c_0+\epsilon(t)\left[c_0+2C_3M_1K_5-\frac{\phi_*C_4}{K_1\beta(1+\beta)(2c_0+1)^\beta}\right]\\
\le&c_0-K_3\epsilon(t)=\bar h'(t)
\end{aligned}
$$

provided that K_1, K_3 and K_5 are suitably small so that

$$
K_1(c_0+2C_3M_1K_5+K_3)\le\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}. \tag{3.24}
$$

Step 2. We show that by choosing K_3, K_5 suitably small and θ sufficiently large, for $t>0$, $x\in[g(t+t_0),\bar h(t)]$,

$$
\overline U_t(t,x)\succeq D\circ\int_{g(t+t_0)}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline U(t,y)\mathrm{d}y-\overline U(t,x)+F(\overline U(t,x)). \tag{3.25}
$$

Using the definition of $\overline U$, we have

$$
\begin{aligned}
\overline U_t(t,x)=&-(1+\epsilon)(\bar h'+\lambda')\Phi'(x-\bar h-\lambda)+\epsilon'\Phi(x-\bar h-\lambda)-\rho_t\\
=&-(1+\epsilon)[c_0+\delta'+\lambda']\Phi'(x-\bar h-\lambda)+\epsilon'\Phi(x-\bar h-\lambda)-\rho_t
\end{aligned}
$$

and from (1.4), we obtain

$$
-(1+\epsilon)c_0\Phi'(x-\bar h-\lambda)
$$

$$
\begin{aligned}
=&(1+\epsilon)\left[D\circ\int_{-\infty}^{\bar h+\lambda}\mathbf{J}(x-y)\circ\Phi(y-\bar h-\lambda)\mathrm{d}y-D\circ\Phi(x-\bar h-\lambda)+F(\Phi(x-\bar h-\lambda))\right]\\
\succeq&(1+\epsilon)\left[D\circ\int_{-\infty}^{\bar h}\mathbf{J}(x-y)\circ\Phi(y-\bar h-\lambda)\mathrm{d}y-D\circ\Phi(x-\bar h-\lambda)+F(\Phi(x-\bar h-\lambda))\right]\\
=&D\circ\int_{-\infty}^{\bar h}\mathbf{J}(x-y)\circ[\overline{U}(t,y)+\rho]\mathrm{d}y-D\circ[\overline{U}(t,x)+\rho]+(1+\epsilon)F(\Phi(x-\bar h-\lambda))\\
=&D\circ\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)\\
&-D\circ\left[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)F(\Phi(x-\bar h-\lambda))\\
\succeq&D\circ\int_{g(t+t_0)}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x))\\
&-D\circ\left[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)F(\Phi(x-\bar h-\lambda))-F(\overline{U}(t,x)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\overline{U}_t(t,x)\succeq&D\circ\int_{g(t+t_0)}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x))\\
&+B(t,x)
\end{aligned}
$$

with

$$
\begin{aligned}
B(t,x):=&-D\circ\left[\rho(t,x)-\int_{-\infty}^{\bar h}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)F(\Phi(x-\bar h-\lambda))-F(\overline{U})\\
&-(1+\epsilon)(\delta'+\lambda')\Phi'(x-\bar h-\lambda)+\epsilon'\Phi(x-\underline{h}-\lambda)-\rho_t.
\end{aligned}
$$

To show (3.25), it remains to choose suitable K_3,K_5 and θ such that $B(t,x)\succeq\mathbf{0}$ for $t>0$ and $x\in[g(t+t_0),\bar h(t)]$.

Claim: There exist small $\tilde\epsilon_0\in(0,\tilde\epsilon/2)$ and some $\tilde J_0>0$ depending on $\tilde\epsilon$ but independent of $\tilde\epsilon_0$, such that

$$
\begin{aligned}
&-D\circ\left[\rho(t,x)-\int_{-\infty}^{\bar h}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)F(\Phi(x-\bar h-\lambda))-F(\overline{U}(t,x))\\
&\succeq\ \tilde J_0\,\rho(t,x)\quad\text{for}\quad x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)].
\end{aligned}
\tag{3.26}
$$

Indeed, for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$,

$$
\begin{aligned}
&D\circ\left[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\rho(t,y)\mathrm{d}y\right]\\
=&K_4\epsilon(t)\left[D\circ V^*-D\circ\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\circ\xi(y-\bar h(t))V^*\mathrm{d}y\right]\\
\preceq&K_4\epsilon(t)\left[D\circ V^*-D\circ\int_{\bar h(t)-\tilde\epsilon}^{\bar h(t)}\mathbf{J}(x-y)\circ V^*\mathrm{d}y\right]\\
=&K_4\epsilon(t)\left[D\circ V^*-D\circ\int_{\bar h(t)-\tilde\epsilon-x}^{\bar h(t)-x}\mathbf{J}(y)\circ V^*\mathrm{d}y\right]
\end{aligned}
$$

$$\preceq D\circ\rho\left[1-\int_{-\tilde\epsilon+\tilde\epsilon_0}^{0}\mathbf{J}(y)\mathrm{d}y\right]\preceq D\circ\rho\left[1-\int_{-\tilde\epsilon/2}^{0}\mathbf{J}(y)\mathrm{d}y\right].$$

On the other hand, for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$, we have

$$\begin{aligned}
&(1+\epsilon)F(\Phi(x-\bar h-\lambda)-F(\overline{U})\\
\succeq& F((1+\epsilon)\Phi(x-\bar h-\lambda))-F(\overline{U})\\
=&F(\overline{U}+\rho)-F(\overline{U})=\rho\Big([\nabla F(\overline{U})]^T+o(1)\mathbf{I}_m\Big)\\
=&K_4\epsilon(t)V^*\Big([\nabla F(\mathbf{0})]^T+o(1)\mathbf{I}_m\Big)\\
=&K_4\epsilon(t)[V^*\tilde D+\tilde\lambda_1V^*+o(1)V^*]\\
=&K_4\epsilon(t)[D\circ V^*+\tilde\lambda_1V^*+o(1)V^*]\\
=&D\circ\rho+\tilde\lambda_1\rho+o(1)\rho.
\end{aligned}$$

since both $\overline{U}(t,x)$ and $\rho(t,x)$ are close to $\mathbf{0}$ for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$ with $\tilde\epsilon_0$ small.

Hence, for such x and $\tilde\epsilon_0$, since $\tilde\lambda_1\ge 0$,

$$\begin{aligned}
&-D\circ\left[\rho(t,x)-\int_{-\infty}^{\bar h(t)}\mathbf{J}(x-y)\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)F(\Phi(x-\bar h(t)))-F(\overline{U}(t,x))\\
\succeq& D\circ\rho\left[-1+\int_{-\tilde\epsilon/2}^{0}\mathbf{J}(y)\mathrm{d}y\right]+D\circ\rho+\tilde\lambda_1\rho+o(1)\rho\\
\succeq&\tilde J_0\,\rho(t,x),\qquad\text{with}\quad \tilde J_0:=\frac12\min_{1\le i\le m}d_i\int_{-\tilde\epsilon/2}^{0}J_i(y)\mathrm{d}y\ \text{if}\ m_0=m.
\end{aligned}$$

This proves (3.26) when $m_0=m$.

If $m_0<m$, we need to modify V^* in the definition of ρ slightly. In this case, for $\tilde\delta>0$ small we define

$$\tilde V^*:=V^*+\tilde\delta D=(v_i^*+\tilde\delta d_i).$$

Since $d_i=0$ for $i=m_0+1,...,m$ and $d_i>0$ for $i=1,...,m_0$, by $(\mathbf{f_1})$ (iv) we see that

$$W=(w_i):=D[\nabla F(\mathbf{0})]^T$$

satisfies $w_i>0$ for $i=m_0+1,...,m$. Let us write

$$W=W^1+W^2=(w_i^1)+(w_i^2)\ \text{with}\ \begin{cases}w_i^1=0\ \text{for}\ i=m_0+1,...,m,\\ w_i^2=0\ \text{for}\ i=1,...,m_0.\end{cases}$$

Then

$$\tilde V^*\Big([\nabla F(\mathbf{0})]^T-\tilde D\Big)=\tilde\lambda_1V^*+\tilde\delta\widetilde{W}^1+\tilde\delta W^2\ \ \text{with}\ \widetilde{W}^1:=W^1-D\tilde D.$$

It is important to observe that the vector $\widetilde{W}^1=(\tilde w_i^1)$ has its last $m-m_0$ components 0, namely $\tilde w_i^1=0$ for $i=m_0+1,...,m$.

Replacing V^* by $\tilde V^*$ in the definition of ρ, we see that the analysis above is not affected, except that, for $\tilde\epsilon_0>0$ small and $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$,

$$\begin{aligned}
&(1+\epsilon)F(\Phi(x-\bar h-\lambda)-F(\overline{U})\\
\succeq&K_4\epsilon(t)\tilde V^*\Big([\nabla F(\mathbf{0})]^T+o(1)\mathbf{I}_m\Big)\\
=&K_4\epsilon(t)\Big([\tilde V^*\tilde D+\tilde\lambda_1V^*+o(1)V^*]+\tilde\delta\widetilde{W}^1+\tilde\delta W^2\Big)\\
=&K_4\epsilon(t)\Big(D\circ\tilde V^*+\tilde\lambda_1V^*+o(1)V^*+\tilde\delta\widetilde{W}^1+\tilde\delta W^2\Big)
\end{aligned}$$

$$\succeq D\circ\rho + K_4\epsilon(t)\Big(o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big).$$

Hence, for such x and $\tilde\epsilon_0$, we now have

$$\begin{aligned}
&-D\circ\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x))\\
&\succeq D\circ\rho\left[-1 + \int_{-\tilde\epsilon/2}^{0}\mathbf{J}(y)\mathrm{d}y\right] + D\circ\rho + K_4\epsilon(t)\Big(o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big)\\
&\succeq K_4\epsilon(t)\Big(\min_{1\le i\le m_0} v_i^*\int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}yD + o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big).
\end{aligned}$$

We now fix $\tilde\delta > 0$ small enough such that

$$-\tilde\delta\widetilde{W}^1 \preceq \frac{1}{2}\min_{1\le i\le m_0} v_i^* d_i \int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}y,$$

and notice that

$$\widehat{W} := \frac{1}{2}\min_{1\le i\le m_0} v_i^* d_i \int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}y + \tilde\delta W^2 \succ\!\succ \mathbf{0}.$$

Therefore there exists $\tilde J_0 > 0$ such that

$$\frac{1}{2}\widehat{W} \succeq \tilde J_0 \tilde V^*.$$

Then

$$\begin{aligned}
&K_4\epsilon(t)\Big(\min_{1\le i\le m_0} v_i^* d_i\int_{-\tilde\epsilon/2}^{0} J_i(y)\mathrm{d}y + o(1)V^* + \tilde\delta\widetilde{W}^1 + \tilde\delta W^2\Big)\\
&\succeq K_4\epsilon(t)\Big(\widehat{W} + o(1)V^*\Big) \succeq K_4\epsilon(t)\frac{1}{2}\widehat{W} \succeq K_4\epsilon(t)\tilde J_0\tilde V^* = \tilde J_0\rho,
\end{aligned}$$

provided that $\tilde\epsilon_0 > 0$ is chosen sufficiently small.

Therefore for $\tilde\epsilon_0 > 0$ small and $x \in [\bar h(t) - \tilde\epsilon_0, \bar h(t)]$, we finally have

$$\begin{aligned}
&-D\circ\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} \mathbf{J}(x-y)\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)F(\Phi(x-\bar h(t))) - F(\overline{U}(t,x))\\
&\succeq \tilde J_0\,\rho(t,x), \quad \text{as desired.}
\end{aligned}$$

With $\tilde\delta > 0$ chosen as above, we will from now on denote

$$\hat V^* := \begin{cases} V^* & \text{if } m_0 = m,\\ \tilde V^* & \text{if } m_0 < m,\end{cases}$$

but keep the notation for ρ unchanged.

Clearly

$$-\rho_t(t,x) = \beta K_4 K_1 (t+\theta)^{-\beta-1}\hat V^* \succeq \mathbf{0}.$$

Recalling $M_1 := \max_{1\le i\le m}\sup_{x\le 0}|\phi_i'(x)|$, we obtain, for $x \in [\bar h(t) - \tilde\epsilon_0, \bar h(t)]$ and small $\tilde\epsilon_0$,

$$\begin{aligned}
B(t,x) &\succeq \tilde J_0 K_4\epsilon(t)\hat V^* + 2(\delta'(t) + \lambda'(t))M_1\mathbf{1} + \epsilon'(t)\mathbf{u}^*\\
&= \tilde J_0 K_4\epsilon(t)\hat V^* + 2\epsilon(t)(-K_3 - K_5\beta(t+\theta)^{-1})M_1\mathbf{1} - \beta(t+\theta)^{-1}\epsilon(t)\mathbf{u}^*\\
&\succeq \epsilon(t)\left[\tilde J_0 K_4\hat V^* - 2(K_3 + K_5\beta\theta^{-1})M_1\mathbf{1} - \beta\theta^{-1}\mathbf{u}^*\right]\\
&= \epsilon(t)\left[\tilde J_0 K_4\hat V^* - 2K_3M_1\mathbf{1} - \theta^{-1}\Big(K_5\beta M_1\mathbf{1} + \beta\mathbf{u}^*\Big)\right]
\end{aligned}$$

$$\succeq \mathbf{0}$$

provided that K_3 is chosen small so that (3.24) holds,

$$\tilde{J}_0 K_4 \hat{V}^* - 2K_3 M_1 \mathbf{1} \succ\succ \mathbf{0}, \tag{3.27}$$

and θ is chosen sufficiently large.

We next estimate $B(t,x)$ for $x \in [g(t+t_0), \bar{h}(t) - \tilde{\epsilon}_0]$. From Claim 2 in the proof of Lemma 3.2, and the Lipschitz continuity of F, there exist positive constants $C_l = C_l(\tilde{\epsilon}_0)$ and C_f such that, for $v = \Phi(x - \bar{h}(t - \lambda(t))) \in [\Phi(-\tilde{\epsilon}_0), \mathbf{u}^*]$,

$$\begin{aligned}
&(1+\epsilon)F(v) - F((1+\epsilon)v - \rho) \\
=&(1+\epsilon)F(v) - F((1+\epsilon)v) + F((1+\epsilon)v) - F((1+\epsilon)v - \rho) \\
\succeq& C_l \epsilon \mathbf{1} - C_f \rho \succeq C_l \epsilon \mathbf{1} - C_f K_4 \epsilon \hat{V}^*
\end{aligned}$$

when $\epsilon = \epsilon(t)$ is small. Hence

$$\begin{aligned}
&(1+\epsilon)F(\Phi(x - \bar{h} - \lambda)) - F(\bar{U}) \\
\succeq& C_l \epsilon \mathbf{1} - C_f K_4 \epsilon \hat{V}^* \quad \text{for} \quad x \in [g(t+t_0), \bar{h}(t) - \tilde{\epsilon}_0],\ 0 < \tilde{\epsilon}_0 \ll 1.
\end{aligned}$$

Clearly,

$$-D \circ \left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} \mathbf{J}(x-y) \circ \rho(t,x) \mathrm{d}y \right] \succeq -K_4 \epsilon(t) D \circ \hat{V}^*,$$

and

$$\rho_t(t,x) = -K_4 \xi' \bar{h}'(t) \epsilon(t) \hat{V}^* + K_4 \xi \epsilon'(t) \hat{V}^* \preceq \xi_* K_4 \epsilon(t) \hat{V}^*$$

with $\xi_* := c_0 \max_{x \in \mathbb{R}} |\xi'(x)|$.

We thus obtain, for $x \in [g(t+t_0), \bar{h}(t) - \tilde{\epsilon}_0]$ and $0 < \tilde{\epsilon}_0 \ll 1$,

$$\begin{aligned}
B(t,x) \succeq& - K_4 \epsilon(t) D \circ \hat{V}^* + (1+\epsilon) F(\phi(x - \bar{h})) - F(\overline{U}) + 2M_1(\delta' + \lambda')\mathbf{1} + \epsilon' \mathbf{u}^* - \rho_t \\
\succeq& C_l \epsilon(t) \mathbf{1} - K_4 \epsilon(t)(D \circ \hat{V}^* + C_f \hat{V}^* + \xi_* \hat{V}^*) + 2M_1(-K_3 \epsilon(t) + K_5 \epsilon'(t))\mathbf{1} + \epsilon'(t) \mathbf{u}^* \\
\succeq& \epsilon(t) \Big[C_l \mathbf{1} - K_4(D \circ \hat{V}^* + C_f \hat{V}^* + \xi_* \hat{V}^*) - 2M_1(K_3 + K_5 \beta (t+\theta)^{-1})\mathbf{1} - \beta(t+\theta)^{-1} \mathbf{u}^* \Big] \\
\succeq& \epsilon(t) \Big[C_l \mathbf{1} - K_4 \Big(D \circ \hat{V}^* + C_f \hat{V}^* + \xi_* \hat{V}^* \Big) - 2M_1 K_3 \mathbf{1} - \theta^{-1} \beta \Big(2M_1 K_5 \mathbf{1} + \mathbf{u}^* \Big) \Big] \\
\succeq& \mathbf{0}
\end{aligned}$$

if we choose K_3 and K_5 small so that (3.24) and (3.27) hold and at the same time, due to (3.19)

$$C_l \mathbf{1} - K_4 \Big(D \circ \hat{V}^* + C_f \hat{V}^* + \xi_* \hat{V}^* \Big) - 2M_1 K_3 \mathbf{1} \succ\succ \mathbf{0},$$

and then choose θ sufficiently large. Hence, (3.25) is satisfied if K_3 and K_5 are chosen small as above, and θ is sufficiently large.

From (3.20), we have

$$\overline{U}(t, g(t+t_0)) \succeq \mathbf{0}, \quad \overline{U}(t, \bar{h}(t)) \succeq \mathbf{0} \quad \text{for} \quad t \geq 0.$$

Together with (3.21), (3.22) and (3.25), this enables us to use the comparison principle to conclude that

$$h(t+t_0) \leq \bar{h}(t),\ U(t+t_0, x) \preceq \overline{U}(t,x) \quad \text{for} \quad t \geq 0,\ x \in [g(t+t_0), \underline{h}(t)],$$

which implies (3.3). The proof of the lemma is now complete. □

3.3. **Proof of Theorem 3.1.** Since ($\mathbf{J^1}$) holds, by Lemma 2.8 and then by (3.1), there exists $C_0 > 0$ such that

$$\begin{aligned} h(t) - c_0 t \geq & - C \left[1 + \int_0^t (1+x)^{-1} dx + \int_0^{\frac{c_0}{2}t} x^2 \hat{J}(x) dx + t \int_{\frac{c_0}{2}t}^{\infty} x \hat{J}(x) dx \right] \\ \geq & - C \left[1 + \int_0^1 \hat{J}(x) dx + \ln(t+1) + C_0 \int_1^{\frac{c_0}{2}t} x^{2-\gamma} dx + C_0 t \int_{\frac{c_0}{2}t}^{\infty} x^{1-\gamma} dx \right]. \end{aligned}$$

Therefore when $\gamma \in (2,3)$ we have, for $t \geq 1$,

$$h(t) - c_0 t \geq - C \left[\tilde{C} + \ln(t+1) + \tilde{C}_1 t^{3-\gamma} \right] \geq -\hat{C}_1 t^{3-\gamma}$$

for some $\hat{C}_1, \tilde{C}, \tilde{C}_1 > 0$, and when $\gamma = 3$, for $t \geq 1$,

$$h(t) - c_0 t \geq - \hat{C}_2 \ln t$$

for some $\hat{C}_2 > 0$. This combined with Lemmas 3.2 and 3.3 gives the desired conclusion of Theorem 3.1. The proof is completed. □

4. Growth rates of accelerated spreading for kernels of type ($\hat{\mathbf{J}}^\gamma$)

Let (U, g, h) be the unique positive solution of (1.1), and assume that spreading happens. Under the assumptions of Theorem B (ii), we have

$$-\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = \infty.$$

Suppose ($\hat{\mathbf{J}}^\gamma$) holds for some $\gamma \in (1,2]$, namely, for $|x| \gg 1$ we have

$$J_i(x) \approx |x|^{-\gamma} \text{ for } i \in \{1, ..., m_0\} \text{ and some } \gamma \in (1,2]. \tag{4.1}$$

Then

$$\int_{\mathbb{R}} J_i(x) \mathrm{d}x < \infty, \quad \int_{\mathbb{R}} |x| J_i(x) \mathrm{d}x = \infty \text{ for } i \in \{1, ..., m_0\}.$$

So **(J1)** is not satisfied.

The purpose of this section is to prove Theorem 1.2, which we restate as

Theorem 4.1. *Assume that* ($\mathbf{J}$) *and* ($\mathbf{f_1}$) $-$ ($\mathbf{f_4}$) *are satisfied. If spreading happens, and additionally* (4.1) *holds, then for large* $t > 0$,

$$\begin{cases} -g(t),\ h(t) \approx\ t^{1/(\gamma-1)} & \text{if } \gamma \in (1,2), \\ -g(t),\ h(t) \approx\ t \ln t & \text{if } \gamma = 2. \end{cases}$$

We will only prove the estimate for $h(t)$, since that for $g(t)$ follows by the change of variable $x \to -x$. Theorem 4.1 will follow directly from the lemmas in Subsections 6.1 and 6.2 below.

4.1. **Upper bound.** To prove the upper bound a slightly weaker condition than (4.1) is enough. We assume that there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{|x|^\gamma + 1} \leq \sum_{i=1}^{m_0} \mu_i J_i(x) \leq \frac{C_2}{|x|^\gamma + 1} \quad \text{for} \quad x \in \mathbb{R} \quad \text{and some } \gamma \in (1,2]. \tag{4.2}$$

Obviously, (4.2) has no restriction for the kernel function J_{i_0} whenever $\mu_{i_0} = 0$, and (4.1) implies (4.2) for the same γ.

Lemma 4.2. *Assume that* $(\mathbf{J})$ *and* $(\mathbf{f_1})-(\mathbf{f_4})$ *hold. If spreading happens, and* (4.2) *is satisfied, then there exits* $C=C(\gamma)>0$ *such that*

$$\begin{cases} h(t)\leq Ct^{1/(\gamma-1)} & \text{if } \gamma\in(1,2),\\ h(t)\leq Ct\ln t & \text{if } \gamma=2. \end{cases} \tag{4.3}$$

Proof. Define, for $t\geq 0$,

$$\bar h(t):=\begin{cases} (Kt+\theta)^{1/(\gamma-1)} & \text{if } \gamma\in(1,2],\\ (Kt+\theta)\ln(Kt+\theta) & \text{if } \gamma=2, \end{cases}$$

and

$$\overline{U}(t,x):=\bar u\mathbf{1},\ \bar u:=\max_{1\leq i\leq m}\{\|u_{i0}\|_\infty,u_i^*\},\quad x\in[-\bar h(t),\bar h(t)],$$

with positive constants θ and K to be determined.

We start by showing

$$\bar h'(t)\geq\sum_{i=1}^{m_0}\mu_i\int_{-\bar h(t)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\quad\text{for}\quad t>0, \tag{4.4}$$

and

$$-\bar h'(t)\leq-\sum_{i=1}^{m_0}\mu_i\int_{-\bar h(t)}^{\bar h(t)}\int_{-\infty}^{-\bar h(t)}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\ \text{ for }\ t>0.$$

Since $\overline{U}(t,x)=\overline{U}(t,-x)$ and $J_i(x)=J_i(-x)$, it suffices to prove (4.4).

By simple calculations and (4.2), for any $k>1$,

$$\begin{aligned}&\sum_{i=1}^{m_0}\mu_i\int_{-k}^{0}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x=\sum_{i=1}^{m_0}\mu_i\int_0^k\int_x^\infty J_i(y)\mathrm{d}y\mathrm{d}x\\ =&\sum_{i=1}^{m_0}\mu_i\int_0^k J_i(y)y\mathrm{d}y+\sum_{i=1}^{m_0}\mu_ik\int_k^\infty J_i(y)\mathrm{d}y\\ \leq&\int_0^k\frac{C_2y}{y^\gamma+1}\mathrm{d}y+k\int_k^\infty\frac{C_2}{y^\gamma+1}\mathrm{d}y\leq\int_0^1C_2\mathrm{d}y+\int_1^k\frac{C_2y}{y^\gamma}\mathrm{d}y+k\int_k^\infty\frac{C_2}{y^\gamma}\mathrm{d}y,\end{aligned}$$

and so

$$\begin{cases}\displaystyle\sum_{i=1}^{m_0}\mu_i\int_{-k}^{0}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\leq C_2+\frac{C_2}{2-\gamma}(k^{2-\gamma}-1)+\frac{C_2k^{2-\gamma}}{\gamma-1} & \text{if } \gamma\in(1,2),\\ \displaystyle\sum_{i=1}^{m_0}\mu_i\int_{-k}^{0}\int_0^\infty J_i(x-y)\mathrm{d}y\mathrm{d}x\leq 2C_2+C_2\ln k & \text{if } \gamma=2.\end{cases} \tag{4.5}$$

A direct calculation gives

$$\int_{-\bar h(t)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x=\bar u\int_{-2\bar h(t)}^{0}\int_0^{+\infty}J_i(x-y)\mathrm{d}y\mathrm{d}x.$$

Hence for $1<\gamma<2$, by (4.5),

$$\begin{aligned}&\sum_{i=1}^{m_0}\mu_i\int_{-\bar h(t)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\\ \leq&\bar u\left[C_2+2^{2-\gamma}\left(\frac{C_2}{2-\gamma}+\frac{C_2}{\gamma-1}\right)(Kt+\theta)^{(2-\gamma)/(\gamma-1)}\right]\\ \leq&\frac{K}{\gamma-1}(Kt+\theta)^{(2-\gamma)/(1-\gamma)}=\bar h'(t)\end{aligned}$$

provided that $K>0$ is large enough. And for $\gamma=2$,

$$\begin{aligned}&\sum_{i=1}^{m_0}\mu_i\int_{-\bar h(t)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J_i(x-y)\bar u_i(t,x)\mathrm{d}y\mathrm{d}x\\ \le&\bar u\big(2C_2+C_2\ln[2(Kt+\theta)\ln(Kt+\theta)]\big)\\ \le&\bar u\big(2C_2+C_2\ln 2(Kt+\theta)+C_2\ln[\ln(Kt+\theta)]\big)\\ \le&K\ln(Kt+\theta)+K=\bar h'(t)\end{aligned}$$

if $K\gg 1$. This finishes the proof of (4.4).

Since $\overline{U}\geq \mathbf{u}^*$ is a constant vector, we have, for $t>0$, $x\in[-\bar h(t),\bar h(t)]$,

$$\overline{U}_t(t,x)\equiv\mathbf{0}\succeq D\circ\int_{-\bar h(t)}^{\bar h(t)}\mathbf{J}(x-y)\circ\overline{U}(t,y)\mathrm{d}y-D\circ\overline{U}(t,x)+F(\overline{U}(t,x)). \tag{4.6}$$

Moreover, $\bar h(0)\geq h_0$ for large θ, and obviously

$$\begin{aligned}&\overline{U}(t,\pm\bar h(t))\succeq\mathbf{0}\text{ for }t\geq 0,\\ &\overline{U}(0,x)\succeq U(0,x)\text{ for }x\in[-h_0,h_0].\end{aligned}$$

Hence we can apply the comparison principle to conclude that

$$\begin{aligned}&[g(t),h(t)]\subset[-\bar h(t),\bar h(t)], && t\geq 0,\\ &U(t,x)\preceq\overline{U}(t,x), && t\geq 0,\ x\in[g(t),h(t)].\end{aligned}$$

Thus (4.3) holds. □

4.2. **Lower bound.** The lower bound is more difficult to obtain, and we will consider the cases $\gamma\in(1,2)$ and $\gamma=2$ separately.

4.2.1. *The case* $\gamma\in(1,2)$. We start with a result from [10].

Lemma 4.3. [10, (2.11)] *If $\tilde J$ satisfies* **(J)**, *then for any $\epsilon>0$, there is $L_\epsilon>0$ such that for all $l>L_\epsilon$ and $\psi_l(x):=l-|x|$,*

$$\int_{-l}^{l}\tilde J(x-y)\psi_l(y)\mathrm{d}y\geq(1-\epsilon)\psi_l(x)\ \text{ in }[-l,l]. \tag{4.7}$$

Lemma 4.4. *Assume that the conditions in Theorem 4.1 are satisfied and $\gamma\in(1,2)$. Then there exits $C=C(\gamma)>0$ such that*

$$h(t)\geq Ct^{1/(\gamma-1)}\ \text{ for } t\gg 1. \tag{4.8}$$

Proof. Define

$$\begin{aligned}&\underline{h}(t):=(K_1t+\theta)^{1/(\gamma-1)},\quad t\geq 0,\\ &\underline{U}(t,x):=K_2\frac{\underline{h}(t)-|x|}{\underline{h}(t)}\Theta,\quad t\geq 0,\ x\in[-\underline{h}(t),\underline{h}(t)],\end{aligned}$$

with positive constants θ and K_1,K_2 to be determined, where the vector $\Theta=(\theta_i)$ is given by Lemma **??**.

Step 1. We show that, for large K_1,

$$\underline{h}'(t)\leq\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{+\infty}J(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x\ \text{ for }\ t>0. \tag{4.9}$$

By simple calculations and (4.2), we obtain

$$\sum_{i=1}^{m_0}\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{+\infty}J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{m_0} \mu_i K_2 \theta_i \int_0^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J_i(x-y) \frac{\underline{h}(t)-x}{\underline{h}(t)} \mathrm{d}y\mathrm{d}x \\
&= \sum_{i=1}^{m_0} \frac{\mu_i K_2 \theta_i}{\underline{h}(t)} \int_{-\underline{h}(t)}^{0} \int_0^{+\infty} J_i(x-y)(-x) \mathrm{d}y\mathrm{d}x \\
&= \sum_{i=1}^{m_0} \frac{\mu_i K_2 \theta_i}{\underline{h}(t)} \int_0^{\underline{h}(t)} \int_x^{+\infty} J_i(y) x \mathrm{d}y\mathrm{d}x \\
&= \sum_{i=1}^{m_0} \frac{\mu_i K_2 \theta_i}{\underline{h}(t)} \left(\int_0^{\underline{h}(t)} \int_0^{y} + \int_{\underline{h}(t)}^{\infty} \int_0^{\underline{h}} \right) J_i(y) x \mathrm{d}x\mathrm{d}y \\
&\geq \sum_{i=1}^{m_0} \mu_i \theta_i \frac{K_2}{2\underline{h}(t)} \int_0^{\underline{h}(t)} J_i(y) y^2 \mathrm{d}y \geq \sum_{i=1}^{m_0} \mu_i \theta_i \frac{K_2 C_1}{2\underline{h}(t)} \int_0^{\underline{h}(t)} \frac{y^2}{y^\gamma+1} \mathrm{d}y \\
&\geq \sum_{i=1}^{m_0} \mu_i \theta_i \frac{K_2 C_1}{4\underline{h}(t)} \int_1^{\underline{h}(t)} y^{2-\gamma} \mathrm{d}y \geq \sum_{i=1}^{m_0} \mu_i \theta_i \frac{K_2 C_1}{4\underline{h}(t)} \frac{\underline{h}(t)^{3-\gamma}}{3-\gamma} \\
&= \hat{C}_0 (K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} \geq \frac{K_1}{\gamma-1} (K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} = \underline{h}'(t)
\end{aligned}
$$

provided that $K_1 \geq \hat{C}_0(\gamma-1)$. This finishes the proof of Step 1.

Step 2. We show that , by choosing K_1, K_2 and θ properly, for $t>0$, $x \in (-\underline{h}(t), \underline{h}(t))$,

$$
\underline{U}_t(t,x) \succeq D \circ \int_{-\underline{h}(t)}^{\underline{h}(t)} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y - D \circ \underline{U}(t,x) + F(\underline{U}(t,x)). \tag{4.10}
$$

From the definition of $\underline{U}$, for $t>0$, $x \in (-\underline{h}(t), \underline{h}(t))$,

$$
\underline{U}_t(t,x) = K_2 \Theta \frac{|x| \underline{h}'(t)}{\underline{h}^2(t)} \preceq K_2 \Theta \frac{\underline{h}'(t)}{\underline{h}(t)} = \frac{K_1 K_2 \Theta}{\gamma-1} \underline{h}(t)^{1-\gamma}.
$$

Claim 1. For $x \in [-\underline{h}(t), \underline{h}(t)]$, there exists a positive constant $\hat{C}_1$ depending only on γ such that

$$
\int_{-\underline{h}(t)}^{\underline{h}(t)} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y \succeq \hat{C}_1 K_2 \Theta \underline{h}(t)^{1-\gamma}. \tag{4.11}
$$

By (4.1), there exists $\tilde{C}_1 > 0$ such that

$$
J_i(x) \geq \frac{\tilde{C}_1}{|x|^\gamma + 1} \text{ for } x \in \mathbb{R},\ i = 1, ..., m_0. \tag{4.12}
$$

Hence

$$
\begin{aligned}
&\int_{-\underline{h}}^{\underline{h}} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y = \int_{-\underline{h}-x}^{\underline{h}-x} \mathbf{J}(y) \circ \underline{U}(t,y+x) \mathrm{d}y \\
\succeq & K_2 \Theta \int_{-\underline{h}-x}^{\underline{h}-x} \frac{\tilde{C}_1}{|y|^\gamma+1} \frac{\underline{h}-|y+x|}{\underline{h}} \mathrm{d}y.
\end{aligned}
$$

Thus, for $x \in [\underline{h}/4, \underline{h}]$,

$$
\begin{aligned}
&\int_{-\underline{h}}^{\underline{h}} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y \succeq K_2 \Theta \int_{-\underline{h}/4}^{0} \frac{\tilde{C}_1}{|y|^\gamma+1} \frac{\underline{h}-|y+x|}{\underline{h}} \mathrm{d}y \\
= & K_2 \Theta \int_{-\underline{h}/4}^{0} \frac{\tilde{C}_1}{|y|^\gamma+1} \frac{\underline{h}-(y+x)}{\underline{h}} \mathrm{d}y \succeq K_2 \Theta \int_{-\underline{h}/4}^{0} \frac{\tilde{C}_1}{|y|^\gamma+1} \frac{-y}{\underline{h}} \mathrm{d}y \\
= & \frac{K_2 \Theta}{\underline{h}} \int_0^{\underline{h}/4} \frac{\tilde{C}_1 y}{y^\gamma+1} \mathrm{d}y \succeq \frac{\tilde{C}_1 K_2 \Theta}{2\underline{h}} \int_1^{\underline{h}/4} y^{1-\gamma} \mathrm{d}y
\end{aligned}
$$

$$\succeq \frac{\tilde{C}_1 K_2 \Theta}{2(2-\gamma)\underline{h}}(\underline{h}/4)^{2-\gamma} = \hat{C}_1 K_2 \Theta \underline{h}^{1-\gamma}.$$

And for $x \in [0, \underline{h}/4]$,

$$\int_{-\underline{h}}^{\underline{h}} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y \succeq K_2 \Theta \int_0^{\underline{h}/4} \frac{\tilde{C}_1}{|y|^\gamma + 1} \frac{\underline{h} - |y+x|}{\underline{h}} \mathrm{d}y$$

$$\succeq K_2 \Theta \int_0^{\underline{h}/4} \frac{\tilde{C}_1}{y^\gamma + 1} \frac{y}{\underline{h}} \mathrm{d}y \succeq \hat{C}_1 K_2 \Theta \underline{h}^{1-\gamma}$$

by repeating the last a few steps in the previous calculations.

This proves (4.11) for $x \in [0, \underline{h}]$. (4.11) also holds for $x \in [-\underline{h}, 0]$ since both $J(x)$ and $\underline{U}(t,x)$ are even in x.

Claim 2. We can choose small K_2 and large θ such that, for $x \in [-\underline{h}(t), \underline{h}(t)]$ and $t \geq 0$,

$$D \circ \int_{-\underline{h}}^{\underline{h}} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y - D \circ \underline{U}(t,x) + F(\underline{U}(t,x)) \succeq F_* \int_{-\underline{h}}^{\underline{h}} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y$$

for some positive constant F_*. Let Θ be defined as in Lemma 2.1 of [11]. It is clear that $\underline{U} \leq K_2 \Theta$, and thus for small $K_2 > 0$ from the definition of Θ,

$$F(\underline{U}(t,x)) = K_2 \frac{\underline{h}(t) - |x|}{\underline{h}(t)} \Theta\Big([\nabla F(\mathbf{0})]^T + o(1)\mathbf{I}_m\Big) \succeq K_2 \frac{\underline{h}(t) - |x|}{\underline{h}(t)} \frac{3}{4} \lambda_1 \Theta = \frac{3}{4} \lambda_1 \underline{U}(t,x),$$

where $\lambda_1 > 0$ is given in Lemma 2.1 of [11]. Moreover, by (4.7), there is $L_1 > 0$ such that for $\theta^{1/(\gamma-1)} \geq L_1$,

$$D \circ \int_{-\underline{h}(t)}^{\underline{h}(t)} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y + \frac{\lambda_1}{4} \underline{U}(t,x) \succeq D \circ \underline{U}(t,x) \quad \text{for} \quad x \in [-\underline{h}(t), \underline{h}(t)].$$

Therefore Claim 2 is valid with $F_* = \lambda_1/2$.

Combining Claim 1 and Claim 2, we obtain

$$D \circ \int_{-\underline{h}}^{\underline{h}} \mathbf{J}(x-y) \circ \underline{U}(t,y) \mathrm{d}y - D \circ \underline{U}(t,x) + F(\underline{U}(t,x))$$

$$\succeq F_* \hat{C}_1 K_2 \Theta \underline{h}(t)^{1-\gamma} \succeq \frac{K_1 K_2 \Theta}{\gamma - 1} \underline{h}(t)^{1-\gamma} \succeq \underline{U}_t(t,x)$$

provided that

$$K_1 \leq F_* \hat{C}_1 (\gamma - 1).$$

This proves (4.10).

Step 3. We prove (4.8) by the comparison principle.

It is clear that

$$\underline{U}(t, \pm \underline{h}(t)) = 0 \quad \text{for} \quad t \geq 0.$$

Since spreading happens for (U, g, h), for fixed θ and small K_1, K_2 as chosen above, there exists a large $t_0 > 0$ such that

$$[-\underline{h}(0), \underline{h}(0)] \subset [g(t_0)/2, h(t_0)/2],$$
$$U(t_0, x) \succeq K_2 \Theta \succeq \underline{U}(0,x) \quad \text{for} \quad x \in [-\underline{h}(0), \underline{h}(0)].$$

Moreover, since $J(x)$ and $\underline{U}(t,x)$ are both even in x, (4.9) implies

$$-\underline{h}'(t) \geq -\mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(x-y) \underline{u}_i(t,x) \mathrm{d}y \mathrm{d}x \quad \text{for} \quad t > 0.$$

These combined with the estimates in Step 1 and Step 2 allow us to apply Lemma **??** to conclude that

$$\begin{aligned}&[-\underline{h}(t),\underline{h}(t)]\subset[g(t+t_0),h(t+t_0)], && t\geq 0,\\ &\underline{U}(t,x)\preceq U(t+t_0,x), && t\geq 0,\ x\in[-\underline{h}(t),\underline{h}(t)].\end{aligned}$$

Hence (4.8) holds. □

4.2.2. *The case* $\gamma=2$. The following simple result will play an important role in our analysis later.

Lemma 4.5. *Let* l_1 *and* l_2 *with* $0<l_1<l_2$ *be two constants, and define*

$$\psi(x)=\psi(x;l_1,l_2):=\min\left\{1,\frac{l_2-|x|}{l_1}\right\},\qquad x\in\mathbb{R}.$$

If $\tilde{J}$ *satisfies* **(J)**, *then for any* $\epsilon>0$, *there is* $L_\epsilon>0$ *such that for all* $l_1>L_\epsilon$ *and* $l_2-l_1>L_\epsilon$,

$$\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y\geq(1-\epsilon)\psi(x)\ \text{ in } [-l_2,l_2]. \tag{4.13}$$

Proof. Since $\int_{\mathbb{R}}\tilde{J}(x)\mathrm{d}x=1$, there exits $B>0$ such that

$$\int_{-B}^{B}\tilde{J}(x)\mathrm{d}x>1-\epsilon/2. \tag{4.14}$$

In the following discussion we always assume that $l_1\gg B$ and $l_2-l_1\gg B$. Clearly, for $x\in[-(l_2-l_1)+B,(l_2-l_1)-B]$, due to

$$\psi(x)=1\ \text{ in } [-(l_2-l_1),l_2-l_1],$$

we have

$$\begin{aligned}&\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y\geq\int_{-(l_2-l_1)}^{l_2-l_1}\tilde{J}(x-y)\psi(y)\mathrm{d}y=\int_{-(l_2-l_1)}^{l_2-l_1}\tilde{J}(x-y)\mathrm{d}y\\ =&\int_{-(l_2-l_1)-x}^{l_2-l_1-x}\tilde{J}(y)\mathrm{d}y\geq\int_{-B}^{B}\tilde{J}(y)\mathrm{d}y\geq 1-\epsilon/2>(1-\epsilon)\psi(x).\end{aligned}$$

It remain to prove (4.13) for $x\in[-l_2,-(l_2-l_1)+B]\cup[(l_2-l_1)-B,l_2]$. By the symmetric property of $\psi(x)$ and $\tilde{J}(x)$ with respect to x, we just need to verify (4.13) for $x\in[(l_2-l_1)-B,l_2]$, which will be carried out according to the following three cases:

(i) $x\in[l_2-l_1-B,l_2-l_1+B]$, (ii) $x\in[l_2-l_1+B,l_2-B]$, (iii) $x\in[l_2-B,l_2]$.

(i) For $x\in[l_2-l_1-B,l_2-l_1+B]$, since $\psi(z)$ is nonincreasing for $z\geq 0$, we have

$$\begin{aligned}&\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y=\int_{-l_2-x}^{l_2-x}\tilde{J}(y)\psi(y+x)\mathrm{d}y\\ \geq&\int_{-2l_2+l_1+B}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y\geq\int_{-B}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y\\ \geq&\int_{-B}^{B}\tilde{J}(y)\psi(y+l_2-l_1+B)\mathrm{d}y.\end{aligned}$$

By the definition of ψ,

$$\psi(y+l_2-l_1+B)=\frac{l_2-(y+l_2-l_1+B)}{l_1}=1-\frac{y+B}{l_1},\quad y\in[-B,B].$$

Hence,

$$\int_{-B}^{B}\tilde{J}(y)\psi(y+l_2-l_1+B)\mathrm{d}y=\int_{-B}^{B}\tilde{J}(y)\mathrm{d}y-\int_{-B}^{B}\tilde{J}(y)\frac{y+B}{l_1}\mathrm{d}y$$

$$\geq 1-\epsilon/2-\|\tilde{J}\|_{L^\infty(\mathbb{R})}\frac{2B^2}{l_1}\geq 1-\epsilon\geq(1-\epsilon)\psi(x)$$

provided

$$l_1\geq\frac{4\|\tilde{J}\|_{L^\infty(\mathbb{R})}B^2}{\epsilon},$$

which then gives

$$\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y\geq(1-\epsilon)\psi(x)\quad\text{for}\quad x\in[l_2-l_1-B,l_2-l_1+B].$$

(ii) For $x\in[l_2-l_1+B,l_2-B]$,

$$\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y=\int_{-l_2-x}^{l_2-x}\tilde{J}(y)\psi(y+x)\mathrm{d}y$$
$$\geq\int_{-2l_2-B+l_1}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y\geq\int_{-B}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y.$$

From the definition of ψ, for $x\in[l_2-l_1+B,l_2-B]$ and $y\in[-B,B]$,

$$\psi(y+x)=\frac{l_2-(y+x)}{l_1}=\frac{l_2-x}{l_1}-\frac{y}{l_1}=\psi(x)-\frac{y}{l_1}.$$

Thus, by (4.14),

$$\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y\geq\int_{-B}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y$$
$$=\psi(x)\int_{-B}^{B}\tilde{J}(y)\mathrm{d}y-\int_{-B}^{B}\tilde{J}(y)\frac{y}{l_1}\mathrm{d}y=\psi(x)\int_{-B}^{B}\tilde{J}(y)\mathrm{d}y\geq(1-\epsilon)\psi(x).$$

(iii) For $x\in[l_2-B,l_2]$,

$$\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y=\int_{-l_2-x}^{l_2-x}\tilde{J}(y)\psi(y+x)\mathrm{d}y$$
$$\geq\int_{-2l_2-B}^{l_2-x}\tilde{J}(y)\psi(y+x)\mathrm{d}y\geq\int_{-B}^{l_2-x}\tilde{J}(y)\psi(y+x)\mathrm{d}y$$
$$=\int_{-B}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y-\int_{l_2-x}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y$$

As in (ii), we see that

$$\int_{-B}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y=\psi(x)\int_{-B}^{B}\tilde{J}(y)\mathrm{d}y\geq(1-\epsilon)\psi(x).$$

By the definition of ψ,

$$\psi(y+x)\leq 0\quad\text{for}\quad x\in[l_2-B,l_2],\ y\in[l_2-x,B],$$

which indicates

$$\int_{-l_2}^{l_2}\tilde{J}(x-y)\psi(y)\mathrm{d}y\geq\int_{-B}^{B}\tilde{J}(y)\psi(y+x)\mathrm{d}y\geq(1-\epsilon)\psi(x).$$

The proof is now complete. □

Lemma 4.6. *If the conditions in Theorem 4.1 are satisfied and $\gamma=2$, then there exits $C>0$ such that*

$$h(t)\geq Ct\ln t\ \textit{for}\ t\gg 1.\tag{4.15}$$

Proof. For fixed $\beta \in (0,1)$, define

$$\begin{cases} \underline{h}(t) := K_1(t+\theta)\ln(t+\theta), & t \geq 0, \\ \underline{U}(t,x) := K_2 \min\left\{1, \dfrac{\underline{h}(t)-|x|}{(t+\theta)^\beta}\right\}\Theta, & t\geq 0,\ x\in[-\underline{h}(t),\underline{h}(t)], \end{cases}$$

for constants $\theta \gg 1$ and $1 \gg K_1 > 0, 1 \gg K_2 > 0$ to be determined, where Θ is given in Lemma **??**. Obviously, for any $t>0$, the function $\partial_t \underline{U}(t,x)$ exists for $x \in [-\underline{h}(t), \underline{h}(t)]$ except when $|x| = \underline{h}(t) - (t+\theta)^\beta$. However, the one-sided partial derivates $\partial_t \underline{U}(t\pm 0, x)$ always exist.

Step 1. We show that by choosing θ and K_1, K_2 suitably,

$$\underline{h}'(t) \leq \sum_{i=1}^{m_0} \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x \qquad \text{for } t>0, \tag{4.16}$$

$$-\underline{h}'(t) \geq -\sum_{i=1}^{m_0} \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x \qquad \text{for } t>0. \tag{4.17}$$

Since $\underline{U}(t,x) = \underline{U}(t,-x)$ and $\mathbf{J}(x) = \mathbf{J}(-x)$, we see that (4.17) follows from (4.16).

By elementary calculations and (4.2), we have

$$\begin{aligned}
&\sum_{i=1}^{m_0} \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x \\
\geq &\sum_{i=1}^{m_0} \mu_i \int_{0}^{\underline{h}(t)-(t+\theta)^\beta} \int_{\underline{h}(t)}^{+\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x \\
= &\sum_{i=1}^{m_0} \mu_i K_2\theta_i \int_{-\underline{h}(t)}^{-(t+\theta)^\beta} \int_{0}^{+\infty} J_i(x-y)\mathrm{d}y\mathrm{d}x = \sum_{i=1}^{m_0} \mu_i K_2\theta_i \int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{x}^{+\infty} J_i(y)\mathrm{d}y\mathrm{d}x \\
= &\sum_{i=1}^{m_0} \mu_i K_2\theta_i \left(\int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{(t+\theta)^\beta}^{y} + \int_{\underline{h}(t)}^{\infty} \int_{(t+\theta)^\beta}^{\underline{h}} \right) J_i(y)\mathrm{d}x\mathrm{d}y \\
\geq &\sum_{i=1}^{m_0} \mu_i K_2\theta_i \int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{(t+\theta)^\beta}^{y} J_i(y)\mathrm{d}x\mathrm{d}y \geq \sum_{i=1}^{m_0} \mu_i C_1 K_2\theta_i \int_{(t+\theta)^\beta}^{\underline{h}(t)} \frac{y-(t+\theta)^\beta}{y^2+1}\mathrm{d}y \\
\geq &\sum_{i=1}^{m_0} \mu_i C_1 K_2\theta_i \int_{(t+\theta)^\beta}^{\underline{h}(t)} \frac{y-(t+\theta)^\beta}{2y^2}\mathrm{d}y \\
= &\sum_{i=1}^{m_0} \mu_i C_1 K_2\theta_i \frac{1}{2}\left(\ln \underline{h}(t) - \beta\ln(t+\theta) + \frac{(t+\theta)^\beta}{\underline{h}(t)} - 1 \right) \\
\geq &\sum_{i=1}^{m_0} \mu_i C_1 K_2\theta_i \frac{1}{2}\left(\ln \underline{h}(t) - \beta\ln(t+\theta) - 1 \right) \\
= &\sum_{i=1}^{m_0} \mu_i C_1 K_2\theta_i \frac{1}{2}\left(\ln K_1 + \ln(t+\theta) + \ln(\ln(t+\theta)) - \beta\ln(t+\theta) - 1 \right) \\
\geq &\sum_{i=1}^{m_0} \frac{\mu_i C_1 K_2\theta_i(1-\beta)}{2}[\ln(t+\theta)+1] \geq K_1\ln(t+\theta) + K_1 = \underline{h}'(t)
\end{aligned}$$

if

$$\begin{cases} \ln(\ln\theta) \geq -\ln K_1 + 2, \\ K_1 \leq K_2 \displaystyle\sum_{i=1}^{m_0} \frac{\mu_i C_1\theta_i(1-\beta)}{2}, \end{cases} \tag{4.18}$$

which then finishes the proof of Step 1.

Step 2. We show that by choosing K_1, K_2 and θ suitably, for $t>0$ and $x\in(-\underline{h}(t),\underline{h}(t))$,

$$\underline{U}_t(t,x) \preceq D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y - D\circ\underline{U}(t,x)+F(\underline{U}(t,x)). \tag{4.19}$$

From the definition of $\underline{U}$, for $t>0$,

$$\underline{U}_t(t,x) = K_1K_2\frac{(1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta}\Theta + \frac{K_2\beta|x|}{(t+\theta)^{1+\beta}}\Theta,\quad \underline{h}(t)-(t+\theta)^\beta<|x|\le\underline{h}(t),$$

$$\underline{U}_t(t,x)=\mathbf{0},\ |x|<\underline{h}(t)-(t+\theta)^\beta.$$

Claim 1. For $x\in[-\underline{h}(t),-\underline{h}(t)+(t+\theta)^\beta]\cup[\underline{h}(t)-(t+\theta)^\beta,\underline{h}(t)]$ and large θ,

$$\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y \succeq \frac{\tilde{C}_1K_2\beta\ln(t+\theta)}{4(t+\theta)^\beta}\Theta, \tag{4.20}$$

where $\tilde{C}_1>0$ is given by (4.12).

A simple calculation yields, for $x\in[\underline{h}(t)-(t+\theta)^\beta,\underline{h}(t)]$,

$$\begin{aligned}\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y &\succeq K_2\Theta\circ\int_{\underline{h}(t)-(t+\theta)^\beta}^{\underline{h}(t)}\mathbf{J}(x-y)\frac{\underline{h}-y}{(t+\theta)^\beta}\mathrm{d}y\\ &=\frac{K_2\Theta}{(t+\theta)^\beta}\circ\int_{\underline{h}(t)-(t+\theta)^\beta-x}^{\underline{h}(t)-x}\mathbf{J}(y)[\underline{h}(t)-(y+x)]\mathrm{d}y.\end{aligned}$$

Hence, for $x\in[\underline{h}(t)-\frac{3}{4}(t+\theta)^\beta,\underline{h}(t)]$, by simple calculations and (4.12),

$$\begin{aligned}\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y &\succeq \frac{K_2\Theta}{(t+\theta)^\beta}\circ\int_{-(t+\theta)^\beta/4}^{0}\mathbf{J}(y)(-y)\mathrm{d}y\\ &=\frac{K_2\Theta}{(t+\theta)^\beta}\circ\int_0^{(t+\theta)^\beta/4}\mathbf{J}(y)y\mathrm{d}y\succeq\frac{\tilde{C}_1K_2\Theta}{(t+\theta)^\beta}\int_0^{(t+\theta)^\beta/4}\frac{y}{y^2+1}\mathrm{d}y\\ &\succeq\frac{\tilde{C}_1K_2\Theta}{2(t+\theta)^\beta}\int_1^{(t+\theta)^\beta/4}y^{-1}\mathrm{d}y=\frac{\tilde{C}_1K_2\Theta}{2(t+\theta)^\beta}[\beta\ln(t+\theta)-\ln 4]\\ &\succeq\frac{\tilde{C}_1K_2\beta\ln(t+\theta)}{4(t+\theta)^\beta}\Theta\end{aligned}$$

if

$$\frac{\beta}{2}\ln\theta\ge\ln 4. \tag{4.21}$$

And for $x\in[\underline{h}(t)-(t+\theta)^\beta,\underline{h}(t)-\frac{3}{4}(t+\theta)^\beta]$,

$$\begin{aligned}\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y &\succeq \frac{K_2\Theta}{(t+\theta)^\beta}\circ\int_0^{3(t+\theta)^\beta/4}\mathbf{J}(y)[\underline{h}(t)-(y+x)]\mathrm{d}y\\ &\succeq\frac{K_2\Theta}{(t+\theta)^\beta}\circ\int_0^{(t+\theta)^\beta/4}\mathbf{J}(y)y\mathrm{d}y\succeq\frac{\tilde{C}_1K_2\beta\ln(t+\theta)}{4(t+\theta)^\beta}\Theta.\end{aligned}$$

This proves (4.20) for $x\in[\underline{h}(t)-(t+\theta)^\beta,\underline{h}(t)]$.

For $x\in[-\underline{h}(t),-\underline{h}(t)+(t+\theta)^\beta]$, (4.11) also holds since both $\mathbf{J}(x)$ and $\underline{U}(t,x)$ are even in x. Claim 1 is thus proved.

Claim 2. We can choose small K_2 and large θ such that, for $x\in[-\underline{h}(t),\underline{h}(t)]$,

$$D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-D\circ\underline{U}(t,x)+F(\underline{U}(t,x))\succeq F_*\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y \tag{4.22}$$

for some $F_*>0$.

For small $K_2>0$, from $\mathbf{0}\preceq \underline{U}\preceq K_2\Theta$ and the definition of Θ in Lemma **??**, we have

$$\begin{aligned}F(\underline{U}(t,x)) =&\underline{U}(t,x)\Big([\nabla F(\mathbf{0})]^T+o(1)\mathbf{I}_m\Big)\\ =&K_2\min\left\{1,\frac{\underline{h}(t)-|x|}{(t+\theta)^\beta}\right\}\Theta\Big([\nabla F(0)]^T+o(1)\mathbf{I}_m\Big)\\ \succeq&K_2\min\left\{1,\frac{\underline{h}(t)-|x|}{(t+\theta)^\beta}\right\}\frac34\lambda_1\Theta=\frac34\lambda_1\underline{U}(t,x),\end{aligned}$$

where $\lambda_1>0$ is given by Lemma 2.1 of [11].

For large θ and $t\geq 0$, we have

$$\underline{h}(t)-(t+\theta)^\beta\geq \theta^\beta(K_1\theta^{1-\beta}\ln\theta-1)\geq\theta^\beta,\quad (t+\theta)^\beta\geq\theta^\beta. \tag{4.23}$$

Hence, by (4.13), there is large $L_1>0$ such that, for $\theta^\beta>L_1$ we have

$$D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y+\frac{\lambda_1}{4}\underline{U}(t,x)\succeq D\circ\underline{U}(t,x)\quad\text{for}\quad x\in[-\underline{h}(t),\underline{h}(t)].$$

Therefore (4.22) holds with $F_*=\lambda_1/2$.

Applying (4.20) and (4.22), we have, for $x\in[-\underline{h}(t),-\underline{h}(t)+(t+\theta)^\beta]\cup[\underline{h}(t)-(t+\theta)^\beta,\underline{h}(t)]$,

$$\begin{aligned}&D\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-\underline{U}(t,x)+F(\underline{U}(t,x))\\ \succeq&\frac{F_*\tilde{C}_1K_2\beta\ln(t+\theta)}{4(t+\theta)^\beta}\Theta\succeq K_1K_2\frac{\ln(t+\theta)+1}{(t+\theta)^\beta}\Theta\\ =&\left[K_1K_2\frac{(1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta}+\frac{K_2\beta\underline{h}(t)}{(t+\theta)^{1+\beta}}\right]\Theta\\ \succeq&\left[\frac{K_1K_2(1-\beta)\ln(t+\theta)+K_1K_2}{(t+\theta)^\beta}+\frac{K_2\beta|x|}{(t+\theta)^{1+\beta}}\right]\Theta\\ =&\underline{U}_t(t,x)\end{aligned}$$

if apart from the earlier requirements, we further have

$$\ln\theta>2\text{ and }K_1\leq\frac{F_*\tilde{C}_1\beta}{2}. \tag{4.24}$$

For $|x|<\underline{h}(t)-(t+\theta)^\beta$,

$$\begin{aligned}&D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y-D\circ\underline{U}(t,x)+F(\underline{U}(t,x))\\ &\succeq F_*\int_{-\underline{h}(t)}^{\underline{h}(t)}\mathbf{J}(x-y)\circ\underline{U}(t,y)\mathrm{d}y\succeq\mathbf{0}=\underline{U}_t(t,x).\end{aligned}$$

Thus (4.19) holds. (Let us stress that it is possible to find K_1, K_2 and large θ such that (4.18), (4.21), (4.23) and (4.24) hold simultaneously.)

Step 3. We finally prove (4.15).

Clearly, $\underline{U}(t,\pm\underline{h}(t))=0$ for $t\geq 0$. Since spreading happens for (U,g,h) and $K_2>0$ is small, there is a large constant $t_0>0$ such that

$$\begin{aligned}&[-\underline{h}(0),\underline{h}(0)]\subset[g(t_0)/2,h(t_0)/2],\\ &\underline{U}(0,x)\preceq K_2\Theta\preceq U(t_0,x)\quad\text{for}\quad x\in[-\underline{h}(0),\underline{h}(0)].\end{aligned}$$

By Remark 2.4 and Lemma 2.5 of [11], we obtain

$$\begin{aligned}&[-\underline{h}(t),\underline{h}(t)]\subset[g(t+t_0),h(t+t_0)], && t\geq 0,\\ &\underline{U}(t,x)\preceq U(t+t_0,x), && t\geq 0,\ x\in[-\underline{h}(t),\underline{h}(t)].\end{aligned}$$

Thus (4.15) holds. This completes the proof of the lemma. □

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