The union-closed sets conjecture

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ORIGINAL PAPER

The Journey of the Union-Closed Sets Conjecture

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Outline

The conjecture

Definition (union-closed families)

A family $\mathcal{A}$ of sets is union-closed if $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

123456

12345 12346 12356 12456 23456

1234 1235 1236 1456 2456 3456

123 145 246 356 456

45 46 56

4 5 6

$\emptyset$

Conjecture (Frankl 1979)

If $\mathcal{A} \neq \{\emptyset\}$ is a finite union-closed family then there exists an element x which is contained in at least half of the member-sets of $\mathcal{A}$.

Known special cases

Throughout, let $n = |\mathcal{A}|$ and $m = |U(\mathcal{A})|$ where $U(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$.

The conjecture is true in the following situations.

- $\mathcal{A}$ has at most 12 elements or at most 50 member-sets.
- $n \geqslant \frac{2}{3} 2^m$
- $n \leqslant 2m$ (and $\mathcal{A}$ is separating)
- $\mathcal{A}$ contains one of a number of subconfigurations (singletons, ...).
- $\mathcal{A}$ has a special structure (for instance coming from a lower semimodular lattice or a subcubic graph).

There is always an element of frequency at least $\dfrac{n-1}{\log_2 n}$ (Knill 1991).

Notation

We write

$$\mathcal{A}_x = \{A \in \mathcal{A} \;:\; x \in A\}, \qquad \mathcal{A}_{\bar{x}} = \{A \in \mathcal{A} \;:\; x \notin A\}$$

The union-closure of a family $\mathcal{S}$ is the family

$$\mathcal{A} = \left\{ \bigcup_{S \in \mathcal{S}'} S \;:\; \mathcal{S}' \subseteq \mathcal{S} \right\}.$$

We say that $\mathcal{A}$ is generated by $\mathcal{S}$.

Every union-closed family $\mathcal{A}$ has a unique minimal generating family $\mathcal{B}$ whose members are called basis sets of $\mathcal{A}$.

The intersection-closed sets conjecture

From a union-closed family we get an intersection-closed family by taking complements.

		123456			
12345	12346	12356	12456	23456	
1234	1235	1236	1456	2456	3456
123	145	246	356	456	
45	46	56			
4	5	6			
		$\emptyset$			

		123456			
12356	12346	12345			
1236	1235	1234			
456	236	135	124	123	
56	46	45	23	13	12
1	2	3	4	5	6
		$\emptyset$			

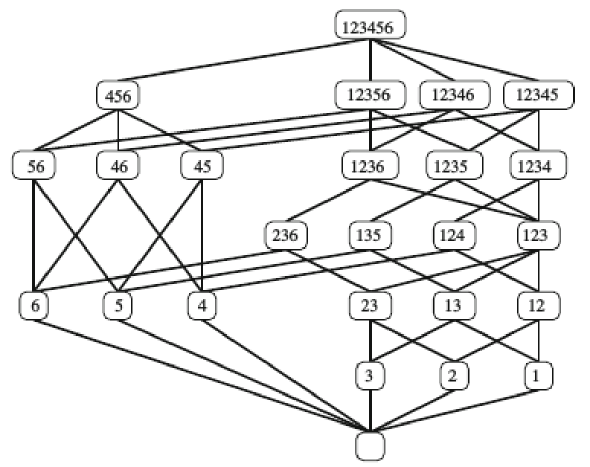
Conjecture (Intersection-closed sets conjecture)

If $\mathcal{A}$ is a finite intersection-closed family of at least two sets then there exists an element x which is contained in at most half of the member-sets of $\mathcal{A}$.

Finite lattices

Definition (lattice)

A (finite) lattice is a finite poset $(L, \leqslant)$ in which every pair $a, b \in L$ has a unique greatest lower bound, denoted by $a \wedge b$ (the meet), and a unique smallest upper bound, denoted by $a \vee b$ (the join).



The lattice formulation

Definition (join-irreducible)

An element a of a lattice L is called join-irreducible if it can not be written as the join $b \vee c$ for two elements $b, c \in L - a$.

Notation (lattice intervals)

For a lattice L and an element $a \in L$, let $[a)$ denote the set $\{b \in L \, : \, b \geqslant a\}$.

Conjecture

Every finite lattice L with at least two elements has a join-irreducible element a with $|[a)| \leqslant \frac{1}{2}|L|$.

Lower semimodular lattices

Definition (lower cover)

Let $x < y$ be two elements of a lattice. Then x is a lower cover of y if $x \leqslant z \leqslant y$ implies $z \in \{x, y\}$.

Definition (lower semimodular)

A lattice L is lower semimodular if $a \wedge b$ is a lower cover of a, whenever b is a lower cover of $a \vee b$.

Theorem (Reinhold 2000)

Lower semimodular lattices satisfy Frankls conjecture.

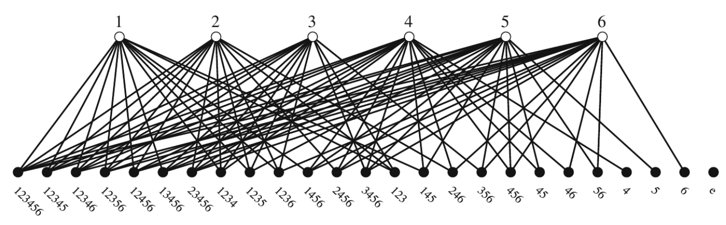
Theorem (Knill 1994)

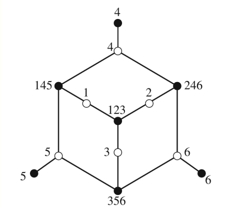
Given a graph $G = (V, E)$ with at least one edge, the intersection-closed family $\{E(X) \ : \ X \subseteq V\}$ satisfies the intersection-closed sets conjecture.

The graph formulation

Conjecture

Any bipartite graph with at least one edge contains in each of its bipartition classes a vertex that lies in at most half of the maximal stable sets.





The graph formulation

Conjecture

Any bipartite graph with at least one edge contains in each of its bipartition classes a vertex that lies in at most half of the maximal stable sets.

Theorem (chordal graphs (Bruhn, Charbit, Schaudt, Telle 2015))

Every chordal bipartite graph satisfies the conjecture.

Theorem (subcubic graphs (Bruhn, Charbit, Schaudt, Telle 2015))

Every bipartite graph with maximum degree at most 3 *satisfies the conjecture.*

Theorem (random graphs (Bruhn, Schaudt 2013))

For every $\delta > 0$ *in almost every random bipartite graph each of the bipartition classes contains a vertex that lies in at most* $1/2 + \delta$ *times the total number of maximal stable sets.*

The Salzborn formulation

Definition (separating families)

A family $\mathcal{A}$ is separating if for any two elements of its universe there is a member-set that contains exactly one of the two.

Definition (normalized families)

A union-closed family $\mathcal{N}$ is called normalized if $\emptyset \in \mathcal{N}$, $\mathcal{N}$ is separating and $|U(\mathcal{N})| = |\mathcal{N}| - 1$.

Conjecture (Salzborn 1989, Wójcik 1999)

Any normalized family $\mathcal{N} \neq \{\emptyset\}$ contains a basis set B with $|B| \geqslant \frac{1}{2}|\mathcal{N}|$.

Two stronger conjectures

Conjecture (Poonen 1992)

Let $\mathcal{A}$ be a separating union-closed family. Unless $\mathcal{A}$ is a power set, it contains an element that appears in strictly more than half of the member-sets.

Conjecture (Poonen 1992)

Let $\mathcal{A}$ be a separating union-closed family on universe $U = U(\mathcal{A})$. If A contains a unique abundant element a then

$$\mathcal{A} = \{\emptyset\} \cup \{B + a \; : \; B \subseteq U - a\}.$$

Frankl-complete families

- If $\{a\} \in \mathcal{A}$ then a is abundant.
- If $\{a, b\} \in \mathcal{A}$ then a or b is abundant.

Theorem (Poonen 1992)

Let $\mathcal{L}$ be a union-closed family with universe $[k]$. The following statements are equivalent:

1. *Every union-closed family $\mathcal{A}$ containing $\mathcal{L}$ has an abundant element in $[k]$.*
2. *There exist reals $c_1, \ldots, c_k \geqslant 0$ with $\sum_{i=1}^{k} c_i = 1$ such that for every union-closed family $\mathcal{K} \subseteq 2^{[k]}$ with $\mathcal{K} = \{K \cup L \ : \ K \in \mathcal{K}, \ L \in \mathcal{L}\}$, we have*

$$\sum_{i=1}^{k} c_i |\mathcal{K}_i| \geqslant \frac{1}{2} |\mathcal{K}|.$$

Some results on FC families.

Let $\mathrm{FC}(k, m)$ be the smallest r such that any r of the k-subsets of $[m]$ generate an FC-family.

Theorem (Morris 2006, Vaughn 2004)
$\lfloor m/2 \rfloor + 1 \leqslant \mathrm{FC}(3, m) \leqslant 2m/3$.

Theorem (Marić, Živković, Vučković 2012)
$\mathrm{FC}(3, 7) = 4$

Small families

Theorem (Živković, Vučković 2012)

The union-closed sets conjecture holds for families on at most 12 elements.

Lemma (Faro 1994; Roberts, Simpson 2010)

If the union closed sets conjecture fails and m *is the smallest size of a universe of a counterexample then any counterexample has at least* $4m - 1$ *member-sets.*

Corollary

The union-closed sets conjecture holds for families of at most 50 sets.

A trivial observation

$\mathcal{A}$ has an element of frequency at least $\frac{1}{2}|\mathcal{A}|$ if the average frequency is at least $\frac{1}{2}|\mathcal{A}|$, i.e., if

$$\frac{1}{|U(\mathcal{A})|}\sum_{u\in U(\mathcal{A})}|\mathcal{A}_u| \geqslant \frac{1}{2}|\mathcal{A}|.$$

But how do we control the average frequency? Double counting!

$$\sum_{u\in U(\mathcal{A})}|\mathcal{A}_u| = \sum_{A\in\mathcal{A}}|A|.$$

Large families

Theorem (Nishimura and Takahashi 1996)
If $|\mathcal{A}| \geqslant 2^m - \frac{1}{2}\sqrt{2^m}$ *then* $\mathcal{A}$ *satisfies the union-closed sets conjecture.*

Theorem (Balla, Bollobas, Eccles 2013)
If $|\mathcal{A}| \geqslant \frac{2}{3}2^m$ *then* $\mathcal{A}$ *satisfies the union-closed sets conjecture.*

Bounds on the average

Theorem (Reimer 2003)

If $\mathcal{A}$ is union-closed then

$$\frac{1}{n}\sum_{A\in\mathcal{A}}|A| \geqslant \frac{\log_2 n}{2}.$$

This gives $\displaystyle \frac{1}{|U(\mathcal{A})|}\sum_{u\in U(\mathcal{A})}|\mathcal{A}_u| \geqslant \frac{\log_2 n}{m}\cdot\frac{n}{2}$.

Theorem (Falgas-Ravry 2011)

If $\mathcal{A}$ is union-closed and separating then

$$\frac{1}{|U(\mathcal{A})|}\sum_{u\in U(\mathcal{A})}|\mathcal{A}_u| \geqslant \frac{m+1}{2}.$$

Limits of averaging

Order the finite subsets of $\mathbb{N}$ by setting $A < B$ if

- $\max A < \max B$, or
- $\max A = \max B$ and $\max(A \Delta B) \in A$

$$\emptyset < 1 < 12 < 2 < 123 < 23 < 13 < 3 < 1234 < 234 \\ < 134 < 34 < 124 < 24 < 14 < 4 < 12345 < \cdots$$

The Hungarian family $\mathcal{H}^n$ is the initial segment of length n.

Theorem (Czédli, Maróti, Schmidt 2009)

For the Hungarian family of size $n = \lfloor \frac{2}{3} 2^m \rfloor$,

$$\frac{1}{m} \sum_{i \in [m]} |\mathcal{H}_i^n| < \frac{|\mathcal{H}^n|}{2}.$$