RATE OF PROPAGATION FOR THE FISHER-KPP EQUATION WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES

YIHONG DU[†] AND WENJIE NI[†]

Abstract. In this paper, we obtain sharp estimates for the rate of propagation of the Fisher-KPP equation with nonlocal diffusion and free boundaries. The nonlocal diffusion operator is given by $\int_{\mathbb{R}} J(x-y)u(t,y)dy - u(t,x)$, and our estimates hold for some typical classes of kernel functions $J(x)$. For example, if for $|x| \gg 1$ the kernel function satisfies $J(x) \sim |x|^{-\gamma}$ with $\gamma > 1$, then it follows from [17] that there is a finite spreading speed when $\gamma > 2$, namely the free boundary $x = h(t)$ satisfies $\lim_{t\to\infty} h(t)/t = c_0$ for some uniquely determined positive constant c_0 depending on J, and when $\gamma \in (1,2]$, $\lim_{t\to\infty} h(t)/t = \infty$; the estimates in the current paper imply that, for $t \gg 1$,

$$c_0 t - h(t) \sim \begin{cases} 1 & \text{when } \gamma > 3 \\ \ln t & \text{when } \gamma = 3, \\ t^{3-\gamma} & \text{when } \gamma \in (2,3), \end{cases}$$

and

$$h(t) \sim \begin{cases} t \ln t & \text{when } \gamma = 2, \\ t^{1/(\gamma-1)} & \text{when } \gamma \in (1,2). \end{cases}$$

Our approach is based on subtle integral estimates and constructions of upper and lower solutions, which rely crucially on guessing correctly the order of growth of the term to be estimated. The techniques developed here lay the ground for extensions to more general situations.

Keywords: Nonlocal diffusion; Free boundary; Rate of propagation.

AMS Subject Classification (2000): 35K57, 35R20

Contents

1. Introduction

In this paper we determine the spreading rate for the Fisher-KPP equation with nonlocal diffusion and free boundaries considered in [11] and [17]. The problem is a "nonlocal diffusion" version of the following free boundary problem with "local diffusion":

$$(1.1)\qquad \begin{cases} u_t - du_{xx} = f(u), & t > 0,\ g(t) < x < h(t), \\ u(t,g(t)) = u(t,h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_x(t,g(t)),\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ g(0) = g_0,\ h(0) = h_0,\ u(0,x) = u_0(x), & g_0 \leq x \leq h_0, \end{cases}$$

Date: July 19, 2021.
[†] School of Science and Technology, University of New England, Armidale, NSW 2351, Australia.
Emails: ydu@une.edu.au (Y. Du), wni2@une.edu.au (W. Ni).

where f is a C^1 function satisfying $f(0) = 0$, $\mu > 0$ and $g_0 < h_0$ are constants, and u_0 is a C^2 function which is positive in (g_0, h_0) and vanishes at $x = g_0$ and $x = h_0$. For Fisher-KPP type of $f(u)$, (1.1) was first studied in [18], as a model for the spreading of a new or invasive species with population density $u(t, x)$, whose population range $(g(t), h(t))$ expands through its boundaries $x = g(t)$ and $x = h(t)$ according to the Stefan conditions $g'(t) = -\mu u_x(t, g(t))$, $h'(t) = -\mu u_x(t, h(t))$. A deduction of these conditions based on some ecological assumptions can be found in [9].

By [18], problem (1.1) admits a unique solution $(u(t, x), g(t), h(t))$ defined for all $t > 0$, and its long-time dynamical behaviour is characterised by a "spreading-vanishing dichotomy": Either $(g(t), h(t))$ is contained in a bounded set of $\mathbb{R}$ for all $t > 0$ and $u(t, x) \to 0$ uniformly as $t \to \infty$ (called the vanishing case), or $(g(t), h(t))$ expands to $\mathbb{R}$ and $u(t, x)$ converges to the unique positive steady state of the ODE $v' = f(v)$ locally uniformly in $x \in \mathbb{R}$ as $t \to \infty$ (the spreading case). Moreover, when spreading occurs,

$$\lim_{t\to\infty} \frac{-g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = k_0 > 0,$$

and k_0 is uniquely determined by a semi-wave problem associated to (1.1).

Problem (1.1) is closely related to the corresponding Cauchy problem

$$\begin{cases} U_t - dU_{xx} = f(U), & t > 0,\ x \in \mathbb{R}, \\ U(0, x) = U_0(x), & x \in \mathbb{R}, \end{cases} \quad \text{where } U_0(x) := \begin{cases} u_0(x), & x \in [g_0, h_0], \\ 0, & x \in \mathbb{R} \setminus [g_0, h_0]. \end{cases} \tag{1.2}$$

Indeed, it follows from [16] that the unique solution (u, g, h) of (1.1) and the unique solution U of (1.2) are related in the following way: For any fixed $T > 0$, as $\mu \to \infty$, $(g(t), h(t)) \to \mathbb{R}$ and $u(t, x) \to U(t, x)$ locally uniformly in $(t, x) \in (0, T] \times \mathbb{R}$. Thus (1.2) may be viewed as the limiting problem of (1.1) (as $\mu \to \infty$).

Problem (1.2) with U_0 a nonnegative function having nonempty compact support has long been used to describe the spreading of a new or invasive species; see, for example, classical works of Fisher [25], Kolmogorov-Petrovski-Piscunov (KPP) [33] and Aronson-Weinberger [2].

In both (1.1) and (1.2), the dispersal of the species is described by the diffusion term du_{xx}, widely called a "local diffusion" operator, which is obtained from the assumption that individuals of the species moves in space according to the rule of Brownian motion. The nonlocal diffusion version of (1.1) considered in [11] has the following form[1]:

$$\begin{cases} u_t = d \displaystyle\int_{\mathbb{R}} J(x-y)u(t,y)dy - du(t,x) + f(u), & t > 0,\ x \in (g(t), h(t)), \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx, & t > 0, \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, & t > 0, \\ u(0, x) = u_0(x),\ h(0) = -g(0) = h_0, & x \in [-h_0, h_0], \end{cases} \tag{1.3}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, which is always assumed to be identically 0 for $x \in \mathbb{R} \setminus [g(t), h(t)]$ [2]; d and μ are positive constants. The initial function $u_0(x)$ satisfies

$$u_0 \in C([-h_0, h_0]),\ u_0(-h_0) = u_0(h_0) = 0 \ \text{ and } \ u_0(x) > 0 \ \text{ in } \ (-h_0, h_0), \tag{1.4}$$

with $[-h_0, h_0]$ representing the initial population range of the species. The basic assumptions on the kernel function $J : \mathbb{R} \to \mathbb{R}$ are

(J): $J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $J \geq 0$, $J(0) > 0$, $\int_{\mathbb{R}} J(x)dx = 1$, J is even.

The nonlocal free boundary problem (1.3) may be viewed as describing the spreading of a new or invasive species with population density $u(t, x)$, whose population range $[g(t), h(t)]$ expands according

[1] The case $f(u) \equiv 0$ was considered in [13], where the long-time dynamics are completely different from the Fisher-KPP case in [11].

[2] Therefore $\int_{\mathbb{R}} J(x-y)u(t,y)dy = \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy$.

to the free boundary conditions

$$\begin{cases} h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx, \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, \end{cases}$$

that is, the expanding rate of the range $[g(t), h(t)]$ is proportional to the outward flux of the population across the boundary of the range (see [11] for further explanations and justification).

One advantage of the nonlocal problem (1.3) over the local problem (1.1) is that the nonlocal diffusion term

$$d\int_{\mathbb{R}} J(x-y)u(t,y)dy - du(t,x)$$

in (1.3) is capable to include spatial dispersal strategies of the species beyond random diffusion modelled by the term du_{xx} in (1.1). Here $J(x-y)$ may be interpreted as the probability that an individual of the species moves from x to y in a time unit.

If f is a Fisher-KPP function, namely it satisfies

(f): $\begin{cases} f \in C^1, \;\; f > 0 = f(0) = f(1) \text{ in } (0,1), \; f'(0) > 0 > f'(1), \\ f(u)/u \text{ is nonincreasing in } u > 0, \end{cases}$

then the long-time dynamical behaviour of (1.3), similar to that of (1.1), is determined by a “spreading-vanishing dichotomy” (see Theorem 1.2 in [11]): As $t \to \infty$, either

(i) Spreading: $\lim_{t\to+\infty}(g(t), h(t)) = \mathbb{R}$ and $\lim_{t\to+\infty} u(t,x) = 1$ locally uniformly in $\mathbb{R}$, or

(ii) Vanishing: $\lim_{t\to+\infty}(g(t), h(t)) = (g_\infty, h_\infty)$ is a finite interval and $\lim_{t\to+\infty} u(t,x) = 0$ uniformly for $x \in [g(t), h(t)]$.

Criteria for spreading and vanishing are also obtained in [11]; see Theorem 1.3 there. In particular, if the size of the initial population range $2h_0$ is larger than a certain critical number, then spreading always happens.

1.1. **Threshold condition, spreading speed, and accelerated spreading.** When spreading happens, the question of spreading speed was considered in [17]. In sharp contrast to the corresponding local diffusion problem (1.1), it was shown in [17] that (1.3) may spread super linearly in time (a phenomenon known as accelerated spreading), depending on whether the following threshold condition is satisfied by the kernel function J,

(J1): $\displaystyle\int_0^\infty xJ(x)dx < +\infty.$

More precisely, we have

Theorem A ([17]). Suppose that **(J)** and **(f)** are satisfied, and spreading happens to the unique solution (u, g, h) of (1.3). Then

$$\lim_{t\to\infty} \frac{h(t)}{t} = -\lim_{t\to\infty} \frac{g(t)}{t} = \begin{cases} c_0 \in (0,\infty) & \text{if } \textbf{(J1)} \text{ holds}, \\ \infty & \text{if } \textbf{(J1)} \text{ does not hold}. \end{cases}$$

As usual, when **(J1)** holds, we call c_0 the spreading speed of (1.3), which is determined by the semi-wave solutions to (1.3). These are pairs $(c, \phi) \in (0, +\infty) \times C^1((-\infty, 0])$ determined by the following two equations:

$$\begin{cases} d\displaystyle\int_{-\infty}^0 J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, \quad -\infty < x < 0, \\ \phi(-\infty) = 1, \quad \phi(0) = 0, \end{cases} \tag{1.5}$$

and

$$c = \mu \int_{-\infty}^0 \int_0^{+\infty} J(x-y)\phi(x)dydx. \tag{1.6}$$

If (c, ϕ) solves (1.5), then we call ϕ a semi-wave with speed c, since the function $v(t,x) := \phi(x-ct)$ satisfies

$$\begin{cases} v_t = d\displaystyle\int_{-\infty}^{ct} J(x-y)v(t,y)dy - dv(t,x) + f(v(t,x)), & t > 0, \; x < ct, \\ v(t,-\infty) = 1, \quad v(t,ct) = 0, & t > 0. \end{cases}$$

However, only the semi-wave satisfying (1.6) meets the free boundary condition along the moving front $x = ct$, and hence useful for determining the long-time dynamical behaviour of (1.3).

The spreading speed c_0 is given by the following result:

Theorem B ([17]). Suppose that **(J)** and **(f)** are satisfied. Then (1.5)-(1.6) has a solution pair $(c,\phi) = (c_0, \phi^{c_0}) \in (0,+\infty) \times C^1((-\infty,0])$ with $\phi^{c_0}(x)$ nonincreasing in x if and only if **(J1)** holds. Moreover, when **(J1)** holds, there exists a unique such solution pair, and $(\phi^{c_0})'(x) < 0$ in $(-\infty, 0]$.

It was also proved in [17] (see Theorem 5.3 there) that as $\mu \to \infty$, the limiting problem of (1.3) is the following nonlocal version of (1.2):

$$\tag{1.7} \begin{cases} u_t = d\displaystyle\int_{\mathbb{R}} J(x-y)u(t,y)dy - du(t,x) + f(u), & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases}$$

Problem (1.7) and its many variations have been extensively studied in the literature; see, for example, [1, 3–5, 7, 12, 14, 15, 23, 24, 26, 31, 32, 35, 38, 40, 43] and the references therein. In particular, if **(J)** and **(f)** are satisfied, and if the nonnegative initial function u_0 has non-empty compact support, then the basic long-time dynamical behaviour of (1.7) is given by

$$\lim_{t\to\infty} u(t,x) = 1 \quad \text{locally uniformly for } x \in \mathbb{R}.$$

Similar to (1.2), the nonlocal Cauchy problem (1.7) does not give a finite population range when $t > 0$. To understand the spreading behaviour of (1.7), one examines the level set

$$E_\lambda(t) := \{x \in \mathbb{R} : u(t,x) = \lambda\} \text{ with fixed } \lambda \in (0,1),$$

by considering the large time behaviour of

$$x^+_\lambda(t) := \sup E_\lambda(t) \quad \text{and} \quad x^-_\lambda(t) = \inf E_\lambda(t).$$

As $t \to \infty$, $|x^\pm_\lambda(t)|$ may go to ∞ linearly in t or super-linearly in t, depending on whether the following threshold condition is satisfied by the kernel function, apart from **(J)**,

(J2): There exists $\lambda > 0$ such that $\displaystyle\int_{\mathbb{R}} J(x)e^{\lambda x}dx < \infty$.

Yagisita [43] has proved the following result on traveling wave solutions to (1.7):

Theorem C ([43]). Suppose that f satisfies **(f)** and J satisfies **(J)**. If additionally J satisfies **(J2)**, then there is a constant $c_* > 0$ such that (1.7) has a traveling wave solution with speed c if and only if $c \geq c_*$.

Condition **(J2)** is often called a "thin tail" condition for J. When f satisfies **(f)**, and J satisfies **(J)** and **(J2)**, it is well known (see, for example, [23, 41]) that

$$\tag{1.8} \lim_{t\to\infty} \frac{|x^\pm_\lambda(t)|}{t} = c_*,$$

with c_* given by Theorem C. On the other hand, if **(f)** and **(J)** hold but **(J2)** is not satisfied, then it follows from Theorem 6.4 of [41] that $|x^\pm_\lambda(t)|$ grows faster than any linear function of t as $t \to \infty$, namely, accelerated spreading happens:

$$\lim_{t\to\infty} \frac{|x^\pm_\lambda(t)|}{t} = \infty.$$

See also [1, 7, 8, 10, 22, 24, 26, 30, 39, 42] and references therein for further progress on accelerated spreading for (1.7) and related problems.

It is easily seen that **(J2)** implies **(J1)**, but the reverse is not true; for example, $J(x) = C(1+x^2)^{-\sigma}$ with $\sigma > 1$ satisfies **(J1)** (for some suitable $C > 0$) but not **(J2)**.

The relationship between $c_0 = c_0(\mu)$ obtained in Theorem B and c_* in Theorem C is given in the following result (see Theorems 5.1 and 5.2 of [17]):

Theorem D ([17]). Suppose that **(J)**, **(J1)** and **(f)** hold. Then $c_0(\mu)$ increases to c_* as $\mu \to \infty$, where we define $c_* = \infty$ when **(J2)** does not hold.

For the local diffusion problem (1.1), sharp estimate for the spreading profile has been obtained in [19]: When spreading happens,

$$\lim_{t\to\infty}[h(t) - k_0 t] = C_1,\ \lim_{t\to\infty}[g(t) + k_0 t] = C_2$$

for some $C_1, C_2 \in \mathbb{R}$ depending on u_0. Moreover, the solution $u(t,x)$ exhibits the corresponding semi-wave profile as $t \to \infty$. This is strikingly different from the situation of (1.2), where a well known logarithmic delay happens [6], namely

$$m(t) := \sup\{x \in \mathbb{R} : U(t,x) = 1/2\} = 2\sqrt{f'(0)d}\, t - \frac{3d}{2\sqrt{f'(0)d}} \ln t + C_0 + o(1) \text{ as } t \to \infty$$

for some $C_0 \in \mathbb{R}$ depending on the initial function U_0. We refer to [27–29, 34, 36, 37] for further advances in research of that direction.

In this paper, we aim to obtain sharp estimates for (1.3) in a similar spirit. It turns out that when (**J1**) holds and so a finite spreading speed c_0 exists for (1.3), the functions $h(t) - c_0 t$ and $g(t) + c_0 t$ need not be bounded as $t \to \infty$. For some rather general classes of J, we will find the exact rate of growth for $h(t) - c_0 t$ and $g(t) + c_0 t$ when (**J1**) holds, and determine the exact rate of growth of $h(t)$ and $g(t)$ when (**J1**) does not hold.

1.2. **Description of the main results.** We now describe our main results precisely. For $\alpha > 1$, we introduce the condition

$$(\mathbf{J}^\alpha): \qquad \int_0^\infty x^{\alpha-1} J(x)dx < \infty.$$

Let us note that $(\mathbf{J^2})$ is equivalent to (**J1**), and if (**J2**) holds, then $(\mathbf{J}^\alpha)$ is satisfied for all $\alpha > 1$.

Theorem 1.1. *In Theorem A, suppose additionally* $(\mathbf{J}^\alpha)$ *holds for some* $\alpha \geq 3$*, and* $f'(v)$ *is locally Lipschitz in* $[0,\infty)$*. Then there exists* $C > 0$ *such that for* $t \gg 1$*,*

$$|h(t) - c_0 t| + |g(t) + c_0 t| \leq C,$$

$$\begin{cases} \phi^{c_0}(x - c_0 t + C) + o(1) \leq u(t,x) \leq \phi^{c_0}(x - c_0 t - C) + o(1) & \text{for } x \in [0, h(t)], \\ \phi^{c_0}(-x + c_0 t + C) + o(1) \leq u(t,x) \leq \phi^{c_0}(-x + c_0 t - C) + o(1) & \text{for } x \in [g(t), 0], \end{cases}$$

where (c_0, ϕ^{c_0}) *is the unique semi-wave pair in Theorem B, and* $o(1) \to 0$ *uniformly as* $t \to \infty$*.*

Further estimates on $g(t)$ and $h(t)$ can be obtained for more specific classes of kernel functions. We will write

$$\eta(t) \sim \xi(t) \ \text{ if } \ C_1\xi(t) \leq \eta(t) \leq C_2\xi(t)$$

for some positive constants $C_1 \leq C_2$ and all t in the concerned range.

Our next two theorems are about kernel functions satisfying, for some $\gamma > 0$,

$$(\hat{\mathbf{J}}^\gamma): \qquad J(x) \sim |x|^{-\gamma} \quad \text{for } |x| \gg 1.$$

Note that for kernel functions satisfying $(\hat{\mathbf{J}}^\gamma)$, condition (**J**) is satisfied only if $\gamma > 1$, and (**J1**) is satisfied only if $\gamma > 2$. Thus accelerated spreading can happen exactly when $\gamma \in (1,2]$. We have the following result on the exact growth rate of $h(t)$ and $g(t)$ in this case:

Theorem 1.2. *In Theorem A, if additionally the kernel function satisfies* $(\hat{\mathbf{J}}^\gamma)$ *for some* $\gamma \in (1,2]$*, then for* $t \gg 1$*,*

$$-g(t),\ h(t) \sim \begin{cases} t \ln t & \text{if } \gamma = 2, \\ t^{1/(\gamma-1)} & \text{if } \gamma \in (1,2). \end{cases}$$

For kernel functions satisfying $(\hat{\mathbf{J}}^\gamma)$, clearly $(\mathbf{J}^\alpha)$ holds if $0 < \alpha < \gamma$. Therefore when $\gamma > 3$ the conclusions in Theorem 1.1 hold. The following theorem is concerned with the remaining case $\gamma \in (2,3]$, which indicates that the result in Theorem 1.1 is sharp.

Theorem 1.3. *In Theorem A, suppose additionally the kernel function satisfies* $(\hat{\mathbf{J}}^\gamma)$ *for some* $\gamma \in (2,3]$*,* $f'(v)$ *is locally Lipschitz in* $[0,\infty)$ *and*

$$[f(v)/v]' < 0 \quad \text{for} \quad v > 0. \tag{1.9}$$

Then for $t \gg 1$*,*

$$c_0 t + g(t),\ c_0 t - h(t) \sim \begin{cases} \ln t & \text{if } \gamma = 3, \\ t^{3-\gamma} & \text{if } \gamma \in (2,3). \end{cases}$$

Note that **(f)** implies $[f(v)/v]' \leq 0$ for $v > 0$, and (1.9) is satisfied, for example, by $f(v) = av - bv^p$ with $a, b > 0$ and $p > 1$.

The proofs of Theorems 1.1 and 1.3 rely on some of the following estimates on the semi-wave solutions of (1.5), which are of independent interests.

Theorem 1.4. *Suppose that f satisfies* (**f**) *and the kernel function satisfies* (**J**), *and $\phi(x)$ is a monotone solution of* (1.5) *for some $c > 0$. Then the following conclusions hold:*

(i) *If* ($\mathbf{J}^\alpha$) *holds for some $\alpha > 1$, then*

$$\int_{-\infty}^{-1} \big[1 - \phi(x)\big]|x|^{\alpha-2}dx < \infty,$$

which implies, by the monotonicity of $\phi(x)$,

$$0 < 1 - \phi(x) \leq C|x|^{1-\alpha} \text{ for some } C > 0 \text{ and all } x < 0.$$

(ii) *If* ($\mathbf{J}^\alpha$) *does not hold for some $\alpha > 1$, then*

$$\int_{-\infty}^{-1} \big[1 - \phi(x)\big]|x|^{\alpha-2}dx = \infty.$$

The conditions on f and ϕ in Theorem 1.4 can be considerably relaxed; see Section 2 for details.

Remark 1.5. *This paper seems the first to establish estimates of the type in Theorems 1.1 and 1.3 for nonlocal diffusion problems, with or without free boundary.*

Remark 1.6. *The proofs of Theorems 1.1, 1.2 and 1.3 are based on subtle constructions of upper and lower solutions. These constructions rely on firstly guessing correctly the order of growth of the term to be estimated, which is perhaps the most difficult part of this research. The techniques developed here lay the ground for extensions to more general situations.*

1.3. **Organisation of the paper.** The rest of the paper is organised as follows. In Section 2, we prove Theorem 1.4, where subtle analysis is used to find out the relationship between the behaviour of the semi-wave solution and that of the kernel function. Theorem 1.1 is proved in Section 3, through careful constructions of upper and lower solutions, based on the estimate obtained in Section 2. Section 4 is devoted to the proof of Theorem 1.3, where we completely determine the growth rate of $c_0 t - h(t)$ when $J(x) \sim |x|^{-\gamma}$ with γ in the range $(2, 3]$; note that the case $\gamma > 3$ is already covered by the more general Theorem 1.1. In Section 5, we prove Theorem 1.2 by giving the exact growth rate of $h(t)$ when $J(x) \sim |x|^{-\gamma}$ with $\gamma \in (1, 2]$.

2. Proof of Theorem 1.4

The purpose of this section is to prove the following two theorems, which imply Theorem 1.4. For possible applications elsewhere, we prove the results under much less restrictions on ϕ and f. We assume that f is C^1 and $f(1) = 0 > f'(1)$, and ϕ satisfies, for some $c > 0$,

$$\text{(2.1)} \qquad \begin{cases} d\displaystyle\int_{-\infty}^{0} J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, \ \phi(x) \in [0,1], \quad -\infty < x < 0, \\ \phi(-\infty) = 1, \quad \phi'(x) \leq 0 \text{ for } x \ll -1. \end{cases}$$

Theorem 2.1. *Suppose that the kernel function satisfies* (**J**) *and* ($\mathbf{J}^\alpha$) *for some $\alpha > 1$, f is C^1 with $f(1) = 0 > f'(1)$. If $\phi(x)$ satisfies* (2.1) *for some $c > 0$, then*

$$\int_{-\infty}^{-1} [1 - \phi(x)]|x|^{\alpha-2}dx < \infty,$$

and therefore, by the monotonicity of $\phi(x)$ near $-\infty$,

$$0 < 1 - \phi(x) \leq C|x|^{1-\alpha} \text{ for some } C > 0 \text{ and all } x \leq -1.$$

The next result shows that Theorem 2.1 is sharp.

Theorem 2.2. *Suppose that f is C^1 with $f(1) = 0 > f'(1)$ and the kernel function satisfies* (**J**). *If* ($\mathbf{J}^\alpha$) *is not satisfied for some $\alpha > 1$, and $\phi(x)$ satisfies* (2.1) *for some $c > 0$, then*

$$\text{(2.2)} \qquad \int_{-\infty}^{-1} [1 - \phi(x)]|x|^{\alpha-2}dx = \infty.$$

The following three lemmas play a crucial role in the proof of Theorem 2.1.

Lemma 2.3. *Suppose that $J(x)$ satisfies $(\mathbf{J})$ and $(\mathbf{J}^\alpha)$ for some $\alpha \geq 2$, and $\psi \in L^1((-\infty, 0])$ is nonnegative and continuous in $(-\infty, 0]$. If ψ is nondecreasing near $-\infty$, and satisfies*

$$\int_{-\infty}^{0} |x|^{\beta-1}\psi(x)dx < \infty \text{ for some } \beta \geq 1, \tag{2.3}$$

then for any $\sigma \in (0, \min\{\beta, \alpha-1\}]$, there exists $C > 0$ such that

$$I = I_M := \int_{-M}^{0} |x|^\sigma \left[\int_{-\infty}^{0} J(x-y)\psi(y)dy - \psi(x)\right] dx \in [-C, C] \text{ for all } M > 0.$$

Proof. For fixed $M > 0$ we have

$$\begin{aligned}
\int_{-M}^{0}\int_{-\infty}^{0} |x|^\sigma J(x-y)\psi(y)dydx &= \int_{0}^{M}\int_{-\infty}^{x} x^\sigma J(y)\psi(y-x)dydx \\
&= \int_{0}^{M}\int_{-\infty}^{0} x^\sigma J(y)\psi(y-x)dydx + \int_{0}^{M}\int_{0}^{x} x^\sigma J(y)\psi(y-x)dydx \\
&= \int_{-\infty}^{0}\int_{0}^{M} x^\sigma J(y)\psi(y-x)dxdy + \int_{0}^{M}\int_{y}^{M} x^\sigma J(y)\psi(y-x)dxdy \\
&= \int_{-\infty}^{0}\int_{-y}^{M-y} (x+y)^\sigma J(y)\psi(-x)dxdy + \int_{0}^{M}\int_{0}^{M-y} (x+y)^\sigma J(y)\psi(-x)dxdy,
\end{aligned}$$

and

$$\int_{-M}^{0} |x|^\sigma \psi(x)dx = \int_{\mathbb{R}}\int_{0}^{M} x^\sigma J(y)\psi(-x)dxdy.$$

Therefore we can write

$$I = \sum_{j=1}^{3} I_j$$

with

$$\begin{aligned}
I_1 &:= \int_{-\infty}^{0}\int_{-y}^{M-y} \left[(x+y)^\sigma - x^\sigma\right] J(y)\psi(-x)dxdy \\
&\quad + \int_{0}^{M}\int_{0}^{M-y} \left[(x+y)^\sigma - x^\sigma\right] J(y)\psi(-x)dxdy, \\
I_2 &:= \int_{-\infty}^{0}\int_{M}^{M-y} x^\sigma J(y)\psi(-x)dxdy - \int_{-\infty}^{0}\int_{0}^{-y} x^\sigma J(y)\psi(-x)dxdy, \\
I_3 &:= -\int_{0}^{M}\int_{M-y}^{M} x^\sigma J(y)\psi(-x)dxdy - \int_{M}^{\infty}\int_{0}^{M} x^\sigma J(y)\psi(-x)dxdy.
\end{aligned}$$

To estimate I_1 we will make use of some elementary inequalities. If $s, t > 0$ and $\sigma \in (0, 1]$, then it is easily checked that

$$(s+t)^\sigma - s^\sigma \leq t^\sigma. \tag{2.4}$$

If $\sigma = n + \theta$ with $n \geq 1$ an integer, and $\theta \in (0, 1]$, then by the mean value theorem

$$\begin{aligned}
(s+t)^\sigma - s^\sigma &= \sigma(s+\zeta t)^{\sigma-1}t \leq \sigma t(s+t)^{\sigma-1} = \sigma t s^{\sigma-1} + \sigma t\left[(s+t)^{\sigma-1} - s^{\sigma-1}\right] \\
&\leq \sum_{k=1}^{n}\left[\Pi_{j=0}^{k-1}(\sigma-j)t^k s^{\sigma-k}\right] + \Pi_{j=0}^{n-1}(\sigma-j)t^n\left[(s^\theta + t^\theta) - s^\theta\right] \\
&\leq \sum_{k=1}^{n}\left[\Pi_{j=0}^{k-1}(\sigma-j)t^k s^{\sigma-k}\right] + \Pi_{j=0}^{n-1}(\sigma-j)t^{n+\theta} \\
&= \sum_{k=1}^{n} c_k t^k s^{\sigma-k} + c_{n+1}t^\sigma
\end{aligned}$$

where $\zeta \in [0, 1]$, and $c_k = c_k(\sigma) > 0$ for $k \in \{1, ..., n+1\}$.

Applying this inequality to $(x+y)^\sigma - x^\sigma$ with $x+y>0$ and $x>0$, we obtain, for the case $\sigma>1$,

$$|(x+y)^\sigma - x^\sigma| \le \sum_{k=1}^{n} c_k |y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma$$

with $\sigma - n = \theta \in (0,1]$ and $n \ge 1$ an integer, $c_k = c_k(\sigma) > 0$ for $k \in \{1, ..., n+1\}$.

Therefore, in the case $\sigma > 1$,

$$\begin{aligned}
|I_1| &\le \int_{-\infty}^{0}\int_{-y}^{M-y}\left[\sum_{k=1}^{n} c_k|y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma\right] J(y)\psi(-x)dxdy \\
&\quad + \int_{0}^{M}\int_{0}^{M-y}\left[\sum_{k=1}^{n} c_k|y|^k x^{\sigma-k} + c_{n+1}|y|^\sigma\right] J(y)\psi(-x)dxdy \\
&\le 2\sum_{k=1}^{n} c_k \int_0^\infty x^{\sigma-k}\psi(-x)dx \int_0^\infty y^k J(y)dy + 2c_{n+1}\int_0^\infty \psi(-x)dx \int_0^\infty y^\sigma J(y)dy \\
&:= C_1.
\end{aligned}$$

Since $1 \le k \le n < \sigma \le \min\{\beta, \alpha - 1\}$, by the assumptions on J and ψ we see that C_1 is a finite number.

If $\sigma \in (0,1]$, then

$$\begin{aligned}
|I_1| &\le \int_{-\infty}^{0}\int_{-y}^{M-y} |y|^\sigma J(y)\psi(-x)dxdy + \int_0^M\int_0^{M-y}|y|^\sigma J(y)\psi(-x)dxdy \\
&\le 2\int_0^\infty \psi(-x)dx\int_0^\infty y^\sigma J(y)dy := \tilde{C}_1 < \infty.
\end{aligned}$$

Since $\psi(x) \ge 0$ is continuous in $x \le 0$ and nondecreasing near $-\infty$, from (2.3) we easily deduce

$$\psi(-x) \le \frac{M_1}{x^\sigma} \text{ for some } M_1 > 0 \text{ and all } x > 0.$$

Due to $(\mathbf{J}^\alpha)$ $(\alpha \ge 2)$, we have

$$\int_0^\infty yJ(y)dy < \infty.$$

Therefore

$$\begin{aligned}
|I_2| &\le \int_{-\infty}^{0}\int_{M}^{M-y} M_1 J(y)dxdy + \int_{-\infty}^{0}\int_{0}^{-y} M_1 J(y)dxdy \\
&= 2M_1\int_0^\infty yJ(y)dy := C_2 < \infty,
\end{aligned}$$

and

$$\begin{aligned}
|I_3| &\le \int_0^M M_1 yJ(y)dy + \int_M^\infty M_1 M J(y)dy \\
&\le M_1\int_0^\infty yJ(y)dy := C_3 < \infty.
\end{aligned}$$

We thus have

$$|I| \le C_1 + \tilde{C}_1 + C_2 + C_3 := C < \infty \text{ for all } M > 0.$$

The proof is complete. □

Lemma 2.4. *Suppose that $J(x)$ satisfies* $(\mathbf{J})$ *and* $(\mathbf{J}^\alpha)$ *for some $\alpha \in (1,2)$. Let ψ be nonnegative, continuous in $(-\infty, 0]$, and be nondecreasing near $-\infty$. Then there exists $C > 0$ such that*

$$S = S_M := \int_{-M}^{0} |x|^{\alpha-1}\left[\int_{-\infty}^{0} J(x-y)\psi(y)dy - \psi(x)\right]dx \le C \text{ for all } M > 0.$$

Proof. As in the proof of Lemma 2.3, we deduce for fixed $M > 0$ and $\sigma > -1$,

$$\int_{-M}^{0}\int_{-\infty}^{0}|x|^\sigma J(x-y)\psi(y)dydx$$

$$= \int_{-\infty}^{0} \int_{-y}^{M-y} (x+y)^\sigma J(y)\psi(-x)dxdy + \int_{0}^{M} \int_{0}^{M-y} (x+y)^\sigma J(y)\psi(-x)dxdy.$$

and

$$\int_{-M}^{0} |x|^\sigma \psi(x)dx = \int_{\mathbb{R}} \int_{0}^{M} |x|^\sigma J(y)\psi(-x)dxdy.$$

Hence

$$S = \sum_{i=1}^{3} \tilde{I}_i$$

with

$$\begin{aligned}
\tilde{I}_1 :=& \int_{-\infty}^{0} \int_{-y}^{M-y} \left[(x+y)^\sigma - x^\sigma\right] J(y)\psi(-x)dxdy \\
&+ \int_{0}^{M} \int_{0}^{M-y} \left[(x+y)^\sigma - x^\sigma\right] J(y)\psi(-x)dxdy, \\
\tilde{I}_2 :=& \int_{-\infty}^{0} \int_{M}^{M-y} x^\sigma J(y)\psi(-x)dxdy - \int_{-\infty}^{0} \int_{0}^{-y} x^\sigma J(y)\psi(-x)dxdy, \\
\tilde{I}_3 :=& -\int_{0}^{M} \int_{M-y}^{M} x^\sigma J(y)\psi(-x)dxdy - \int_{M}^{\infty} \int_{0}^{M} x^\sigma J(y)\psi(-x)dxdy.
\end{aligned}$$

Take $\sigma = \alpha - 2$. It is clear that $\tilde{I}_3 \le 0$. For $\tilde{I}_1$, since $\sigma < 0$,

$$(x+y)^\sigma - x^\sigma < 0 \quad \text{when } x > 0 \quad \text{and } y > 0,$$

and hence, by $(\mathbf{J}^\alpha)$ and $\sigma + 1 = \alpha - 1 \in (0,1)$,

$$\begin{aligned}
\tilde{I}_1 \le& \int_{-\infty}^{0} \int_{-y}^{M-y} \left[(x+y)^\sigma - x^\sigma\right] J(y)\psi(-x)dxdy \\
\le& \|\psi\|_\infty \int_{-\infty}^{0} \int_{-y}^{M-y} \left[(x+y)^\sigma - x^\sigma\right] J(y)dxdy \\
=& \frac{\|\psi\|_\infty}{\sigma+1} \int_{-\infty}^{0} [M^{\sigma+1} - (M-y)^{\sigma+1} + (-y)^{\sigma+1}] J(y)\mathrm{d}y \\
\le& \frac{\|\psi\|_\infty}{\sigma+1} \int_{-\infty}^{0} (-y)^{\sigma+1} J(y)\mathrm{d}y = \frac{\|\psi\|_\infty}{\sigma+1} \int_{0}^{\infty} y^{\sigma+1} J(y)\mathrm{d}y := C_1 < \infty.
\end{aligned}$$

Moreover, by $(\mathbf{J}^\alpha)$, $\sigma + 1 = \alpha - 1 \in (0,1)$ and (2.4),

$$\begin{aligned}
\tilde{I}_2 \le& \int_{-\infty}^{0} \int_{M}^{M-y} x^\sigma J(y)\psi(-x)\mathrm{d}x\mathrm{d}y \le \|\psi\|_\infty \int_{-\infty}^{0} \int_{M}^{M-y} x^\sigma J(y)\mathrm{d}x\mathrm{d}y \\
=& \frac{\|\psi\|_\infty}{\sigma+1} \int_{-\infty}^{0} [(M-y)^{\sigma+1} - M^{\sigma+1}] J(y)\mathrm{d}y \\
\le& \frac{\|\psi\|_\infty}{\sigma+1} \int_{0}^{\infty} y^{\sigma+1} J(y)\mathrm{d}y := C_2 < \infty.
\end{aligned}$$

Therefore,

$$S \le C_1 + C_2 := C < \infty \text{ for all } M > 0.$$

The proof is complete. □

Let $\phi(x)$ be a solution of (2.1) with some $c > 0$, and define

$$\psi(x) := 1 - \phi(x),\ g(u) := -f(1-u).$$

Then ψ satisfies

$$\begin{cases} 0 = d\displaystyle\int_{-\infty}^{0} J(x-y)\psi(y)\mathrm{d}y - d\psi(x) + d\int_{0}^{\infty} J(x-y)\mathrm{d}y + c\psi'(x) + g(\psi(x)) \text{ for } x < 0, \\ \psi(-\infty) = 0, \quad \psi'(x) \ge 0 \text{ for } x \ll -1. \end{cases} \tag{2.5}$$

Since $g'(0) = f'(1) < 0$, there exist $\epsilon > 0$ sufficiently small and some $b > 0$ such that

$$g(u) \le -bu \text{ for } u \in [0, \epsilon].$$

As $\psi(-\infty) = 0$ and $\psi(x) \ge 0$ for $x < 0$, we thus have $0 \le \psi(x) < \epsilon$ for $x \ll -1$, and so

$$g(\psi(x)) \le -b\psi(x) \quad \text{for } x \ll -1. \tag{2.6}$$

Lemma 2.5. *Suppose* (**J**) *is satisfied,* f *is* C^1 *with* $f(1) = 0 > f'(1)$*. If* ($\mathbf{J}^\alpha$) *holds for some* $\alpha \ge 2$*, then the above defined* ψ *satisfies*

$$\int_{-\infty}^{0} \psi(x) \mathrm{d}x < \infty.$$

Proof. A simple calculation gives

$$\int_{-\infty}^{0} J(z-w)\psi(w)\mathrm{d}w - \psi(z) + \int_{0}^{\infty} J(z-w)\mathrm{d}w = -\int_{-\infty}^{0} J(z-w)\phi(w)dw + \phi(z).$$

Integrating the equation satisfied by ψ over the interval (x, y) with $x < y \ll -1$, and making use of (2.6), we obtain

$$\begin{aligned} &c(\psi(y) - \psi(x)) + d\int_x^y \left[\int_{-\infty}^{0} J(z-w)\psi(w)\mathrm{d}w - \psi(z)\mathrm{d}z + \int_0^\infty J(z-w)\mathrm{d}w\right]\mathrm{d}z \\ &= -\int_x^y g(\psi(z))\mathrm{d}z \ge b\int_x^y \psi(z)\mathrm{d}z. \end{aligned}$$

We extend ϕ to a C^1 function $\tilde\phi$ over $\mathbb{R}$ satisfying $\tilde\phi(x) = 0$ for $x > 1$ and $|\tilde\phi(x)| \le 2\|\phi\|_\infty$ for $x \in [0, 1]$. Then, due to ($\mathbf{J}^\alpha$), we have,

$$\begin{aligned} &\left|\int_x^y \left(\int_{-\infty}^{0} J(z-w)\phi(w)\mathrm{d}w - \phi(z)\right)\mathrm{d}z\right| \\ =&\left|\int_x^y \left(\int_{\mathbb{R}} J(z-w)\tilde\phi(w)\mathrm{d}w - \phi(z)\right)\mathrm{d}z - \int_x^y\int_0^1 J(z-w)\tilde\phi(w)\mathrm{d}w\mathrm{d}z\right| \\ \le&\left|\int_x^y\int_{\mathbb{R}} J(w)(\tilde\phi(z+w) - \tilde\phi(z))\mathrm{d}w\mathrm{d}z\right| + 2\|\phi\|_\infty \\ =&\left|\int_x^y\int_{\mathbb{R}} J(w)\int_0^1 w\tilde\phi'(z+sw)\mathrm{d}s\mathrm{d}w\mathrm{d}z\right| + 2\|\phi\|_\infty \\ =&\left|\int_{\mathbb{R}} wJ(w)\int_0^1 [\tilde\phi(y+sw) - \tilde\phi(x+sw)]\mathrm{d}s\mathrm{d}w\right| + 2\|\phi\|_\infty \\ \le& 2\|\tilde\phi\|_\infty\int_{\mathbb{R}} |y|J(y)\mathrm{d}y + 2\|\phi\|_\infty =: M < \infty. \end{aligned}$$

Thus, for $x < y \ll -1$,

$$b\int_x^y \psi(z)\mathrm{d}z \le c(\psi(y) - \psi(x)) + dM \le c\|\psi\|_\infty + dM,$$

which implies $\int_{-\infty}^0 \psi(z)\mathrm{d}z < \infty$. □

Proof of Theorem 2.1: **Case 1**. $\alpha \ge 2$.

We want to show

$$\int_{-\infty}^{0} \psi(x)|x|^{\alpha-2}\mathrm{d}x < \infty.$$

By Lemma 2.5 we have

$$\int_{-\infty}^{0} \psi(x)dx < \infty.$$

So there is nothing to prove if $\alpha = 2$, and we only need to consider the case $\alpha > 2$.

Suppose $\alpha > 2$ and

$$\int_{-\infty}^{0} |x|^{\gamma-1}\psi(x)dx < \infty \text{ for some } \gamma \ge 1. \tag{2.7}$$

Then by Lemma 2.3, for any β satisfying $0<\beta\le\min\{\gamma,\alpha-1\}$,

$$\int_{-M}^{0}\left[\int_{-\infty}^{0}J(x-y)\psi(y)\mathrm{d}y-\psi(x)\right]|x|^{\beta}\mathrm{d}x\le C \text{ for some } C>0 \text{ and all } M>0. \tag{2.8}$$

Moreover, if we fix $M_0>1$ so that (2.6) holds for $x\le -M_0$, then for $M>M_0$ and β as above, we have

$$\begin{aligned}
&b\int_{-M}^{-M_0}\psi(x)|x|^{\beta}dx\le-\int_{-M}^{-M_0}g(\psi(x))|x|^{\beta}dx\\
&=c\int_{-M}^{-M_0}\psi'(x)|x|^{\beta}dx+d\int_{-M}^{-M_0}\left[\int_{-\infty}^{0}J(x-y)\psi(y)\mathrm{d}y-\psi(x)\right]|x|^{\beta}\mathrm{d}x\\
&\quad+d\int_{-M}^{-M_0}\int_{0}^{\infty}|x|^{\beta}J(x-y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

By (2.8),

$$\begin{aligned}
&d\int_{-M}^{-M_0}\left[\int_{-\infty}^{0}J(x-y)\psi(y)\mathrm{d}y-\psi(x)\right]|x|^{\beta}\mathrm{d}x\\
&\le dC-d\int_{-M_0}^{0}\left[\int_{-\infty}^{0}J(x-y)\psi(y)\mathrm{d}y-\psi(x)\right]|x|^{\beta}\mathrm{d}x\\
&:=C_1<\infty \text{ for all } M>M_0.
\end{aligned}$$

Moreover, if we assume additionally that $\beta\le\alpha-2$, then we have

$$\begin{aligned}
&\int_{-M}^{-M_0}\int_{0}^{\infty}|x|^{\beta}J(x-y)\mathrm{d}y\mathrm{d}x\\
&\le\int_{0}^{M}\int_{0}^{\infty}x^{\beta}J(x+y)dydx=\int_{0}^{M}\int_{x}^{\infty}x^{\beta}J(y)dydx\\
&\le\int_{0}^{\infty}\int_{x}^{\infty}x^{\beta}J(y)dydx=\frac{1}{\beta+1}\int_{0}^{\infty}y^{\beta+1}J(y)dy:=C_2<\infty.
\end{aligned}$$

Therefore, for $\beta\in(0,\min\{\gamma,\alpha-2\}]$ and $M>M_0$,

$$\begin{aligned}
&b\int_{-M}^{-M_0}\psi(x)|x|^{\beta}dx\le\ c\int_{-M}^{-M_0}\psi'(x)|x|^{\beta}dx+C_1+dC_2\\
&\le c\int_{1}^{M}x^{\beta}\psi'(-x)dx+C_3\le c\int_{1}^{M}x^{\gamma}\psi'(-x)dx+C_3\\
&\le c\psi(-1)+c\int_{1}^{M}\gamma x^{\gamma-1}\psi(-x)dx+C_3:=C_4<\infty \text{ by (2.7)}.
\end{aligned}$$

It follows that

$$\int_{-\infty}^{0}\tilde{\psi}(x)|x|^{\beta}dx<\infty. \tag{2.9}$$

Thus we have proved that (2.7) implies (2.9) for any $\beta\in(0,\min\{\gamma,\alpha-2\}]$.

If we write $\alpha-2=n+\theta$ with $n\ge 0$ an integer and $\theta\in(0,1]$. Then by the above conclusion and an induction argument we see that (2.9) holds with $\beta=n$. Thus (2.7) holds for $\gamma=n+1$. So applying the above conclusion once more we see that (2.9) holds for every $\beta\in(0,\min\{n+1,\alpha-2\}]=(0,\alpha-2]$, as desired.

Case 2. $\alpha\in(1,2)$.

Let $\beta=\alpha-2$. As in Case 1, for $M>M_0$,

$$\begin{aligned}
&b\int_{-M}^{-M_0}\psi(x)|x|^{\beta}dx\\
&\le c\int_{-M}^{-M_0}\psi'(x)|x|^{\beta}dx+d\int_{-M}^{-M_0}\left[\int_{-\infty}^{0}J(x-y)\psi(y)\mathrm{d}y-\psi(x)\right]|x|^{\beta}\mathrm{d}x\\
&\quad+d\int_{-M}^{-M_0}\int_{0}^{\infty}|x|^{\beta}J(x-y)\mathrm{d}y\mathrm{d}x
\end{aligned}$$

$$\leq c\int_{-M}^{-M_0}\psi'(x)|x|^\beta dx+\widetilde{C}_1+d\int_{-M}^{-M_0}\int_0^\infty |x|^\beta J(x-y)\mathrm{d}y\mathrm{d}x,$$

where $\widetilde{C}_1>0$ is obtained by making use of Lemma 2.4. By ($\mathbf{J}^\alpha$) and $\beta+1=\alpha-1$,

$$\int_{-M}^{-M_0}\int_0^\infty |x|^\beta J(x-y)\mathrm{d}y\mathrm{d}x\leq\int_0^\infty\int_x^\infty x^\beta J(y)\mathrm{d}y\mathrm{d}x$$
$$=\frac{1}{\alpha-1}\int_0^\infty y^{\alpha-1}J(y)\mathrm{d}y:=\widetilde{C}_2<\infty.$$

Due to $\beta<0$, we have

$$\int_{-M}^{-M_0}\psi'(x)|x|^\beta dx=\int_{M_0}^{M}\psi'(-x)x^\beta dx$$
$$=\psi(-M_0)M_0^\beta-\psi(-M)M^\beta+\beta\int_{M_0}^M\psi(-x)x^{\beta-1}dx$$
$$\leq\psi(-M_0)M_0^\beta:=\widetilde{C}_3<\infty.$$

Hence

$$b\int_{-M}^{-M_0}\psi(x)|x|^\beta dx\leq\widetilde{C}_1+\widetilde{C}_2d+c\widetilde{C}_3<\infty$$

for all $M>M_0$, which implies

$$\int_{-\infty}^{-1}\psi(x)|x|^{\alpha-2}dx<\infty.$$

The proof is completed. □

Proof of Theorem 2.2: We have

$$|g(\psi(x))|\leq L\psi(x)\text{ for some }L>0\text{ and all }x<0.$$

Now for $M>M_0\gg1$ and $\beta=\alpha-2$,

$$L\int_{-M}^{-M_0}\psi(x)|x|^\beta dx\geq-\int_{-M}^{-M_0}g(\psi(x))|x|^\beta dx$$
$$=c\int_{-M}^{-M_0}\psi'(x)|x|^\beta dx+d\int_{-M}^{-M_0}\left[\int_{-\infty}^0 J(x-y)\psi(y)\mathrm{d}y-\psi(x)\right]|x|^\beta\mathrm{d}x$$
$$+d\int_{-M}^{-M_0}\int_0^\infty|x|^\beta J(x-y)\mathrm{d}y\mathrm{d}x$$
$$\geq-d\int_{-M}^{-M_0}\psi(x)|x|^\beta\mathrm{d}x+d\int_{-M}^{-M_0}\int_0^\infty|x|^\beta J(x-y)\mathrm{d}y\mathrm{d}x$$

Therefore, with $\widetilde{L}:=L+d$, we have

$$\widetilde{L}\int_{-M}^{-M_0}\psi(x)|x|^\beta dx\geq d\int_{-M}^{-M_0}\int_0^\infty|x|^\beta J(x-y)\mathrm{d}y\mathrm{d}x=d\int_{M_0}^M\int_x^\infty x^\beta J(y)dydx$$
$$=d\Big[\int_{M_0}^M\int_{M_0}^\infty-\int_{M_0}^M\int_{M_0}^x\Big]x^\beta J(y)dydx$$
$$=\frac{d}{\beta+1}\Big[\int_{M_0}^\infty(M^{\beta+1}-M_0^{\beta+1})J(y)dy+\int_{M_0}^M(y^{\beta+1}-M^{\beta+1})J(y)dy\Big]$$
$$\geq\frac{d}{\beta+1}\Big[\int_{M_0}^M y^{\beta+1}J(y)dy-M_0^{\beta+1}\int_{M_0}^\infty J(y)dy\Big]\to\infty\text{ as }M\to\infty,$$

since $\beta+1=\alpha-1$. Therefore (2.2) holds, as we wanted. □

3. Proof of Theorem 1.1

Let us first observe that it suffices to estimate $h(t)-c_0t$, since that for $g(t)+c_0t$ follows by a simple change of the initial function: $(\tilde u(t,x),\tilde g(t),\tilde h(t)) := (u(t,-x),-h(t),-g(t))$ is the unique solution of (1.3) with initial function $\tilde u_0(x) := u_0(-x)$.

Theorem 1.1 will follow easily from the following two lemmas and their proofs, where more general and stronger conclusions are proved.

Lemma 3.1. *In Theorem A, if additionally* $(\mathbf{J}^\alpha)$ *holds for some* $\alpha\geq 2$*, and* f' *is locally Lipschitz in* $[0,\infty)$*, then there exists* $C>0$ *such that for* $t\geq 0$*,*

$$h(t)-c_0t\geq -C\left[1+\int_0^t(1+x)^{1-\alpha}dx+\int_0^{\frac{c_0}{2}t}x^2J(x)dx+t\int_{\frac{c_0}{2}t}^\infty xJ(x)dx\right],$$

where $c_0>0$ *is given in Theorem B.*

Proof. Let (c_0,ϕ^{c_0}) be the unique semi-wave pair in Theorem B. To simplify notations we will write $\phi^{c_0}(x)=\phi(x)$. By $(\mathbf{J}^\alpha)$ $(\alpha\geq 2)$ and Theorem 2.1 there is $C>0$ such that

$$\int_0^\infty J(y)y^{\alpha-1}\mathrm{d}y\leq C,\quad 0<1-\phi(x)\leq \frac{C}{x^{\alpha-1}}\quad\text{for}\quad x<0. \tag{3.1}$$

Define

$$\begin{cases} \underline{h}(t):=c_0t+\delta(t), & t\geq 0,\\ \underline{u}(t,x):=(1-\epsilon(t))[\phi(x-\underline{h}(t))+\phi(-x-\underline{h}(t))-1], & t\geq 0, x\in[-\underline{h}(t),\underline{h}(t)],\end{cases}$$

where $\epsilon(t):=(t+\theta)^{1-\alpha}$ and

$$\delta(t):=K_1-K_2\int_0^t\epsilon(\tau)\mathrm{d}\tau-2\mu\int_0^t\int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)}\int_0^\infty J(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau,$$

with θ, K_1 and K_2 large positive constants to be determined.

For any $M>0$,

$$\begin{aligned}\int_{-\infty}^{-M}\int_0^\infty J(x-y)\mathrm{d}y\mathrm{d}x&=\int_M^\infty\int_x^\infty J(y)\mathrm{d}y\mathrm{d}x\\ &=\int_M^\infty\int_M^y J(y)\mathrm{d}x\mathrm{d}y=\int_M^\infty(y-M)J(y)\mathrm{d}y\leq\int_M^\infty yJ(y)\mathrm{d}y.\end{aligned}$$

Hence, due to $\int_0^\infty yJ(y)\mathrm{d}y<\infty$ (because $\alpha\geq 2$), we have

$$\begin{aligned}2\mu\int_0^t\int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)}\int_0^\infty J(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau&\leq 2\mu\int_0^t\int_{-\infty}^{-\frac{c_0}{2}\theta}\int_0^\infty J(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau\\ &\leq\left[2\mu\int_{\frac{c_0}{2}\theta}^\infty yJ(y)\mathrm{d}y\right]t\leq\frac{c_0}{4}t\end{aligned}$$

provided that $\theta>0$ is large enough, say $\theta\geq\theta_0$.

For any given small $\epsilon_0>0$, due to $\phi(-\infty)=1$ there is $K_0=K_0(\epsilon_0)>0$ such that

$$1-\epsilon_0\leq\phi(x)\text{ for }x\leq -K_0,$$

which implies that

$$\phi(x-\underline{h}(t)),\phi(-x-\underline{h}(t))\in[1-\epsilon_0,1]\quad\text{for}\quad x\in[-\underline{h}(t)+K_0,\underline{h}(t)-K_0], \tag{3.2}$$

where we have assumed $\underline{h}(0)=K_1>K_0$.

Clearly

$$K_2\int_0^t(\tau+\theta)^{1-\alpha}\mathrm{d}\tau\leq K_2\theta^{1-\alpha}t\leq\frac{c_0}{4}t$$

provided $\theta\geq(4K_2/c_0)^{1/(\alpha-1)}$. Therefore

$$\underline{h}(t)\geq\frac{c_0}{2}t+K_1\geq\frac{c_0}{2}(t+\theta)>K_0\text{ for all }t\geq 0\text{ provided that} \tag{3.3}$$

$$K_1\geq\frac{c_0}{2}\theta\text{ and }\theta\geq\max\left\{(4K_2/c_0)^{1/(\alpha-1)},\theta_0,2K_0/c_0\right\}. \tag{3.4}$$

Define

$$\epsilon_1 := \inf_{x\in[-K_0,0]} |\phi'(x)| > 0.$$

Then

$$\begin{cases} \phi'(x-\underline{h}(t)) \le -\epsilon_1 & \text{for } x\in[\underline{h}(t)-K_0, \underline{h}(t)],\\ \phi'(-x-\underline{h}(t)) \le -\epsilon_1 & \text{for } x\in[-\underline{h}(t), -\underline{h}(t)+K_0]. \end{cases} \tag{3.5}$$

Claim 1: For suitably chosen $\theta,\ K_1,\ K_2$, we have

$$\underline{h}'(t) \le \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y, \qquad t>0 \tag{3.6}$$

and

$$-\underline{h}'(t) \ge -\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(x-y)\underline{u}(t,x)\mathrm{d}y, \qquad t>0.$$

Due to $\underline{u}(t,x)=\underline{u}(t,-x)$ and $J(x)=J(-x)$, we just need to verify (3.6). We calculate

$$\begin{aligned}
&\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x\\
=&(1-\epsilon(t))\mu \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x\\
&+(1-\epsilon(t))\mu \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J(x-y)[\phi(-x-2\underline{h}(t))-1]\mathrm{d}y\mathrm{d}x\\
=&(1-\epsilon(t))c_0-(1-\epsilon(t))\mu \int_{-\infty}^{-2\underline{h}(t)} \int_{0}^{\infty} J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x\\
&-(1-\epsilon(t))\mu \int_{-2\underline{h}(t)}^{0} \int_{0}^{\infty} J(x-y)[1-\phi(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

From (3.3), for $t\ge 0$,

$$\begin{aligned}
&(1-\epsilon(t))\mu \int_{-\infty}^{-2\underline{h}(t)} \int_{0}^{\infty} J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x\\
&+(1-\epsilon(t))\mu \int_{-2\underline{h}(t)}^{-\underline{h}(t)} \int_{0}^{\infty} J(x-y)[1-\phi(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x\\
\le& 2\mu \int_{-\infty}^{-\underline{h}(t)} \int_{0}^{\infty} J(x-y)\mathrm{d}y\mathrm{d}x \le 2\mu \int_{-\infty}^{-\frac{c_0}{2}(t+\theta)} \int_{0}^{\infty} J(x-y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

And by (3.1), we have, for $t>0$,

$$\begin{aligned}
&(1-\epsilon(t))\mu \int_{-\underline{h}(t)}^{0} \int_{0}^{\infty} J(x-y)[1-\phi(-x-2\underline{h}(t))]\mathrm{d}y\mathrm{d}x\\
\le&\mu[1-\phi(-\underline{h}(t))] \int_{-\underline{h}(t)}^{0} \int_{0}^{\infty} J(x-y)\mathrm{d}y\mathrm{d}x\\
\le&\mu\frac{C}{h(t)^{\alpha-1}} \int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)\mathrm{d}y\mathrm{d}x\\
=&\mu\frac{C}{h(t)^{\alpha-1}} \int_{0}^{\infty} yJ(y)\mathrm{d}y \le \mu\frac{C^2}{(c_0/2)^{\alpha-1}(t+\theta)^{\alpha-1}} \le \frac{K_2-c_0}{(t+\theta)^{\alpha-1}}
\end{aligned}$$

if

$$K_2 \ge c_0 + \frac{C^2}{(c_0/2)^{\alpha-1}}\mu. \tag{3.7}$$

Hence, when θ, K_1 and K_2 are chosen such that (3.4) and (3.7) hold, then

$$\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x$$

$$
\begin{aligned}
&\ge (1-\epsilon(t))c_0 - 2\mu \int_{-\infty}^{-\frac{c_0}{2}(t+\theta)} \int_0^\infty J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x - \frac{K_2-c_0}{(t+\theta)^{\alpha-1}} \\
&= c_0 - K_2\epsilon(t) - 2\mu \int_{-\infty}^{-\frac{c_0}{2}(t+\theta)} \int_0^\infty J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x \\
&= h'(t) \quad \text{for all } t>0,
\end{aligned}
$$

which finishes the proof of (3.6).

Claim 2: With $\theta,\ K_1,\ K_2$ chosen such that (3.4) and (3.7) hold, and K_2 suitably further enlarged (see (3.8) below), $\theta_0 \gg 1$ and $0<\epsilon_0 \ll 1$, we have, for all $t>0$ and $x\in(-\underline{h}(t),\underline{h}(t))$,

$$
\underline{u}_t(t,x) \le d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x)).
$$

A simple calculation gives

$$
\begin{aligned}
\underline{u}_t =& -\epsilon'(t)[\phi(x-\underline{h}(t)) + \phi(-x-\underline{h}(t)) - 1] \\
& - (1-\epsilon(t))h'(t)[\phi'(x-\underline{h}(t)) + \phi'(-x-\underline{h}(t))] \\
=&(\alpha-1)(t+\theta)^{-\alpha}[\phi(x-\underline{h}(t)) + \phi(-x-\underline{h}(t)) - 1] \\
& - (1-\epsilon(t))[c_0+\delta'(t)][\phi'(x-\underline{h}(t)) + \phi'(-x-\underline{h}(t))],
\end{aligned}
$$

and using the equation satisfied by ϕ we deduce

$$
\begin{aligned}
& - (1-\epsilon(t))c_0[\phi'(x-\underline{h}(t)) + \phi'(-x-\underline{h}(t))] \\
=&(1-\epsilon)\Big[d\int_{-\infty}^{\underline{h}(t)} J(x-y)\phi(y-\underline{h}(t))\mathrm{d}y - d\phi(x-\underline{h}(t)) \\
&\qquad + d\int_{-\underline{h}(t)}^{\infty} J(-x-y)\phi(-y-\underline{h}(t))\mathrm{d}y - d\phi(-x-\underline{h}(t))\Big] \\
& + (1-\epsilon(t))\Big[f(\phi(x-\underline{h}(t))) + f(\phi(-x-\underline{h}(t)))\Big] \\
=&\ d\left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - \underline{u}(t,x)\right] \\
& + (1-\epsilon(t))\Big[d\int_{-\infty}^{-\underline{h}(t)} J(x-y)\phi(y-\underline{h}(t)) - 1]\mathrm{d}y \\
&\qquad + d\int_{\underline{h}(t)}^{\infty} J(-x-y)\phi(-y-\underline{h}(t))\mathrm{d}y - 1]\mathrm{d}y\Big] \\
& + (1-\epsilon(t))\Big[f(\phi(x-\underline{h}(t))) + f(\phi(-x-\underline{h}(t)))\Big] \\
\le&\ d\left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - \underline{u}(t,x)\right] \\
& + (1-\epsilon(t))\Big[f(\phi(x-\underline{h}(t))) + f(\phi(-x-\underline{h}(t)))\Big].
\end{aligned}
$$

Hence

$$
\underline{u}_t \le d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - \underline{u}(t,x) + f(\underline{u}(t,x)) + A_1(t,x) + A_2(t,x),
$$

where

$$
\begin{aligned}
A_1(t,x) :=&\ (\alpha-1)(t+\theta)^{-\alpha}[\phi(x-\underline{h}(t)) + \phi(-x-\underline{h}(t)) - 1], \\
A_2(t,x) :=& - (1-\epsilon(t))\delta'(t)[\phi'(x-\underline{h}(t)) + \phi'(-x-\underline{h}(t))] \\
& + (1-\epsilon(t))[f(\phi(x-\underline{h}(t))) + f(\phi(-x-\underline{h}(t)))] - f(\underline{u}(t,x)).
\end{aligned}
$$

To finish the proof of Claim 2, it remains to check that

$$
A_1(t,x) + A_2(t,x) \le 0 \ \text{ for } \ t>0,\ x\in(-\underline{h}(t),\underline{h}(t)).
$$

We next prove this inequality for x in the following three intervals, separately:

$$I_1(t) := [\underline{h}(t) - K_0, \underline{h}(t)],\ I_2(t) := [-\underline{h}(t), -\underline{h}(t) + K_0],\ I_3(t) := [-\underline{h}(t) + K_0, \underline{h}(t) - K_0].$$

For $x \in I_1(t)$, by (3.1),

$$0 \geq \phi(-x - \underline{h}(t)) - 1 \geq \phi(K_0 - 2\underline{h}(t)) - 1 \geq \phi(-\underline{h}(t)) - 1 \geq \frac{-C}{h(t)^{\alpha-1}}$$

Then by (**f**),

$$f(\phi(-x - \underline{h}(t))) = f(\phi(-x - \underline{h}(t))) - f(1) \leq L\frac{C}{h(t)^{\alpha-1}}$$

and

$$\begin{aligned} f(\underline{u}(t,x)) \geq &(1 - \epsilon(t))f\Big(\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1\Big) \\ \geq &(1 - \epsilon(t))\big[f(\phi(x - \underline{h}(t))) - L\frac{C}{h(t)^{\alpha-1}}\big]. \end{aligned}$$

Thus from the definition of $\delta(t)$, (3.3) and (3.5), we deduce

$$\begin{aligned} A_2(t,x) \leq &(1 - \epsilon(t))\Big[\delta'(t)[\phi'(x - \underline{h}(t)) + \phi'(-x - \underline{h}(t))] + f(\phi(x - \underline{h}(t))) \\ &+ f(\phi(-x - \underline{h}(t))) - f\Big(\phi(x - \underline{h}(t)) + \phi(-x - \underline{h}(t)) - 1\Big)\Big] \\ \leq &(1 - \epsilon(t))\left[-\delta'(t)\epsilon_1 + 2L\frac{C}{h(t)^{\alpha-1}}\right] \leq (1 - \epsilon(t))\left[-K_2(t+\theta)^{1-\alpha}\epsilon_1 + \frac{2LC}{h(t)^{\alpha-1}}\right] \\ \leq &(1 - \epsilon(t))(t+\theta)^{1-\alpha}\big[-K_2\epsilon_1 + 2LC(2/c_0)^{\alpha-1}\big]. \end{aligned}$$

Moreover,

$$A_1(t,x) \leq (\alpha - 1)(t+\theta)^{-\alpha} \leq 2(1 - \epsilon(t))(\alpha - 1)(t+\theta)^{-\alpha},$$

where by enlarging θ_0 we have assumed that $\epsilon(t) \leq \theta_0^{1-\alpha} < 1/2$. Hence

$$A_1(t,x) + A_2(t,x) \leq (1 - \epsilon)(t+\theta)^{1-\alpha}\Big[-K_2\epsilon_1 + 2LC(2/c_0)^{\alpha-1} + 2(\alpha - 1)\theta_0^{-1}\Big] \leq 0$$

if additionally

$$K_2 \geq \epsilon_1^{-1}\Big[2LC(2/c_0)^{\alpha-1} + 2(\alpha - 1)\theta_0^{-1}\Big]. \tag{3.8}$$

This proves the desired inequality for $x \in I_1(t)$.

Since $A_1(t,x) + A_2(t,x)$ is even in x, the desired inequality is also valid for $x \in I_2(t) = -I_1(t)$. It remains to prove the desired inequality for $x \in I_3(t)$.

The case $x \in I_3(t)$ requires some preparations. Define, for $0 < \epsilon \ll 1$,

$$g(u,v) := (1 - \epsilon)[f(u) + f(v)] - f((1 - \epsilon)(u + v - 1)), \quad u, v \in \mathbb{R}.$$

For $u, v \in [0,1]$, we may apply the mean value theorem to the function

$$\xi(t) := g(1 + t(u-1), 1 + t(v-1))$$

to obtain

$$\xi(1) = \xi(0) + \xi'(\zeta) \text{ for some } \zeta \in [0,1].$$

Denote

$$\tilde{u} := 1 + \zeta(u-1), \quad \tilde{v} := 1 + \zeta(v-1).$$

Then the above identity is equivalent to

$$\begin{aligned} g(u,v) = &g(1,1) + \partial_u g(\tilde{u},\tilde{v})(u-1) + \partial_v g(\tilde{u},\tilde{v})(v-1) \\ = &- f(1-\epsilon) + (1-\epsilon)f'(\tilde{u})(u-1) + (1-\epsilon)f'(\tilde{v})(v-1) \\ &- (1-\epsilon)f'\big((1-\epsilon)(\tilde{u}+\tilde{v}-1)\big)(u-1) \\ &- (1-\epsilon)f'\big((1-\epsilon)(\tilde{u}+\tilde{v}-1)\big)(v-1). \end{aligned}$$

Let us note that $\tilde{u} \in [u,1]$ and $\tilde{v} \in [v,1]$. Since f' is locally Lipschitz, there is C_1 such that

$$|f'(u) - f'(v)| \leq C_1|u - v| \quad \text{for} \quad u, v \in [0,1].$$

It follows that

$$\begin{aligned}&(1-\epsilon)f'(\tilde u)(u-1)-(1-\epsilon)f'\big((1-\epsilon)(\tilde u+\tilde v-1)\big)(u-1)\\=&(1-\epsilon)\Big[f'(\tilde u)-f'\big((1-\epsilon)(\tilde u+\tilde v-1)\big)\Big](u-1)\\ \le&(1-\epsilon)b_1(1-u),\end{aligned}$$

where

$$\begin{aligned}b_1:=&\ C_1|\tilde u-(1-\epsilon)(\tilde u+\tilde v-1)|\\=&\ C_1|\epsilon\tilde u-(1-\epsilon)(\tilde v-1)|\\ \le&\ C_1(\epsilon+1-v).\end{aligned}$$

Similarly,

$$\begin{aligned}&(1-\epsilon)f'(\tilde v)(v-1)-(1-\epsilon)f'\big((1-\epsilon)(\tilde u+\tilde v-1)\big)(v-1)\\ \le&\ (1-\epsilon)b_2(1-v),\end{aligned}$$

where

$$b_2:=C_1|\epsilon\tilde v-(1-\epsilon)(\tilde u-1)|\le\ C_1(\epsilon+1-u).$$

Thus

$$\begin{aligned}g(u,v)\le&-f(1-\epsilon)+(1-\epsilon)b_1(1-u)+(1-\epsilon)b_2(1-v)\\ \le&-f(1-\epsilon)+C_1(\epsilon+1-v)(1-u)+C_1(\epsilon+1-u)(1-v)\\ =&\ \epsilon\Big[f'(1)+o(1)+C_1(1-u+1-v)\Big]+2C_1(1-u)(1-v),\end{aligned}$$

where $o(1)\to 0$ as $\epsilon\to 0$.

For our discussions below, it is convenient to introduce the notations

$$p(t,x):=1-\phi(x-\underline{h}(t)),\quad q(t,x):=1-\phi(-x-\underline{h}(t)).$$

Then by (3.2) we have

$$p(t,x),\ q(t,x)\in[0,\epsilon_0]\text{ for }x\in I_3(t),t>0. \tag{3.9}$$

Moreover, since $\min\{x-\underline{h}(t),-x-\underline{h}(t)\}\le-\underline{h}(t)$ always holds, by (3.1) and (3.3), if we denote $C_2:=C(c_0/2)^{1-\alpha}$, then

$$p(t,x)q(t,x)\le\frac{C\epsilon_0}{\underline{h}(t)^{\alpha-1}}\le C_2\epsilon_0\epsilon(t)\ \text{ for }x\in I_3(t),\ t>0. \tag{3.10}$$

Now due to $\delta'(t)<0$ and $\phi'<0$, we have, by (3.9) and (3.10),

$$\begin{aligned}A_2(t,x)\le&\ g(1-p,1-q)\\ \le&\ \epsilon(t)\Big[f'(1)+o(1)+C_1(p+q)\Big]+2C_1pq\\ \le&\ \epsilon(t)\Big[f'(1)+o(1)+C_3\epsilon_0\Big]\quad\text{ for }x\in I_3(t),\ t>0,\end{aligned}$$

with $C_3:=2(C_1+C_1C_2)$. Since

$$A_1(t,x)\le(\alpha-1)(t+\theta)^{-\alpha}\le(\alpha-1)\theta_0^{-1}\epsilon(t)$$

and $f'(1)<0$, we thus obtain

$$A_1+A_2\le\epsilon(t)\Big(f'(1)+\Big[o(1)+C_3\epsilon_0+(\alpha-1)\theta_0^{-1}\Big]\Big)<0\quad\text{ for }x\in I_3(t),\ t>0$$

provided that θ_0 is sufficiently large and ϵ_0 is sufficiently small. The proof of Claim 2 is now complete.

Claim 3: There exists $t_0>0$ such that

$$\begin{cases}g(t+t_0)\le-\underline{h}(t),\ h(t+t_0)\ge\underline{h}(t)\text{ for }t\ge 0,\\ u(t+t_0,x)\ge\underline{u}(t,x)\text{ for }t\ge 0,\ x\in[-\underline{h}(t),\underline{h}(t)].\end{cases} \tag{3.11}$$

It is clear that

$$\underline{u}(t,\pm\underline{h}(t))=(1-\epsilon(t))[\phi(-2\underline{h}(t))-1]\le 0\text{ for }\ t\ge 0.$$

Since spreading happens for (u,g,h), there exists a large constant $t_0>0$ such that

$$g(t_0)<-K_1=-\underline{h}(0)\text{ and }\underline{h}(0)=K_1<h(t_0),$$

$$u(t_0,x)\geq (1-\theta^{1-\alpha})\geq \underline{u}(0,x) \quad \text{for} \quad x\in[-\underline{h}(0),\underline{h}(0)].$$

which, together with the inequalities proved in Claims 1 and 2, allows us to apply the comparison principle in [11] to conclude that (3.11) is valid.

Claim 4: There exists $C>0$ such that

$$\delta(t)\geq -C\left[1+\int_0^t (1+x)^{1-\alpha}dx+\int_0^{\frac{c_0}{2}t}x^2J(x)dx+t\int_{\frac{c_0}{2}t}^{\infty}xJ(x)dx\right].$$

Clearly

$$\int_0^t \epsilon(\tau)\mathrm{d}\tau=\int_0^t (x+\theta)^{1-\alpha}\mathrm{d}x<\int_0^t (x+1)^{1-\alpha}\mathrm{d}x.$$

By changing order of integrations we have

$$\int_0^t\int_{-\infty}^{-\frac{c_0}{2}(\tau+\theta)}\int_0^{\infty}J(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau\leq \int_0^t\int_{-\infty}^{-\frac{c_0}{2}\tau}\int_0^{\infty}J(x-y)\mathrm{d}y\mathrm{d}x\mathrm{d}\tau$$

$$=\int_0^t\int_{\frac{c_0}{2}\tau}^{\infty}\left[y-\frac{c_0}{2}\tau\right]J(y)\mathrm{d}y\mathrm{d}\tau\leq \int_0^t\int_{\frac{c_0}{2}\tau}^{\infty}yJ(y)\mathrm{d}y\mathrm{d}\tau$$

$$=\frac{c_0}{2}\int_0^{\frac{c_0}{2}t}y^2J(y)dy+t\int_{\frac{c_0}{2}t}^{\infty}yJ(y)dy.$$

The desired inequality now follows directly from the definition of $\delta(t)$. □

Next we prove an upper bound for $h(t)-c_0t$. Let us note that we do not need the condition ($\mathbf{J}^\alpha$) in the following result.

Lemma 3.2. *Under the assumptions of Theorem A, if* (**J1**) *holds, and additionally f' is locally Lipschitz in $[0,\infty)$, then there exits $C>0$ such that*

$$h(t)-c_0t\leq C \quad \textit{for all} \;\; t>0. \tag{3.12}$$

Proof. As in the proof of Lemma 3.1, (c_0,ϕ^{c_0}) denotes the unique semi-wave pair in Theorem B, and to simplify notations we write $\phi^{c_0}(x)=\phi(x)$.

For fixed $\beta>1$, and some large constants $\theta>0$ and $K_1>0$ to be determined, define

$$\begin{cases} \bar{h}(t):=c_0+\delta(t), \quad t\geq 0,\\ \bar{u}(t,x):=(1+\epsilon(t))\phi(x-\bar{h}(t)), \quad t\geq 0,\; x\leq \bar{h}(t),\end{cases}$$

where $\epsilon(t):=(t+\theta)^{-\beta}$ and

$$\delta(t):=K_1+\frac{c_0}{1-\beta}[(t+\theta)^{1-\beta}-\theta^{1-\beta}].$$

By comparing $u(t,x)$ with a suitable ODE solution, we see that there is a large constant $t_0>0$ such that

$$u(t+t_0,x)\leq 1+\frac{1}{2}\epsilon(0) \quad \text{for} \quad t\geq 0,\; x\in[g(t+t_0),h(t+t_0)].$$

Due to $\phi(-\infty)=1$, we may choose sufficiently large $K_1>0$ such that $\underline{h}(0)=K_1>2h(t_0)$, $-\underline{h}(0)=-K_1<2g(t_0)$, and also

$$\bar{u}(0,x)=(1+\epsilon(0))\phi(-K_1/2)\geq 1+\frac{1}{2}\epsilon(0)\geq u(t_0,x) \quad \text{for} \;\; x\in[g(t_0),h(t_0)]. \tag{3.13}$$

Claim 1: We have

$$\bar{h}'(t)\geq \mu\int_{g(t+t_0)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty}J(x-y)\bar{u}(t,x)\mathrm{d}y \quad \text{for} \;\; t>0.$$

A direct calculation shows

$$\mu\int_{g(t+t_0)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty}J(x-y)\bar{u}(t,x)\mathrm{d}y\leq \mu\int_{-\infty}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty}J(x-y)\bar{u}(t,x)\mathrm{d}y$$

$$=(1+\epsilon(t))\mu\int_{-\infty}^0\int_0^{+\infty}J(x-y)\phi(x)\mathrm{d}y=(1+\epsilon(t))c_0=\bar{h}'(t),$$

as desired.

Claim 2: If $\theta > 0$ is sufficiently large, then for $t > 0$ and $x \in (g(t+t_0), \underline{h}(t))$, we have

$$\overline{u}_t(t,x) \geq d\int_{g(t+t_0)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - d\overline{u}(t,x) + f(\overline{u}(t,x)). \tag{3.14}$$

We calculate

$$\begin{aligned}\overline{u}_t(t,x) =& -(1+\epsilon(t))[c_0+\delta'(t)]\phi'(x-\bar{h}(t)) + \epsilon'(t)\phi(x-\underline{h}(t))\\ =& -(1+\epsilon(t))c_0\phi'(x-\bar{h}(t)) - (1+\epsilon(t))\delta'(t)\phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\phi(x-\underline{h}(t))\\ \geq& \ d\int_{g(t_0+t)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - d\overline{u}(t,x) + f(\overline{u}(t,x)) + A(t,x)\end{aligned}$$

with

$$\begin{aligned}A(t,x) :=&(1+\epsilon(t))f(\phi(x-\bar{h}(t))) - f((1+\epsilon(t))\phi(x-\bar{h}(t)))\\ &-(1+\epsilon(t))\delta'(t)\phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\phi(x-\underline{h}(t)).\end{aligned}$$

To prove the claim, we need to show

$$A(t,x) \geq 0 \quad \text{for } x \in [g(t_0+t), \bar{h}(t)] \text{ and } t > 0.$$

Let ϵ_0, ϵ_1 and K_0 be given as in the proof of Lemma 3.1. For $x \in [\bar{h}(t) - K_0, \bar{h}(t)]$ and $t > 0$, by (3.5), we have

$$\begin{aligned}A(t,x) \geq& -(1+\epsilon)\delta'(t)\phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\phi(x-\underline{h}(t))\\ =& -(1+\epsilon)c_0(t+\theta)^{-\beta}\phi'(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\phi(x-\underline{h}(t))\\ \geq& \ c_0(t+\theta)^{-\beta}\epsilon_1 - \beta(t+\theta)^{-\beta-1}\\ \geq& \ (t+\theta)^{-\beta-1}\big[c_0\theta\epsilon_1 - \beta\big] \geq 0,\end{aligned}$$

provided θ is large enough.

We next estimate $A(t,x)$ for $x \in [g(t+t_0), \underline{h}(t) - K_0]$. Define, for $0 < \epsilon \ll 1$ and $u, v \geq 0$,

$$g(u) := (1+\epsilon)f(u) - f((1+\epsilon)u).$$

Then for $u, v \in [0,1]$,

$$\begin{aligned}g(u) =&g(1) + g'(\tilde{u})(u-1)\\ =& -f(1+\epsilon) + (1+\epsilon)f'(\tilde{u})(u-1) - (1+\epsilon)f'((1+\epsilon)\tilde{u})(u-1)\\ =& -f(1+\epsilon) + (1+\epsilon)\Big[f'(\tilde{u}) - f'((1+\epsilon)\tilde{u})\Big](u-1)\end{aligned}$$

for some $\tilde{u} \in [u,1]$. Since f' is locally Lipschitz, there exists $C_1 > 0$ such that

$$|f'(u) - f'(v)| \leq C_1|u-v| \quad \text{for} \quad u, v \in [0,2].$$

Therefore

$$\begin{aligned}g(u) \geq& -f(1+\epsilon) - (1+\epsilon)\epsilon C_1(1-u)\\ \geq& -\epsilon f'(1) + o(\epsilon) - 2C_1\epsilon(1-u).\end{aligned}$$

By (3.2) we have

$$-\epsilon_0 \leq \phi(x-\bar{h}(t)) - 1 < 0 \quad \text{for} \quad x \in [g(t_0+t), \underline{h}(t) - K_0],\ t > 0. \tag{3.15}$$

Using (3.2), $\delta' > 0$, $\phi' \leq 0$ and $\epsilon(t) = (t+\theta)^{-\beta} \leq \theta^{-\beta}$, we obtain

$$\begin{aligned}A(t,x) \geq&(1+\epsilon(t))f(\phi(x-\bar{h}(t))) - f((1+\epsilon)\phi(x-\bar{h}(t))) - \beta(t+\theta)^{-\beta-1}\phi(x-\underline{h}(t))\\ =& \ g(\phi(x-\bar{h}(t)) - \beta(t+\theta)^{-\beta-1}\phi(x-\underline{h}(t))\\ \geq& \ \epsilon(t)\Big[-f'(1) + o(1) - 2\epsilon_0 C_1 - \beta\theta^{-\beta-1}\Big]\\ >& \ 0 \quad \text{for} \quad x \in [g(t_0+t), \underline{h}(t) - K_0],\ t > 0,\end{aligned}$$

provided θ is large enough and $\epsilon_0 > 0$ is small enough, since $f'(1) < 0$. We have now proved (3.14).

Due to the inequalities proved in Claims 1 and 2, (3.13) and

$$\overline{u}(t, g(t+t_0)) > 0, \quad \overline{u}(t, \bar{h}(t)) = (1+\epsilon)\phi(\bar{h}(t) - \bar{h}(t)) = 0 \quad \text{for} \quad t \geq 0,$$

we are now able to apply the comparison principle to conclude that

$$\begin{aligned} &h(t+t_0) \le \bar{h}(t), && t\ge 0,\\ &u(t+t_0,x) \le \overline{u}(t,x), && t\ge 0,\ x\in[g(t+t_0),\underline{h}(t)]. \end{aligned}$$

The desired inequality (3.12) follows directly from $\delta(t) \le K_1 + \frac{c_0}{\beta-1}\theta^{1-\beta}$ and $h(t+t_0)\le \bar{h}(t)$. The proof is complete. □

Proof of Theorem 1.1. Since $\alpha \ge 3$, from the definitions of $\bar{h}(t)$ and $\underline{h}(t)$ in the proofs of Lemmas 3.1 and 3.2, it is easily seen that

$$C_0 := \sup_{t>0}\left[|\bar{h}(t)-c_0t| + |\underline{h}(t)-c_0t|\right] < \infty.$$

Hence for large fixed $\theta>0$ and all large t, say $t\ge t_0$,

$$[g(t),h(t)] \supset [-\underline{h}(t-t_0),\underline{h}(t-t_0)] \supset [-c_0t+C, c_0t-C] \text{ with } C := C_0 + c_0t_0,$$

and

$$u(t,x) \ge \underline{u}(t,x) \ge (1-\epsilon(t))\left[\phi^{c_0}(x-c_0t+C) + \phi^{c_0}(-x-c_0t+C) - 1\right]$$

for $x\in[-c_0t+C, c_0t-C]$, where $\epsilon(t) = (t+\theta)^{1-\alpha}$. This inequality for $u(t,x)$ also holds for $x\in[g(t),h(t)]$ if we assume that $\phi^{c_0}(x)=0$ for $x>0$, since when x lies outside of $[-c_0t+C, c_0t-C]$ the right side is negative.

From the proof of Lemma 3.2 we see that the following analogous inequalities hold:

$$g(t) \ge -\bar{h}(t-t_0),\ u(t,x) \le (1+\epsilon(t))\phi^{c_0}(-x-\bar{h}(t-t_0))$$

for $t>t_0$ and $x\in[g(t),h(t)]$. We thus have

$$[g(t),h(t)] \subset [-\bar{h}(t-t_0),\bar{h}(t-t_0)] \subset [-c_0t-C, c_0t+C],$$

and

$$u(t,x) \le \overline{u}(t,x) \le (1-\epsilon(t))\min\left\{\phi^{c_0}(x-c_0t-C), \phi^{c_0}(-x-c_0t-C)\right\}$$

for $t>t_0$ and $x\in[g(t),h(t)]$.

Finally we note that as $t\to\infty$,

$$\begin{cases} \phi^{c_0}(-x-c_0t\pm C)\to 1 & \text{uniformly in } [0,\infty),\\ \phi^{c_0}(x-c_0t\pm C)\to 1 & \text{uniformly in } (-\infty,0], \end{cases}$$

and the conclusions for $u(t,x)$ in Theorem 1.1 thus follow directly. □

4. Proof of Theorem 1.3

In this section we determine the growth rate of $c_0t-h(t)$ and $c_0t+g(t)$ when the kernel function satisfies, for some $\gamma\in(2,3]$,

$$J(x) \sim |x|^{-\gamma} \quad \text{for } |x|\gg 1. \tag{4.1}$$

Namely ($\hat{\mathbf{J}}^\gamma$) holds with $\gamma\in(2,3]$. As before, we will only estimate $c_0t-h(t)$, since the estimate for $c_0t+g(t)$ follows by making the variable change $x\to -x$ in the initial function.

The upper bound for $c_0t-h(t)$ follows directly from Lemma 3.1, so we only need to obtain a suitable lower bound. It turns out that the case $f'(0)\ge d$ is more difficult to treat than the case $f'(0)<d$. Therefore we will consider the case $f'(0)<d$ first, and then handle the more difficult case $f'(0)\ge d$ by adequate modifications of the proof for the first case.

4.1. The case $f'(0)<d$.

Lemma 4.1. *Suppose that the assumptions in Theorem 1.3 are satisfied and $f'(0)<d$. Then there exists $\sigma=\sigma(\gamma)>0$ such that for all large $t>0$,*

$$c_0t-h(t) \ge \begin{cases} \sigma\, t^{3-\gamma} & \text{if } \gamma\in(2,3),\\ \sigma \ln t & \text{if } \gamma=3. \end{cases} \tag{4.2}$$

Proof. Let $\beta:=\gamma-2\in(0,1]$, and (c_0,ϕ) be the semi-wave pair in Theorem B. Define

$$\epsilon(t) := K_1(t+\theta)^{-\beta},\quad \delta(t) := K_2 - K_3\int_0^t \epsilon(\tau)\mathrm{d}\tau$$

and

$$\begin{cases} \bar{h}(t) := c_0t+\delta(t), & t\ge 0,\\ \overline{u}(t,x) := (1+\epsilon(t))\phi(x-\bar{h}(t)) + \rho(t,x), & t\ge 0,\ x\le \bar{h}(t), \end{cases}$$

where

$$\rho(t,x) := K_4\xi(x-\overline{h}(t))\epsilon(t),$$

with $\xi \in C^2(\mathbb{R})$ satisfying

$$0 \le \xi(x) \le 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\epsilon},\ \xi(x) = 0 \text{ for } |x| > 2\tilde{\epsilon}, \tag{4.3}$$

and the positive constants θ, K_1, K_2, K_3, K_4, $\tilde{\epsilon}$ are to be determined.

We are going to show that, it is possible to choose these constants and some $t_0 > 0$ such that

$$\overline{u}_t \ge d\int_{g(t+t_0)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - \overline{u} + f(\overline{u}) \qquad \text{for } t>0,\ x \in (g(t+t_0), \overline{h}(t)), \tag{4.4}$$

$$\bar{h}'(t) \ge \mu\int_{g(t+t_0)}^{\bar{h}(t)}\int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{u}(t,x)\mathrm{d}y \qquad \text{for } t>0, \tag{4.5}$$

$$\overline{u}(t, g(t+t_0)) \ge 0, \quad \overline{u}(t,\bar{h}(t)) \ge 0 \qquad \text{for } t \ge 0, \tag{4.6}$$

$$\overline{u}(0,x) \ge u(t_0,x),\ \overline{h}(0) \ge h(t_0) \qquad \text{for } x \in [g(t_0), h(t_0)]. \tag{4.7}$$

If these inequalities are proved, then by the comparison principle, we obtain

$$\overline{h}(t) \ge h(t+t_0),\ \overline{u}(t,x) \ge u(t+t_0,x) \text{ for } t>0,\ x \in [g(t+t_0), h(t+t_0)],$$

and the desired inequality for $c_0t - h(t)$ follows easily from the definition of $\overline{h}(t)$.

Therefore, to complete the proof, it suffices to prove the above inequalities. We divide the arguments below into several steps.

Firstly, by Theorem A, there is $C_1 > 1$ such that

$$-g(t), h(t) \le (c_0+1)t + C_1 \quad \text{for} \quad t \ge 0. \tag{4.8}$$

Let us also note that (4.6) holds trivially.

Step 1. Choose $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (4.7) holds.

For later analysis, we need to find $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ so that (4.7) holds and at the same time they have less than linear growth in θ.

Since $f'(1) < 0$, there exists small $\epsilon_* > 0$ such that for any $k \in (0, \epsilon_*]$,

$$f(1+k) \le \frac{f'(1)}{2}k < 0 < -\frac{f'(1)}{2}k \le f(1-k).$$

It follows that, for $\tilde{\sigma} := f'(1)/2$,

$$\overline{w}(t) = 1 + \epsilon_* e^{\tilde{\sigma}t}, \quad \underline{w}(t) = 1 - \epsilon_* e^{\tilde{\sigma}t}$$

are a pair of upper and lower solutions of the ODE $w' = f(w)$ with initial data $w(0) \in [1-\epsilon_*, 1+\epsilon_*]$. By (**f**), the unique solution of the ODE

$$W' = F(W),\ W(0) = \|u_0\|_\infty$$

satisfies $\lim_{t\to\infty} W(t) = 1$. Hence there exists $t_* > 0$ such that

$$W(t_*) \in [1-\epsilon_*, 1+\epsilon_*].$$

Using the above defined upper solution $\overline{w}(t)$ we obtain

$$W(t+t_*) \le 1 + \epsilon_* e^{\tilde{\sigma}t} \text{ for } t \ge 0.$$

By the comparison principle we deduce

$$u(t+t_*, x) \le W(t+t_*) \le 1 + \epsilon_* e^{\tilde{\sigma}t} \text{ for } t \ge 0,\ x \in [g(t+t_*), h(t+t_*)].$$

Hence

$$u(t_0,x) \le (1 + \frac{\epsilon(0)}{2}) \text{ for } x \in [g(t_0), h(t_0)]$$

provided that

$$t_0 = t_0(\theta) := \frac{\beta}{|\tilde{\sigma}|}\ln\theta + \frac{\ln(2\tilde{\epsilon}_*/K_1)}{|\tilde{\sigma}|} + t_*.$$

By (4.1), for any fixed $\omega_* \in (\beta, \gamma - 1)$, we have

$$\int_{\mathbb{R}} J(x)|x|^{\omega_*}\mathrm{d}x < \infty.$$

Then by Theorem 1.4, there is C_2 such that

$$1-\phi(x)\leq \frac{C_2}{|x|^{\omega_*}} \text{ for } x\leq -1.$$

Hence, for $K>1$ we have

$$\begin{aligned}(1+\epsilon(0))\phi(-K)-(1+\epsilon(0)/2) &\geq (1+\epsilon(0))\big[1-C_2K^{-\omega_*}\big]-(1+\epsilon(0)/2)\\ &= K_1\theta^{-\beta}/2-C_2K^{-\omega_*}(1+K_1\theta^{-\beta})\geq 0\end{aligned}$$

provided that

$$K^{\omega_*}\geq 2C_2+\frac{2C_2}{K_1}\theta^{\beta}.$$

Therefore, for all $K_1\in(0,1]$, $\theta\geq 1$ and $K\geq (4C_2/K_1)^{1/\omega_*}\theta^{\beta/\omega_*}$, we have

$$(1+\epsilon(0))\phi(-K)-(1+\epsilon(0)/2)\geq 0.$$

Now define

$$K_2(\theta):=2\max\left\{(4C_2/K_1)^{1/\omega_*}\theta^{\beta/\omega_*},(c_0+1)t_0(\theta)+C_1\right\}. \tag{4.9}$$

Then for $K_2=K_2(\theta)$ we have

$$\overline{h}(0)=K_2>K_2/2\geq (c_0+1)t_0+C_1\geq h(t_0),$$

and for $x\in[g(t_0),h(t_0)]$,

$$\overline{u}(0,x)=(1+\epsilon(0))\phi(x-K_2)\geq (1+\epsilon(0))\phi(-K_2/2)\geq (1+\epsilon(0)/2).$$

Thus (4.7) holds if t_0 and K_2 are chosen as above, for any $\theta\geq 1$, $K_1\in(0,1]$.

Step 2. We verify that (4.5) holds if θ, K_1,K_3 and K_4 are chosen suitably.

Denote

$$C_3:=\mu\int_{-\infty}^{0}\int_0^{+\infty}J(x-y)\mathrm{d}y\mathrm{d}x=\mu\int_0^{+\infty}J(y)y\mathrm{d}y. \tag{4.10}$$

A direct calculation shows, writing $\epsilon(t)=\epsilon$,

$$\begin{aligned}&\mu\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x\\ =&\mu\int_{-\infty}^{\bar h(t)}\int_{\bar h(t)}^{+\infty}J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x-\mu\int_{-\infty}^{g(t+t_0)}\int_{\bar h(t)}^{+\infty}J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x\\ =&\mu\int_{-\infty}^{0}\int_0^{+\infty}J(x-y)[(1+\epsilon)\phi(x)+\rho(t,x+\overline{h}(t))]\mathrm{d}y\mathrm{d}x\\ &-\mu\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}J(x-y)[(1+\epsilon)\phi(x)+\rho(t,x+\overline{h}(t))]\mathrm{d}y\mathrm{d}x\\ \leq&(1+\epsilon)c_0+C_3K_4\epsilon-\mu\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}J(x-y)(1+\epsilon)\phi(x)\mathrm{d}y\mathrm{d}x\\ \leq&(1+\epsilon)c_0+C_3K_4\epsilon-\mu\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x.\end{aligned}$$

By elementary calculus, for any $k>1$,

$$\int_{-\infty}^{-k}\int_0^{\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x=\int_{-\infty}^{-k}\int_{-x}^{\infty}\frac{1}{y^{2+\beta}}\mathrm{d}y\mathrm{d}x=\beta^{-1}(1+\beta)^{-1}k^{-\beta}. \tag{4.11}$$

Due to (4.1) and (4.8), there exists $C_4>0$ such that

$$\begin{aligned}
&\mu\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty} J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x\\
&\geq C_4\phi(g(t+t_0)-\bar h(t))\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x\\
&\geq \phi_* C_4\int_{-\infty}^{g(t+t_0)-\bar h(t)}\int_0^{+\infty}\frac{1}{|x-y|^{2+\beta}}\mathrm{d}y\mathrm{d}x=\frac{\phi_*C_4}{\beta(1+\beta)}(|g(t+t_0)|+\bar h(t))^{-\beta}\\
&\geq \frac{\phi_*C_4}{\beta(1+\beta)}[(c_0+1)(t+t_0)+C_1+c_0t+K_2]^{-\beta}\\
&=\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}\left[t+\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)}\right]^{-\beta},
\end{aligned}\tag{4.12}$$

where $\phi_*:=\phi(-1)\leq\phi(-K_2)\leq\phi(g(t+t_0)-\bar h(t))$. Therefore, for all large $\theta>0$ so that

$$\theta>\frac{(c_0+1)t_0+C_1+K_2}{(2c_0+1)},\tag{4.13}$$

which is possible since $t_0(\theta)$ and $K_2(\theta)$ grow slower than linearly in θ, we have

$$\begin{aligned}
&\mu\int_{g(t+t_0)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty} J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x\\
&\leq (1+\epsilon(t))c_0+C_3K_4\epsilon(t)-\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}(t+\theta)^{-\beta}\\
&=c_0+\epsilon(t)\left[c_0+C_3K_4-\frac{\phi_*C_4}{K_1\beta(1+\beta)(2c_0+1)^\beta}\right]\\
&\leq c_0-K_3\epsilon(t)=h'(t)
\end{aligned}$$

provided that K_1,K_3 and K_4 are small enough so that

$$K_1(c_0+C_3K_4+K_3)\leq\frac{\phi_*C_4}{\beta(1+\beta)(2c_0+1)^\beta}.\tag{4.14}$$

Therefore (4.5) holds if we first fix K_1,K_3,K_4 small so that (4.14) holds, and then choose θ large such that (4.13) is satisfied.

Step 3. We show that (4.4) holds when K_3 and K_4 are chosen suitably small and θ is large.

We have

$$\bar u_t(t,x)=-(1+\epsilon(t))[c_0+\delta'(t)]\phi'(x-\bar h(t))+\epsilon'(t)\phi(x-\underline{h}(t))+\rho_t(t,x),$$

and, writing $\epsilon(t)=\epsilon$ to simplify the notation,

$$\begin{aligned}
&-(1+\epsilon)c_0\phi'(x-\bar h(t))\\
&=(1+\epsilon)\left[d\int_{-\infty}^{\bar h(t)}J(x-y)\phi(y-\bar h(t))\mathrm{d}y-d\phi(x-\bar h(t))+f(\phi(x-\bar h(t)))\right]\\
&=d\int_{-\infty}^{\bar h(t)}J(x-y)[\bar u(t,y)-\rho(t,y)]\mathrm{d}y-d[\bar u(t,x)-\rho(t,x)]+(1+\epsilon)f(\phi(x-\bar h(t)))\\
&\geq d\int_{g(t+t_0)}^{\bar h(t)}J(x-y)\bar u(t,y)\mathrm{d}y-d\bar u(t,x)+f(\bar u(t,x))\\
&\quad+d\left[\rho(t,x)-\int_{-\infty}^{\bar h(t)}J(x-y)\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)f(\phi(x-\bar h(t)))-f(\bar u(t,x)).
\end{aligned}$$

Hence

$$\bar u_t(t,x)\geq d\int_{g(t+t_0)}^{\bar h(t)}J(x-y)\bar u(t,y)\mathrm{d}y-d\bar u(t,x)+f(\bar u(t,x))+A(t,x)$$

with

$$A(t,x) := d\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} J(x-y)\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)f(\phi(x-\bar h(t))) - f(\overline{u}(t,x))$$
$$- (1+\epsilon)\delta'(t)\phi'(x-\bar h(t)) + \epsilon'(t)\phi(x-\underline{h}(t)) + \rho_t(t,x).$$

Therefore to complete this step, it suffices to show that we can choose K_3, K_4 and θ such that $A(t,x)\geq 0$. We will do that for $x\in[\bar h(t)-\tilde\epsilon, \bar h(t)]$ and for $x\in[g(t_0+t), \bar h(t)-\tilde\epsilon]$ separately.

Claim 1. If $\tilde\epsilon>0$ in (4.3) is sufficiently small and θ is sufficiently large, then

$$\begin{aligned} &d\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} J(x-y)\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)f(\phi(x-\bar h(t))) - f(\overline{u}(t,x)) \\ &\geq \frac{d-f'(0)}{2}\rho(t,x) > 0 \quad \text{ for } \quad x\in[\bar h(t)-\tilde\epsilon, \bar h(t)]. \end{aligned} \tag{4.15}$$

We have, for $x\in[\bar h(t)-\tilde\epsilon, \bar h(t)]$,

$$d\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} J(x-y)\rho(t,y)\mathrm{d}y\right] = K_4\epsilon(t)\left[d - d\int_{-\infty}^{0} J(x-\bar h(t)-y)\xi(y)\mathrm{d}y\right]$$
$$\geq K_4\epsilon(t)\left[d - d\int_{-2\tilde\epsilon}^{0} J(x-\bar h(t)-y)\mathrm{d}y\right] = K_4\epsilon(t)\left[d - d\int_{\bar h(t)-x-2\tilde\epsilon}^{\bar h(t)-x} J(y)\mathrm{d}y\right]$$
$$\geq K_4\epsilon(t)\left[d - d\int_{-2\tilde\epsilon}^{\tilde\epsilon} J(y)\mathrm{d}y\right] \geq K_4\epsilon(t)\left[d - \frac{d-f'(0)}{4}\right] = \left[d - \frac{d-f'(0)}{4}\right]\rho(t,x),$$

provided $\tilde\epsilon\in(0,\epsilon_1]$ for some small $\epsilon_1>0$.

On the other hand, for $x\in[\bar h(t)-\tilde\epsilon, \bar h(t)]$, by (**f**) we obtain

$$\begin{aligned}(1+\epsilon)f(\phi(x-\bar h(t))) - f(\overline{u}(t,x)) &\geq f((1+\epsilon)\phi(x-\bar h(t))) - f(\overline{u}(t,x)) \\ &= f(\overline{u}(t,x)-\rho(t,x)) - f(\overline{u}(t,x)),\end{aligned}$$

and due to $0<K_4\ll 1$,

$$0\leq \overline{u}(t,x)\leq (1+\epsilon)\phi(\tilde\epsilon) + K_4\epsilon \leq 2\phi(\tilde\epsilon) + \theta^{-\beta}.$$

So $\overline{u}(t,x)$ and $\rho(t,x)$ are small for small $\tilde\epsilon$ and large θ. It follows that

$$f(\overline{u}(t,x)-\rho(t,x)) - f(\overline{u}(t,x)) = -\rho(t,x)[f'(\overline{u}(t,x)) + o(1)]$$
$$= -\rho(t,x)[f'(0)+o(1)] \geq -\left[f'(0) + \frac{d-f'(0)}{4}\right]\rho(t,x)$$

for $x\in[\bar h(t)-\tilde\epsilon, \bar h(t)]$, provided that $\tilde\epsilon$ is small and θ is large. Hence, (4.15) holds.

Denote

$$M := \sup_{x\leq 0}|\phi'(x)|.$$

For $x\in[\bar h(t)-\tilde\epsilon, \bar h(t)]$, by (4.15) we have

$$\begin{aligned} A(t,x) &\geq \frac{d-f'(0)}{2}\rho(t,x) - (1+\epsilon(t))\delta'(t)\phi'(x-\bar h(t)) + \epsilon'(t)\phi(x-\underline{h}(t)) + \rho_t(t,x) \\ &\geq \epsilon(t)\left[\frac{d-f'(0)}{2}K_4 - 2K_3M - \beta(t+\theta)^{-1} - K_4\beta(t+\theta)^{-1}\right] \\ &\geq \epsilon(t)\left[\frac{d-f'(0)}{2}K_4 - 2K_3M - \theta^{-1}\beta\big(1+K_4\big)\right] \\ &\geq 0 \end{aligned}$$

provided that we first fix K_3 and K_4 so that (4.14) holds and at the same time

$$\frac{d-f'(0)}{2}K_4 - 2K_3M > 0, \tag{4.16}$$

and then choose θ sufficiently large.

Next, for fixed small $\tilde\epsilon>0$, we estimate $A(t,x)$ for $x\in[g(t+t_0), \bar h(t)-\tilde\epsilon]$.

Claim 2. For any given $1 \gg \eta > 0$, there is $c_1 = c_1(\eta)$ such that

$$(1+\epsilon)f(v) - f((1+\epsilon)v) \geq c_1\epsilon \ \text{ for } \ v \in [\eta, 1] \text{ and } 0 < \epsilon \ll 1. \tag{4.17}$$

Indeed, by (1.9) there exists $c_1 > 0$ depending on η such that

$$f(v) - vf'(v) \geq 2c_1 \ \text{ for } \ v \in [\eta, 1].$$

Since

$$\lim_{\epsilon \to 0} \frac{(1+\epsilon)f(v) - f((1+\epsilon)v)}{\epsilon} = f(v) - vf'(v) \geq 2c_1$$

uniformly for $v \in [\eta, 1]$, there exists $\epsilon_0 > 0$ small so that

$$\frac{(1+\epsilon)f(v) - f((1+\epsilon)v)}{\epsilon} \geq c_1$$

for $v \in [\eta, 1]$ and $\epsilon \in (0, \epsilon_0]$. This proves Claim 2.

By Claim 2 and $f \in C^1$, there exist a positive constant C_f such that, for $v = \phi(x - \bar{h}(t)) \in [\phi(-\tilde{\epsilon}), 1]$,

$$\begin{aligned}
&(1+\epsilon)f(v) - f((1+\epsilon)v + \rho) \\
&= (1+\epsilon)f(v) - f((1+\epsilon)v) + f((1+\epsilon)v) - f((1+\epsilon)v + \rho) \\
&\geq c_1\epsilon - C_f K_4 \epsilon
\end{aligned}$$

when $\epsilon = \epsilon(t)$ is small.

We also have

$$d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\mathrm{d}y\right] \geq -d\int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\mathrm{d}y \geq -dK_4\epsilon(t),$$

and

$$\begin{aligned}
\rho_t(t,x) =& -\xi'\bar{h}'K_4\epsilon(t) + \xi K_4\epsilon'(t) \geq -\xi_* K_4\epsilon(t) - K_4\beta(t+\theta)^{-1}\epsilon(t) \\
\geq& -(\xi_* + \beta\theta^{-1})K_4\epsilon(t),
\end{aligned}$$

with $\xi_* := c_0 \max_{x\in\mathbb{R}} |\xi'(x)|$.

Using these we obtain, for $x \in [g(t_0 + t), \bar{h}(t) - \tilde{\epsilon}]$,

$$\begin{aligned}
A(t,x) \geq& \ -dK_4\epsilon(t) + (1+\epsilon)f(\phi(x - \bar{h}(t))) - f(\bar{u}(t,x)) + 2M\delta'(t) + \epsilon'(t) + \rho_t(t,x) \\
\geq& \ \epsilon(t)\Big[c_1 - K_4(C_f + d) - 2MK_3 - \beta(t+\theta)^{-1} - (\xi_* + \beta\theta^{-1})K_4\Big] \\
\geq& \ \epsilon(t)\Big[c_1 - K_4(C_f + d) - 2MK_3 - \xi_* K_4 - \beta\theta^{-1}\big(1 + K_4\big)\Big] \\
\geq& \ 0
\end{aligned}$$

provided that we first choose K_3 and K_4 small such that

$$c_1 - K_4(C_f + d) - 2MK_3 - \xi_* K_4 > 0$$

while keeping both (4.14) and (4.16) hold, and then choose $\theta > 0$ sufficiently large.

Therefore, (4.4) holds when K_3, K_4 and θ are chosen as above. The proof of the lemma is now complete. □

4.2. The case $f'(0) \geq d$.

Lemma 4.2. *In Lemma 4.1, if $f'(0) \geq d$, then* (4.2) *still holds.*

Proof. This is a modification of the proof of Lemma 4.1, where in the definition of $\bar{u}$, we add a new term $\lambda(t)$ and change $\rho(t,x)$ to $-\rho(t,x)$; see details below.

We will use similar notations. Let $\beta = \gamma - 2 \in (0, 1]$, and for fixed $\tilde{\epsilon} > 0$, let $\xi \in C^2(\mathbb{R})$ satisfy

$$0 \leq \xi(x) \leq 1, \quad \xi(x) = 1 \text{ for } |x| < \tilde{\epsilon}, \ \xi(x) = 0 \text{ for } |x| > 2\tilde{\epsilon}.$$

Define

$$\begin{cases}
\bar{h}(t) := c_0 t + \delta(t), & t \geq 0, \\
\overline{u}(t,x) := (1+\epsilon(t))\phi\big(x - \bar{h}(t) - \lambda(t)\big) - \rho(t,x), & t \geq 0, \ x \leq \bar{h}(t),
\end{cases}$$

where

$$\epsilon(t) := K_1(t+\theta)^{-\beta}, \ \delta(t) := K_2 - K_3\int_0^t \epsilon(\tau)\mathrm{d}\tau,$$

$$\rho(t,x) := K_4\xi(x-\bar{h}(t))\epsilon(t),\ \ \lambda(t) := K_5\epsilon(t),$$

and the positive constants θ, $\tilde{\epsilon}$ and K_1, K_2, K_3, K_4, K_5 are to be determined.

Let

$$C_{\tilde{\epsilon}} := \min_{x\in[-2\tilde{\epsilon},0]} |\phi'(x)|.$$

Then for $x \in [\bar{h}(t) - 2\tilde{\epsilon}, \bar{h}(t)]$,

$$\bar{u}(t,x) \geq \phi\big(-\lambda(t)\big) - \rho(t,x) \geq C_{\tilde{\epsilon}}\lambda(t) - K_4\epsilon(t) \geq \epsilon(t)(C_{\tilde{\epsilon}}K_5 - K_4) > 0$$

if

$$K_4 = C_{\tilde{\epsilon}}K_5/2, \tag{4.18}$$

which combined with $\xi(x) = 0$ for $|x| \geq 2\tilde{\epsilon}$ implies

$$\overline{u}(t,x) \geq 0 \text{ for } \ t \geq 0,\ x \leq \bar{h}(t). \tag{4.19}$$

Let $t_0 = t_0(\theta)$ and $K_2 = K_2(\theta)$ be given by Step 1 in the proof of Lemma 4.1. Then $[g(t_0), h(t_0)] \subset (-\infty, K_2/2)$, and due to $\rho(0,x) = 0$ for $x \leq h(t_0) < K_2/2 < K_2 = \bar{h}(0)$, we have

$$\begin{aligned}\overline{u}(0,x) &= (1+\epsilon(0))\phi(x - K_2 - \lambda(0)) \geq (1+\epsilon(0))\phi(-K_2/2)\\ &\geq 1 + \epsilon(0)/2 \geq u(t_0,x) \ \text{ for } \ x \in [g(t_0), h(t_0)].\end{aligned} \tag{4.20}$$

Step 1. We verify that by choosing K_1, K_3 and K_5 suitably small,

$$\bar{h}'(t) \geq \mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{u}(t,x)\mathrm{d}y\mathrm{d}x \ \text{ for all } \ t > 0. \tag{4.21}$$

By direct calculations we have

$$\begin{aligned}&\mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{u}(t,x)\mathrm{d}y\mathrm{d}x\\
\leq\ &\mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(x-y)(1+\epsilon)\phi(x-\bar{h}(t)-\lambda(t))\mathrm{d}y\mathrm{d}x\\
=\ &(1+\epsilon)\mu \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)\phi(x-\lambda(t))\mathrm{d}y\mathrm{d}x\\
&\quad - (1+\epsilon)\mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_{0}^{+\infty} J(x-y)\phi(x-\lambda(t))\mathrm{d}y\mathrm{d}x\\
\leq\ &(1+\epsilon)c_0 + (1+\epsilon)\mu \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)[\phi(x-\lambda(t)) - \phi(x)]\mathrm{d}y\mathrm{d}x\\
&\quad - (1+\epsilon)\mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_{0}^{+\infty} J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x\end{aligned}$$

Let $M_1 := \sup_{x\leq 0} |\phi'(x)|$ and C_3 be given by (4.10). Then

$$(1+\epsilon)\mu \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)[\phi(x-\lambda(t)) - \phi(x)]\mathrm{d}y\mathrm{d}x \leq 2C_3M_1\lambda(t).$$

By (4.12),

$$\begin{aligned}&\mu \int_{-\infty}^{g(t+t_0)-\bar{h}(t)} \int_{0}^{+\infty} J(x-y)\phi(x)\mathrm{d}y\mathrm{d}x\\
\geq\ &\frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}\left[t + \frac{(c_0+1)t_0 + C_1 + K_2}{(2c_0+1)}\right]^{-\beta}.\end{aligned}$$

Therefore, as in the proof of Lemma 4.1, for sufficiently large θ so that

$$\theta > \frac{(c_0+1)t_0 + C_1 + K_2}{(2c_0+1)} \tag{4.22}$$

holds, we have

$$\begin{aligned}
&\mu \int_{g(t+t_0)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{u}(t,x)\mathrm{d}y\mathrm{d}x \\
&\le (1+\epsilon)c_0 + 2C_3M_1\lambda(t) - \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}(t+\theta)^{-\beta} \\
&= c_0 + \epsilon(t)\left[c_0 + 2C_3M_1K_5 - \frac{\phi_* C_4}{K_1\beta(1+\beta)(2c_0+1)^\beta}\right] \\
&\le c_0 - K_3\epsilon(t) = \bar{h}'(t)
\end{aligned}$$

provided that K_1, K_3 and K_5 are suitably small so that

$$K_1(c_0 + 2C_3M_1K_5 + K_3) \le \frac{\phi_* C_4}{\beta(1+\beta)(2c_0+1)^\beta}. \tag{4.23}$$

Step 2. We show that by choosing K_3, K_5 suitably small and θ sufficiently large, for $t > 0$ and $x \in [g(t+t_0), \bar{h}(t)]$,

$$\overline{u}_t(t,x) \ge d\int_{g(t+t_0)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - \overline{u}(t,x) + f(\overline{u}(t,x)). \tag{4.24}$$

Using the definition of $\overline{u}$, we have

$$\begin{aligned}
\overline{u}_t(t,x) &= -(1+\epsilon)(\bar{h}' + \lambda')\phi'(x-\bar{h}-\lambda) + \epsilon'\phi(x-\bar{h}-\lambda) - \rho_t \\
&= -(1+\epsilon)[c_0 + \delta' + \lambda']\phi'(x-\bar{h}-\lambda) + \epsilon'\phi(x-\bar{h}-\lambda) - \rho_t
\end{aligned}$$

and

$$\begin{aligned}
&-(1+\epsilon)c_0\phi'(x-\bar{h}-\lambda) \\
&= (1+\epsilon)\left[d\int_{-\infty}^{\bar{h}+\lambda} J(x-y)\phi(y-\bar{h}-\lambda)\mathrm{d}y - d\phi(x-\bar{h}-\lambda) + f(\phi(x-\bar{h}-\lambda))\right] \\
&\ge (1+\epsilon)\left[d\int_{-\infty}^{\bar{h}} J(x-y)\phi(y-\bar{h}-\lambda)\mathrm{d}y - d\phi(x-\bar{h}-\lambda) + f(\phi(x-\bar{h}-\lambda))\right] \\
&= d\int_{-\infty}^{\bar{h}} J(x-y)[\overline{u}(t,y)+\rho]\mathrm{d}y - d[\overline{u}(t,x)+\rho] + (1+\epsilon)f(\phi(x-\bar{h}-\lambda)) \\
&= \int_{-\infty}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - d\overline{u}(t,x) \\
&\quad - d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)f(\phi(x-\bar{h}-\lambda)) \\
&\ge d\int_{g(t+t_0)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - d\overline{u}(t,x) + f(\overline{u}(t,x)) \\
&\quad - d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}(t)} J(x-y)\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)f(\phi(x-\bar{h}-\lambda)) - f(\overline{u}(t,x)).
\end{aligned}$$

Hence

$$\overline{u}_t(t,x) \ge d\int_{g(t+t_0)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - d\overline{u}(t,x) + f(\overline{u}(t,x)) + B(t,x)$$

with

$$\begin{aligned}
B(t,x) := &-d\left[\rho(t,x) - \int_{-\infty}^{\bar{h}} J(x-y)\rho(t,y)\mathrm{d}y\right] + (1+\epsilon)f(\phi(x-\bar{h}-\lambda)) - f(\overline{u}(t,x)) \\
&-(1+\epsilon)(\delta' + \lambda')\phi'(x-\bar{h}-\lambda) + \epsilon'\phi(x-\underline{h}-\lambda) - \rho_t.
\end{aligned}$$

To show (4.24), it remains to choose suitable K_3, K_5 and θ such that $B(t,x) \ge 0$ for $t > 0$ and $x \in [g(t+t_0), \bar{h}(t)]$.

Claim: There exist small $\tilde\epsilon_0\in(0,\tilde\epsilon/2)$ and some $\tilde J_0>0$ depending on $\tilde\epsilon$ but independent of $\tilde\epsilon_0$, such that

$$\begin{aligned}&-d\left[\rho(t,x)-\int_{-\infty}^{\bar h}J(x-y)\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)f(\phi(x-\bar h-\lambda))-f(\overline{u}(t,x))\\&\geq\ \tilde J_0\,\rho(t,x)\ \text{ for }\ x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)].\end{aligned}\tag{4.25}$$

Indeed, for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$,

$$\begin{aligned}&d\left[\rho(t,x)-\int_{-\infty}^{\bar h(t)}J(x-y)\rho(t,y)\mathrm{d}y\right]=K_2\epsilon(t)\left[d-d\int_{-\infty}^{\bar h(t)}J(x-y)\xi(y-\bar h(t))\mathrm{d}y\right]\\&\leq K_2\epsilon(t)\left[d-d\int_{\bar h(t)-\tilde\epsilon}^{\bar h(t)}J(x-y)\mathrm{d}y\right]=K_2\epsilon(t)\left[d-d\int_{\bar h(t)-\tilde\epsilon-x}^{\bar h(t)-x}J(x-y)\mathrm{d}y\right]\\&\leq d\rho\left[1-\int_{-\tilde\epsilon+\tilde\epsilon_0}^{0}J(y)\mathrm{d}y\right]\leq d\rho\left[1-\int_{-\tilde\epsilon/2}^{0}J(y)\mathrm{d}y\right].\end{aligned}$$

On the other hand, for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$, we have

$$\begin{aligned}(1+\epsilon)f(\phi(x-\bar h-\lambda)-f(\overline{u})&\geq f((1+\epsilon)\phi(x-\bar h-\lambda))-f(\overline{u})\\&=f(\overline{u}+\rho)-f(\overline{u})=\rho\Big(f'(\overline{u})+o(1)\Big)=\Big(f'(0)+o(1)\Big)\rho\end{aligned}$$

since both $\overline{u}(t,x)$ and $\rho(t,x)$ are close to 0 for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$ with $\tilde\epsilon_0$ small.

Hence, for such x and $\tilde\epsilon_0$, since $f'(0)\geq d$,

$$\begin{aligned}&-d\left[\rho(t,x)-\int_{-\infty}^{\bar h(t)}J(x-y)\rho(t,y)\mathrm{d}y\right]+(1+\epsilon)f(\phi(x-\bar h(t)))-f(\overline{u}(t,x))\\&\geq d\rho\left[-1+\int_{-\tilde\epsilon/2}^{0}J(y)\mathrm{d}y\right]+f'(0)\rho+o(1)\rho\\&\geq\tilde J_0\,\rho(t,x),\qquad\text{with }\ \tilde J_0:=\frac{d}{2}\int_{-\tilde\epsilon/2}^{0}J(y)\mathrm{d}y.\end{aligned}$$

This proves (4.25).

Clearly

$$-\rho_t(t,x)=\beta K_4K_1(t+\theta)^{-\beta-1}\geq 0.$$

Denoting $M_1:=\sup_{x\leq 0}|\phi'(x)|$, we obtain, for $x\in[\bar h(t)-\tilde\epsilon_0,\bar h(t)]$ and small $\tilde\epsilon_0$,

$$\begin{aligned}B(t,x)&\geq\tilde J_0K_2\epsilon(t)+2(\delta'(t)+\lambda'(t))M_1+\epsilon'(t)\\&=\tilde J_0K_2\epsilon(t)+2\epsilon(t)(-K_3-K_5\beta(t+\theta)^{-1})M_1-\beta(t+\theta)^{-1}\epsilon(t)\\&\geq\epsilon(t)\left[\tilde J_0K_2-2(K_3+K_5\beta\theta^{-1})M_1-\beta\theta^{-1}\right]\\&=\epsilon(t)\left[\tilde J_0K_2-2K_3M_1-\theta^{-1}\Big(K_5\beta M_1+\beta\Big)\right]\\&\geq 0\end{aligned}$$

provided that K_3 is chosen small so that (4.23) holds,

$$\tilde J_0K_2-2K_3M_1>0,\tag{4.26}$$

and θ is chosen sufficiently large.[3]

We next estimate $B(t,x)$ for $x\in[g(t+t_0),\bar h(t)-\tilde\epsilon_0]$. From Claim 2 in the proof of Lemma 4.1, and the Lipschitz continuity of f, there exist positive constants $C_l=C_l(\tilde\epsilon_0)$ and C_f such that, for $v=\phi(x-\bar h(t-\lambda(t)))\in[\phi(-\tilde\epsilon_0),1]$,

$$(1+\epsilon)f(v)-f((1+\epsilon)v-\rho)$$

[3] In fact, by the choice of $K_2=K_2(\theta)$ in (4.9), for fixed K_3, (4.26) always holds for large enough θ.

$$
\begin{aligned}
&= (1+\epsilon)f(v) - f((1+\epsilon)v) + f((1+\epsilon)v) - f((1+\epsilon)v-\rho)\\
&\geq C_l\epsilon - C_f\rho \geq C_l\epsilon - C_fK_4\epsilon
\end{aligned}
$$

when $\epsilon=\epsilon(t)$ is small. Hence

$$
\begin{aligned}
&(1+\epsilon(t))f(\phi(x-\bar h(t)-\lambda(t))) - f(\bar u(t,x))\\
&\geq C_l\epsilon(t) - C_fK_4\epsilon(t) \ \text{ for } \quad x\in[g(t+t_0),\bar h(t)-\tilde\epsilon_0],\ 0<\tilde\epsilon_0\ll 1.
\end{aligned}
$$

Clearly,

$$
-d\left[\rho(t,x) - \int_{-\infty}^{\bar h(t)} J(x-y)\rho(t,y)\mathrm{d}y\right] \geq -dK_4\epsilon(t),
$$

and

$$
\rho_t(t,x) = -K_4\xi'\bar h'(t)\epsilon(t) + K_4\xi\epsilon'(t) \leq \xi_*K_4\epsilon(t)
$$

with $\xi_* := c_0\max_{x\in\mathbb{R}}|\xi'(x)|$.

We thus obtain, for $x\in[g(t+t_0),\bar h(t)-\tilde\epsilon_0]$ and $0<\tilde\epsilon_0\ll 1$,

$$
\begin{aligned}
B(t,x) \geq&\ -K_4\epsilon(t)d + (1+\epsilon)f(\phi(x-\bar h)) - f(\overline{u}) + 2M_1(\delta'+\lambda') + \epsilon' - \rho_t\\
\geq&\ C_l\epsilon(t) - K_4\epsilon(t)(d+C_f+\xi_*) + 2M_1[-K_3\epsilon(t)+K_5\epsilon'(t)] + \epsilon'(t)\\
\geq&\ \epsilon(t)\Big[C_l - K_4(d+C_f+\xi_*) - 2M_1(K_3+K_5\beta(t+\theta)^{-1}) - \beta(t+\theta)^{-1}\Big]\\
\geq&\ \epsilon(t)\Big[C_l - K_4\Big(d+C_f+\xi_*\Big) - 2M_1K_3 - \theta^{-1}\beta\Big(2M_1K_5+1\Big)\Big]\\
\geq&\ 0
\end{aligned}
$$

if we choose K_3 and K_5 small so that (4.23) and (4.26) hold and at the same time, due to (4.18)

$$
C_l - K_4\Big(d+C_f+\xi_*\Big) - 2M_1K_3 > 0,
$$

and then choose θ sufficiently large. Hence, (4.24) is satisfied if K_3 and K_5 are chosen small as above, and θ is sufficiently large.

From (4.19), we have

$$
\overline{u}(t,g(t+t_0))\geq 0,\quad \overline{u}(t,\bar h(t))\geq 0 \ \text{ for } \ t\geq 0.
$$

Together with (4.20), (4.21) and (4.24), this enables us to use the comparison principle to conclude that

$$
h(t+t_0)\leq \bar h(t),\ u(t+t_0,x)\leq \overline{u}(t,x) \ \text{ for } \ t\geq 0,\ x\in[g(t+t_0),\underline{h}(t)],
$$

which implies (4.2). The proof of the lemma is now complete. □

4.3. **Proof of Theorem 1.3.** By Lemma 3.1 and then by (4.1), there exists $C_0>0$ such that

$$
\begin{aligned}
h(t)-c_0t \geq&-C\left[1+\int_0^t(1+x)^{1-\gamma}dx + \int_0^{\frac{c_0}{2}t}x^2J(x)dx + t\int_{\frac{c_0}{2}t}^{\infty}xJ(x)dx\right]\\
\geq&-C\left[1+\frac{1}{\gamma-2}+\int_0^1J(x)dx + C_0\int_1^{\frac{c_0}{2}t}x^{2-\gamma}dx + C_0t\int_{\frac{c_0}{2}t}^{\infty}x^{1-\gamma}dx\right].
\end{aligned}
$$

Therefore when $\gamma\in(2,3)$ we have

$$
h(t)-c_0t \geq -C\left[\tilde C + \ln(t+1) + \tilde C_1t^{3-\gamma}\right] \geq -\hat C_1t^{3-\gamma} \ \text{ for all } t\gg 1 \text{ and some } \hat C_1,\tilde C,\tilde C_1>0,
$$

and when $\gamma=3$,

$$
h(t)-c_0t\geq -\hat C_2\ln t \ \text{ for all } t\gg 1 \text{ and some } \hat C_2>0.
$$

These combined with Lemmas 4.1 and 4.2 yield the desired conclusion of Theorem 1.3. □

5. Proof of Theorem 1.2

Throughout this section, we assume that J satisfies $(\mathbf{J})$ and $(\hat{\mathbf{J}}^\gamma)$ for some $\gamma \in (1,2]$. So there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{|x|^\gamma+1} \le J(x) \le \frac{C_2}{|x|^\gamma+1} \quad \text{for} \quad x\in\mathbb{R} \quad \text{and some } \gamma\in(1,2]. \tag{5.1}$$

Clearly now **(J1)** is not satisfied.

The purpose of this section is to prove Theorem 1.2, and as before we will only prove the estimate for $h(t)$, since that for $g(t)$ follows by the change of variable $x\to -x$. Theorem 1.2 will follow directly from the lemmas in Subsections 5.1 and 5.2 below.

5.1. Upper bound.

This is the easy part of the proof.

Lemma 5.1. *Assume that* $(\mathbf{J})$ *and* $(\mathbf{f})$ *hold. If spreading happens, and* (5.1) *is satisfied, then there exits* $C=C(\gamma)>0$ *such that*

$$h(t) \le \begin{cases} Ct^{1/(\gamma-1)} & \text{if } \gamma\in(1,2),\\ Ct\ln t & \text{if } \gamma=2. \end{cases} \tag{5.2}$$

Proof. Define, for $t\ge 0$,

$$\bar h(t) := \begin{cases} (Kt+\theta)^{1/(\gamma-1)} & \text{if } \gamma\in(1,2],\\ (Kt+\theta)\ln(Kt+\theta) & \text{if } \gamma=2, \end{cases}$$

$$\overline{u}(t,x)=\overline{u} := \max\{\|u_0\|_\infty,1\}, \quad x\in[-\bar h(t),\bar h(t)],$$

with positive constants θ and K to be determined.

We start by showing

$$\bar h'(t) \ge \mu\int_{-\bar h(t)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty} J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x \quad \text{for} \quad t>0, \tag{5.3}$$

and

$$-\bar h'(t) \le -\mu\int_{-\bar h(t)}^{\bar h(t)}\int_{-\infty}^{-\bar h(t)} J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x \quad \text{for} \quad t>0.$$

Since $\overline{u}(t,x)=\overline{u}(t,-x)$ and $J(x)=J(-x)$, it suffices to prove (5.3).

By simple calculations and (5.1), for any $k>1$,

$$\begin{aligned}\int_{-k}^0\int_0^\infty J(x-y)\mathrm{d}y\mathrm{d}x &= \int_0^k\int_x^\infty J(y)\mathrm{d}y\mathrm{d}x = \int_0^k J(y)y\mathrm{d}y + \mu k\int_k^\infty J(y)\mathrm{d}y\\ &\le \int_0^k \frac{C_2 y}{y^\gamma+1}\mathrm{d}y + k\int_k^\infty\frac{C_2}{y^\gamma+1}\mathrm{d}y \le \int_0^1 C_2\mathrm{d}y + \int_1^k\frac{C_2 y}{y^\gamma}\mathrm{d}y + k\int_k^\infty \frac{C_2}{y^\gamma}\mathrm{d}y,\end{aligned}$$

and so

$$\int_{-k}^0\int_0^\infty J(x-y)\mathrm{d}y\mathrm{d}x \le \begin{cases} C_2+\frac{C_2}{2-\gamma}(k^{2-\gamma}-1)+\frac{C_2k^{2-\gamma}}{\gamma-1} & \text{if } \gamma\in(1,2),\\ 2C_2+C_2\ln k & \text{if } \gamma=2. \end{cases} \tag{5.4}$$

Clearly

$$\int_{-\bar h(t)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty} J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x = \bar u\int_{-2\bar h(t)}^0\int_0^{+\infty}J(x-y)\mathrm{d}y\mathrm{d}x.$$

Hence for $1<\gamma<2$, by (5.4),

$$\begin{aligned}&\mu\int_{-\bar h(t)}^{\bar h(t)}\int_{\bar h(t)}^{+\infty} J(x-y)\bar u(t,x)\mathrm{d}y\mathrm{d}x\\ &\le \mu\bar u\left[C_2+2^{2-\gamma}\left(\frac{C_2}{2-\gamma}+\frac{C_2}{\gamma-1}\right)(Kt+\theta)^{(2-\gamma)/(\gamma-1)}\right]\\ &\le \frac{K}{\gamma-1}(Kt+\theta)^{(2-\gamma)/(\gamma-1)} = \bar h'(t)\end{aligned}$$

provided that $K > 0$ is large enough. And for $\gamma = 2$,

$$\mu \int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(x-y)\bar{u}(t,x)\mathrm{d}y\mathrm{d}x \le \mu\bar{u}\Big(2C_2 + C_2 \ln\big[2(Kt+\theta)\ln(Kt+\theta)\big]\Big)$$
$$\le \mu\bar{u}\big(2C_2 + C_2 \ln 2(Kt+\theta) + C_2 \ln[\ln(Kt+\theta)]\big) \le K\ln(Kt+\theta) + K = \bar{h}'(t)$$

if $K \gg 1$. This finishes the proof of (5.3).

Since $\overline{u} \ge 1$ is a constant, we have, for $t > 0$, $x \in [-\bar{h}(t), \bar{h}(t)]$,

$$\overline{u}_t(t,x) \equiv 0 \ge d\int_{-\bar{h}(t)}^{\bar{h}(t)} J(x-y)\overline{u}(t,y)\mathrm{d}y - d\overline{u}(t,x) + f(\overline{u}(t,x)). \tag{5.5}$$

Moreover, $\bar{h}(0) \ge h_0$ for large θ, and obviously

$$\begin{aligned}&\overline{u}(t, \pm\bar{h}(t)) \ge 0 \text{ for } t \ge 0,\\ &\overline{u}(0,x) \ge u(0,x) \text{ for } x \in [-h_0, h_0].\end{aligned}$$

Hence we can apply the comparison principle to conclude that

$$\begin{aligned}&[g(t+t_0), h(t+t_0)] \subset [-\bar{h}(t), \bar{h}(t)], && t \ge 0,\\ &u(t+t_0, x) \le \overline{u}(t,x), && t \ge 0,\ x \in [g(t+t_0), h(t+t_0)].\end{aligned}$$

Thus (5.2) holds. □

5.2. **Lower bound.** We will consider the cases $\gamma \in (1,2)$ and $\gamma = 2$ separately.

5.2.1. *The case $\gamma \in (1,2)$.* We start with a result from [20].

Lemma 5.2. [20, (2.11)] *If J satisfies* **(J)**, *then for any $\epsilon > 0$, there is $L_\epsilon > 0$ such that for all $l > L_\epsilon$, the function $\psi_l(x) := l - |x|$ satisfies*

$$\int_{-l}^{l} J(x-y)\psi_l(y)\mathrm{d}y \ge (1-\epsilon)\psi_l(x) \text{ in } [-l, l]. \tag{5.6}$$

Lemma 5.3. *Assume that the conditions in Theorem 1.2 are satisfied and $\gamma \in (1,2)$. Then there exits $C = C(\gamma) > 0$ such that*

$$h(t) \ge Ct^{1/(\gamma-1)} \text{ for } t \gg 1. \tag{5.7}$$

Proof. Define

$$\begin{aligned}&\underline{h}(t) := (K_1 t + \theta)^{1/(\gamma-1)}, \quad t \ge 0,\\ &\underline{u}(t,x) := K_2 \frac{\underline{h}(t) - |x|}{\underline{h}(t)}, \quad t \ge 0,\ x \in [-\underline{h}(t), \underline{h}(t)],\end{aligned}$$

with positive constants θ and K_1, K_2 to be determined.

Step 1. We show that, for large K_1,

$$\underline{h}'(t) \le \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \quad \text{for} \quad t > 0. \tag{5.8}$$

By simple calculations and (5.1), we obtain

$$\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \ge \mu K_2 \int_0^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(x-y)\frac{\underline{h}(t) - x}{\underline{h}(t)}\mathrm{d}y\mathrm{d}x$$
$$= \frac{\mu K_2}{\underline{h}(t)} \int_{-\underline{h}(t)}^{0} \int_0^{+\infty} J(x-y)(-x)\mathrm{d}y\mathrm{d}x = \frac{\mu K_2}{\underline{h}(t)} \int_0^{\underline{h}(t)} \int_x^{+\infty} J(y)x\mathrm{d}y\mathrm{d}x$$
$$= \frac{\mu K_2}{\underline{h}(t)} \left(\int_0^{\underline{h}(t)} \int_0^y + \int_{\underline{h}(t)}^{\infty} \int_0^{\underline{h}(t)} \right) J(y)x\mathrm{d}x\mathrm{d}y \ge \frac{\mu K_2}{2\underline{h}(t)} \int_0^{\underline{h}(t)} J(y)y^2\mathrm{d}y$$
$$\ge \frac{\mu K_2 C_1}{2\underline{h}(t)} \int_0^{\underline{h}(t)} \frac{y^2}{y^\gamma + 1}\mathrm{d}y \ge \frac{\mu K_2 C_1}{4\underline{h}(t)} \int_1^{\underline{h}(t)} y^{2-\gamma}\mathrm{d}y \ge \frac{\mu K_2 C_1}{8\underline{h}(t)} \frac{\underline{h}(t)^{3-\gamma}}{3-\gamma}$$
$$= \hat{C}_0 (K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} \ge \frac{K_1}{\gamma - 1}(K_1 t + \theta)^{(2-\gamma)/(\gamma-1)} = \underline{h}'(t)$$

provided that $0 < K_1 \le \hat{C}_0(\gamma-1)$ and $\theta \gg 1$. This finishes the proof of Step 1.

Step 2. We show that, by choosing K_1, K_2 and θ properly, for $t>0$ and $x \in (-\underline{h}(t), \underline{h}(t))$,

$$\underline{u}_t(t,x) \ge d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x)). \tag{5.9}$$

From the definition of $\underline{u}$, for $t>0$ and $x \in (-\underline{h}(t), \underline{h}(t))$,

$$\underline{u}_t(t,x) = K_2 \frac{|x|\underline{h}'(t)}{\underline{h}^2(t)} \le K_2 \frac{\underline{h}'(t)}{\underline{h}(t)} = \frac{K_1K_2}{\gamma-1}\underline{h}(t)^{1-\gamma}.$$

Claim 1. For $x \in [-\underline{h}(t), \underline{h}(t)]$, there exists a positive constant $\hat{C}_1$ depending only on γ such that

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y \ge \hat{C}_1 K_2 \underline{h}(t)^{1-\gamma}. \tag{5.10}$$

By (5.1), writing $\underline{h}(t) = \underline{h}$ for simplicity of notation, we have

$$\int_{-\underline{h}}^{\underline{h}} J(x-y)\underline{u}(t,y)\mathrm{d}y = \int_{-\underline{h}-x}^{\underline{h}-x} J(y)\underline{u}(t,y+x)\mathrm{d}y \ge K_2\int_{-\underline{h}-x}^{\underline{h}-x} \frac{C_1}{|y|^\gamma+1}\frac{\underline{h}-|y+x|}{\underline{h}}\mathrm{d}y.$$

Thus, for $x \in [\underline{h}/4, \underline{h}]$,

$$\begin{aligned}
&\int_{-\underline{h}}^{\underline{h}} J(x-y)\underline{u}(t,y)\mathrm{d}y \ge K_2\int_{-\underline{h}/4}^{0} \frac{C_1}{|y|^\gamma+1}\frac{\underline{h}-|y+x|}{\underline{h}}\mathrm{d}y\\
&= K_2\int_{-\underline{h}/4}^{0} \frac{C_1}{|y|^\gamma+1}\frac{\underline{h}-(y+x)}{\underline{h}}\mathrm{d}y \ge K_2\int_{-\underline{h}/4}^{0} \frac{C_1}{|y|^\gamma+1}\frac{-y}{\underline{h}}\mathrm{d}y\\
&= \frac{K_2}{\underline{h}}\int_0^{\underline{h}/4} \frac{C_1 y}{y^\gamma+1}\mathrm{d}y \ge \frac{C_1K_2}{2\underline{h}}\int_1^{\underline{h}/4} y^{1-\gamma}\mathrm{d}y\\
&\ge \frac{C_1K_2}{4(2-\gamma)\underline{h}}(\underline{h}/4)^{2-\gamma} =: \hat{C}_1K_2\underline{h}^{1-\gamma}.
\end{aligned}$$

And for $x \in [0, \underline{h}/4]$,

$$\begin{aligned}
&\int_{-\underline{h}}^{\underline{h}} J(x-y)\underline{u}(t,y)\mathrm{d}y \ge K_2\int_0^{\underline{h}/4} \frac{C_1}{|y|^\gamma+1}\frac{\underline{h}-|y+x|}{\underline{h}}\mathrm{d}y\\
&\ge K_2\int_0^{\underline{h}/4} \frac{C_1}{y^\gamma+1}\frac{y}{\underline{h}}\mathrm{d}y \ge \hat{C}_1K_2\underline{h}^{1-\gamma}
\end{aligned}$$

by repeating the last a few steps in the previous calculations.

This proves (5.10) for $x \in [0, \underline{h}]$. It also holds for $x \in [-\underline{h}, 0]$ since both $J(x)$ and $\underline{u}(t,x)$ are even in x.

Claim 2. We can choose small K_2 and large θ such that, for $x \in [-\underline{h}(t), \underline{h}(t)]$ and $t \ge 0$,

$$d\int_{-\underline{h}}^{\underline{h}} J(x-y)\underline{u}(t,y)\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x)) \ge F_*\int_{-\underline{h}}^{\underline{h}} J(x-y)\underline{u}(t,y)\mathrm{d}y$$

for some positive constant F_*.

It is clear that $0 \le \underline{u}(t,x) \le K_2$, and thus for small $K_2 > 0$,

$$f(\underline{u}(t,x)) = \Big[f'(0) + o(1)\Big]\underline{u}(t,x) \ge \frac{3}{4}f'(0)\underline{u}(t,x).$$

Moreover, by (5.6), there is $L_1 > 0$ such that for $\theta^{1/(\gamma-1)} \ge L_1$,

$$d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y + \frac{f'(0)}{4}\underline{u}(t,x) \ge d\underline{u}(t,x) \quad \text{for} \quad x \in [-\underline{h}(t), \underline{h}(t)].$$

Therefore Claim 2 is valid with $F_* = f'(0)/2$.

Combining Claim 1 and Claim 2, we obtain

$$\begin{aligned}
&d\int_{-\underline{h}}^{\underline{h}} J(x-y)\underline{u}(t,y)\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x))\\
&\ge F_*\hat{C}_1K_2\underline{h}(t)^{1-\gamma} \ge \frac{K_1K_2}{\gamma-1}\underline{h}(t)^{1-\gamma} \ge \underline{u}_t(t,x)
\end{aligned}$$

provided that

$$K_1 \le F_* \hat{C}_1(\gamma - 1).$$

This proves (5.9).

Step 3. We prove (5.7) by the comparison principle.

It is clear that

$$\underline{u}(t, \pm\underline{h}(t)) = 0 \quad \text{for} \quad t \ge 0.$$

Since spreading happens for (u, g, h), for fixed $\theta \gg 1$ and small K_1, K_2 as chosen above, there exists a large $t_0 > 0$ such that

$$\begin{aligned} &[-\underline{h}(0), \underline{h}(0)] \subset [g(t_0)/2, h(t_0)/2], \\ &u(t_0, x) \ge K_2 \ge \underline{u}(0, x) \quad \text{for} \quad x \in [-\underline{h}(0), \underline{h}(0)]. \end{aligned}$$

Moreover, since $J(x)$ and $\underline{u}(t, x)$ are both even in x, (5.8) implies

$$-\underline{h}'(t) \ge \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \quad \text{for} \quad t > 0.$$

These combined with the estimates in Step 1 and Step 2 allow us to apply the comparison principle to conclude that

$$\begin{aligned} &[-\underline{h}(t), \underline{h}(t)] \subset [g(t+t_0), h(t+t_0)], && t \ge 0, \\ &\underline{u}(t, x) \ge u(t+t_0, x), && t \ge 0,\ x \in [-\underline{h}(t), \underline{h}(t)]. \end{aligned}$$

Hence (5.7) holds. □

5.2.2. *The case $\gamma = 2$.* The following simple result will play an important role in our analysis later.

Lemma 5.4. *Let l_1 and l_2 with $0 < l_1 < l_2$ be two constants, and define*

$$\psi(x) = \psi(x; l_1, l_2) := \min\left\{1, \frac{l_2 - |x|}{l_1}\right\}, \qquad x \in \mathbb{R}.$$

If J satisfies **(J)***, then for any $\epsilon > 0$, there is $L_\epsilon > 0$ such that for all $l_1 > L_\epsilon$ and $l_2 - l_1 > L_\epsilon$,*

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y \ge (1-\epsilon)\psi(x) \ \text{in}\ [-l_2, l_2]. \tag{5.11}$$

Proof. Since $\int_{\mathbb{R}} J(x)\mathrm{d}x = 1$, there exits $B > 0$ such that

$$\int_{-B}^{B} J(x)\mathrm{d}x > 1 - \epsilon/2. \tag{5.12}$$

In the following discussion we always assume that $l_1 \gg B$ and $l_2 - l_1 \gg B$. Clearly, for $x \in [-(l_2 - l_1) + B, (l_2 - l_1) - B]$, due to

$$\psi(x) = 1 \ \text{in}\ [-(l_2 - l_1), l_2 - l_1],$$

we have

$$\begin{aligned} &\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y \ge \int_{-(l_2-l_1)}^{l_2-l_1} \tilde{J}(x-y)\psi(y)\mathrm{d}y = \int_{-(l_2-l_1)}^{l_2-l_1} \tilde{J}(x-y)\mathrm{d}y \\ &= \int_{-(l_2-l_1)-x}^{l_2-l_1-x} J(y)\mathrm{d}y \ge \int_{-B}^{B} \tilde{J}(y)\mathrm{d}y \ge 1 - \epsilon/2 > (1-\epsilon)\psi(x). \end{aligned}$$

It remain to prove (5.11) for $x \in [-l_2, -(l_2-l_1)+B] \cup [(l_2-l_1)-B, l_2]$. By the symmetric property of $\psi(x)$ and $J(x)$ with respect to x, we just need to verify (5.11) for $x \in [(l_2 - l_1) - B, l_2]$, which will be carried out according to the following three cases:

(i) $x \in [l_2 - l_1 - B, l_2 - l_1 + B]$, (ii) $x \in [l_2 - l_1 + B, l_2 - B]$, (iii) $x \in [l_2 - B, l_2]$.

(i) For $x \in [l_2 - l_1 - B, l_2 - l_1 + B]$, since $\psi(z)$ is nonincreasing for $z \ge 0$, we have

$$\begin{aligned} &\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y = \int_{-l_2-x}^{l_2-x} J(y)\psi(y+x)\mathrm{d}y \\ &\ge \int_{-2l_2+l_1+B}^{B} J(y)\psi(y+x)\mathrm{d}y \ge \int_{-B}^{B} J(y)\psi(y+x)\mathrm{d}y \end{aligned}$$

$$\geq \int_{-B}^{B} J(y)\psi(y+l_2-l_1+B)\mathrm{d}y.$$

By the definition of ψ, for $y \in [-B, B]$, we have

$$\psi(y+l_2-l_1+B) = \frac{l_2-(y+l_2-l_1+B)}{l_1} = 1 - \frac{y+B}{l_1}.$$

Hence,

$$\int_{-B}^{B} J(y)\psi(y+l_2-l_1+B)\mathrm{d}y = \int_{-B}^{B} J(y)\mathrm{d}y - \int_{-B}^{B} J(y)\frac{y+B}{l_1}\mathrm{d}y$$

$$\geq 1-\epsilon/2 - \|J\|_{L^\infty(\mathbb{R})}\frac{2B^2}{l_1} \geq 1-\epsilon \geq (1-\epsilon)\psi(x)$$

provided

$$l_1 \geq \frac{4\|J\|_{L^\infty(\mathbb{R})}B^2}{\epsilon},$$

which then gives

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y \geq (1-\epsilon)\psi(x) \quad \text{for} \quad x \in [l_2-l_1-B, l_2-l_1+B].$$

(ii) For $x \in [l_2-l_1+B, l_2-B]$,

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y = \int_{-l_2-x}^{l_2-x} J(y)\psi(y+x)\mathrm{d}y$$

$$\geq \int_{-2l_2-B+l_1}^{B} J(y)\psi(y+x)\mathrm{d}y \geq \int_{-B}^{B} J(y)\psi(y+x)\mathrm{d}y.$$

From the definition of ψ, for $x \in [l_2-l_1+B, l_2-B]$ and $y \in [-B, B]$,

$$\psi(y+x) = \frac{l_2-(y+x)}{l_1} = \frac{l_2-x}{l_1} - \frac{y}{l_1} = \psi(x) - \frac{y}{l_1}.$$

Thus, by (5.12),

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y \geq \int_{-B}^{B} J(y)\psi(y+x)\mathrm{d}y$$

$$=\psi(x)\int_{-B}^{B} J(y)\mathrm{d}y - \int_{-B}^{B} J(y)\frac{y}{l_1}\mathrm{d}y = \psi(x)\int_{-B}^{B} J(y)\mathrm{d}y \geq (1-\epsilon)\psi(x).$$

(iii) For $x \in [l_2-B, l_2]$,

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y = \int_{-l_2-x}^{l_2-x} J(y)\psi(y+x)\mathrm{d}y$$

$$\geq \int_{-2l_2-B+l_1}^{l_2-x} J(y)\psi(y+x)\mathrm{d}y \geq \int_{-B}^{l_2-x} J(y)\psi(y+x)\mathrm{d}y$$

$$= \int_{-B}^{B} J(y)\psi(y+x)\mathrm{d}y - \int_{l_2-x}^{B} J(y)\psi(y+x)\mathrm{d}y$$

As in (ii), we see that

$$\int_{-B}^{B} J(y)\psi(y+x)\mathrm{d}y = \psi(x)\int_{-B}^{B} J(y)\mathrm{d}y \geq (1-\epsilon)\psi(x).$$

By the definition of ψ,

$$\psi(y+x) \leq 0 \quad \text{for} \quad x \in [l_2-B, l_2],\ y \in [l_2-x, B],$$

which indicates

$$\int_{-l_2}^{l_2} J(x-y)\psi(y)\mathrm{d}y \geq \int_{-B}^{B} J(y)\psi(y+x)\mathrm{d}y \geq (1-\epsilon)\psi(x).$$

The proof is now complete. □

Lemma 5.5. *If the conditions in Theorem 1.2 are satisfied and $\gamma = 2$, then there exits $C > 0$ such that*

$$h(t) \geq Ct \ln t \text{ for } t \gg 1. \tag{5.13}$$

Proof. Fix $\beta \in (0,1)$ and define

$$\begin{cases} \underline{h}(t) := K_1(t+\theta)\ln(t+\theta), & t \geq 0, \\ \underline{u}(t,x) := K_2 \min\left\{1, \dfrac{\underline{h}(t)-|x|}{(t+\theta)^\beta}\right\}, & t \geq 0,\ x \in [-\underline{h}(t), \underline{h}(t)], \end{cases}$$

with constants $\theta \gg 1$ and $1 \gg K_1 > 0, 1 \gg K_2 > 0$ to be determined. Obviously, for any $t > 0$, the function $\partial_t \underline{u}(t,x)$ exists for $x \in [-\underline{h}(t), \underline{h}(t)]$ except when $|x| = \underline{h}(t) - (t+\theta)^\beta$. However, the one-sided partial derivates $\partial_t \underline{u}(t \pm 0, x)$ always exist.

Step 1. We show that by choosing θ and K_1, K_2 suitably,

$$\underline{h}'(t) \leq \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \qquad \text{for } t > 0, \tag{5.14}$$

$$-\underline{h}'(t) \geq -\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \qquad \text{for } t > 0. \tag{5.15}$$

Since $\underline{u}(t,x) = \underline{u}(t,-x)$ and $J(x) = J(-x)$, we see that (5.15) follows from (5.14).

By elementary calculations and (5.1), we have

$$\begin{aligned}
&\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \\
&\geq \mu \int_{0}^{\underline{h}(t)-(t+\theta)^\beta} \int_{\underline{h}(t)}^{+\infty} J(x-y)\underline{u}(t,x)\mathrm{d}y\mathrm{d}x \\
&= \mu K_2 \int_{-\underline{h}(t)}^{-(t+\theta)^\beta} \int_{0}^{+\infty} J(x-y)\mathrm{d}y\mathrm{d}x = \mu K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{x}^{+\infty} J(y)\mathrm{d}y\mathrm{d}x \\
&= \mu K_2 \left(\int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{(t+\theta)^\beta}^{y} + \int_{\underline{h}(t)}^{\infty} \int_{(t+\theta)^\beta}^{\underline{h}(t)} \right) J(y)\mathrm{d}x\mathrm{d}y \\
&\geq \mu K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \int_{(t+\theta)^\beta}^{y} J(y)\mathrm{d}x\mathrm{d}y \geq \mu C_1 K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \frac{y-(t+\theta)^\beta}{y^2+1}\mathrm{d}y \\
&\geq \mu C_1 K_2 \int_{(t+\theta)^\beta}^{\underline{h}(t)} \frac{y-(t+\theta)^\beta}{2y^2}\mathrm{d}y \\
&= \mu C_1 K_2 \frac{1}{2}\left(\ln \underline{h}(t) - \beta \ln(t+\theta) + \frac{(t+\theta)^\beta}{\underline{h}(t)} - 1\right) \\
&\geq \mu C_1 K_2 \frac{1}{2}\left(\ln \underline{h}(t) - \beta\ln(t+\theta) - 1\right) \\
&= \mu C_1 K_2 \frac{1}{2}\left(\ln K_1 + \ln(t+\theta) + \ln(\ln(t+\theta)) - \beta \ln(t+\theta) - 1\right) \\
&\geq \frac{\mu C_1 K_2 (1-\beta)}{2}[\ln(t+\theta)+1] \geq K_1 \ln(t+\theta) + K_1 = \underline{h}'(t)
\end{aligned}$$

provided

$$\ln(\ln\theta) \geq -\ln K_1 + 2 \text{ and } 0 < K_1 \leq \frac{\mu C_1 K_2(1-\beta)}{2}, \tag{5.16}$$

which then finishes the proof of Step 1.

Step 2. We show that by choosing K_1, K_2 and θ suitably, for $t > 0$ and $x \in [-\underline{h}(t), \underline{h}(t)]$ with $|x| \neq \underline{h}(t) - (t+\theta)^\beta$,

$$\underline{u}_t(t,x) \leq d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x)). \tag{5.17}$$

From the definition of $\underline{u}$, for $t > 0$,

$$\underline{u}_t(t,x) = \begin{cases} K_1K_2\frac{(1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta} + \frac{K_2\beta|x|}{(t+\theta)^{1+\beta}} & \text{if } \underline{h}(t)-(t+\theta)^\beta < |x| \le \underline{h}(t), \\ 0 & \text{if } |x| < \underline{h}(t)-(t+\theta)^\beta. \end{cases}$$

Claim 1. For $x \in [-\underline{h}(t), -\underline{h}(t)+(t+\theta)^\beta] \cup [\underline{h}(t)-(t+\theta)^\beta, \underline{h}(t)]$ and large θ,

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y \ge \frac{C_1K_2\beta\ln(t+\theta)}{4(t+\theta)^\beta}, \tag{5.18}$$

where $C_1 > 0$ is given by (5.1).

A simple calculation yields, for $x \in [\underline{h}(t)-(t+\theta)^\beta, \underline{h}(t)]$,

$$\begin{aligned} &\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y \ge K_2\int_{\underline{h}(t)-(t+\theta)^\beta}^{\underline{h}(t)} J(x-y)\frac{\underline{h}(t)-y}{(t+\theta)^\beta}\mathrm{d}y \\ &= \frac{K_2}{(t+\theta)^\beta}\int_{\underline{h}(t)-(t+\theta)^\beta-x}^{\underline{h}(t)-x} J(y)[\underline{h}(t)-(y+x)]\mathrm{d}y. \end{aligned}$$

Hence, for $x \in [\underline{h}(t)-\frac{3}{4}(t+\theta)^\beta, \underline{h}(t)]$, by simple calculations and (5.1),

$$\begin{aligned} &\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y \ge \frac{K_2}{(t+\theta)^\beta}\int_{-(t+\theta)^\beta/4}^{0} J(y)(-y)\mathrm{d}y \\ &= \frac{K_2}{(t+\theta)^\beta}\int_0^{(t+\theta)^\beta/4} J(y)y\mathrm{d}y \ge \frac{C_1K_2}{(t+\theta)^\beta}\int_0^{(t+\theta)^\beta/4}\frac{y}{y^2+1}\mathrm{d}y \\ &\ge \frac{C_1K_2}{2(t+\theta)^\beta}\int_1^{(t+\theta)^\beta/4} y^{-1}\mathrm{d}y = \frac{C_1K_2}{2(t+\theta)^\beta}[\beta\ln(t+\theta)-\ln 4] \\ &\ge \frac{C_1K_2\beta\ln(t+\theta)}{4(t+\theta)^\beta} \end{aligned}$$

provided that

$$\frac{\beta}{2}\ln\theta \ge \ln 4. \tag{5.19}$$

And for $x \in [\underline{h}(t)-(t+\theta)^\beta, \underline{h}(t)-\frac{3}{4}(t+\theta)^\beta]$,

$$\begin{aligned} &\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y \ge \frac{K_2}{(t+\theta)^\beta}\int_0^{3(t+\theta)^\beta/4} J(y)[\underline{h}(t)-(y+x)]\mathrm{d}y \\ &\ge \frac{K_2}{(t+\theta)^\beta}\int_0^{(t+\theta)^\beta/4} J(y)y\mathrm{d}y \ge \frac{C_1K_2\beta\ln(t+\theta)}{4(t+\theta)^\beta}. \end{aligned}$$

This proves (5.18) for $x \in [\underline{h}(t)-(t+\theta)^\beta, \underline{h}(t)]$.

For $x \in [-\underline{h}(t), -\underline{h}(t)+(t+\theta)^\beta]$, (5.10) also holds since both $J(x)$ and $\underline{u}(t,x)$ are even in x. Claim 1 is thus proved.

Claim 2. We can choose small K_2 and large θ such that, for $x \in [-\underline{h}(t), \underline{h}(t)]$,

$$d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x)) \ge F_*\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y \tag{5.20}$$

for some $F_* > 0$.

For small $K_2 > 0$, from $0 \le \underline{u}(t,x) \le K_2$ we obtain

$$f(\underline{u}(t,x)) \ge \frac{3}{4}f'(0)\underline{u}(t,x).$$

For large θ and $t \ge 0$, we have

$$\underline{h}(t)-(t+\theta)^\beta \ge \theta^\beta(K_1\theta^{1-\beta}\ln\theta - 1) \ge \theta^\beta. \tag{5.21}$$

Hence, by (5.11), there is large $L_1 > 0$ such that, for $\theta^\beta > L_1$ it holds

$$d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y + \frac{f'(0)}{4}\underline{u}(t,x) \ge d\underline{u}(t,x) \quad \text{for } x \in [-\underline{h}(t), \underline{h}(t)].$$

Therefore (5.20) holds with $F_* = f'(0)/2$.

Applying (5.18) and (5.20), we have, for $x \in [-\underline{h}(t), -\underline{h}(t) + (t+\theta)^\beta) \cup (\underline{h}(t) - (t+\theta)^\beta, \underline{h}(t)]$,

$$
\begin{aligned}
& d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - \underline{u}(t,x) + f(\underline{u}(t,x)) \\
\geq\ & \frac{F_* C_1 K_2 \beta \ln(t+\theta)}{4(t+\theta)^\beta} \geq K_1 K_2 \frac{\ln(t+\theta)+1}{(t+\theta)^\beta} \\
=\ & \left[K_1 K_2 \frac{(1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta} + \frac{K_2 \beta \underline{h}(t)}{(t+\theta)^{1+\beta}}\right] \\
\geq\ & \left[K_1 K_2 \frac{1-\beta)\ln(t+\theta)+1}{(t+\theta)^\beta} + \frac{K_2 \beta |x|}{(t+\theta)^{1+\beta}}\right] \\
=\ & \underline{u}_t(t,x)
\end{aligned}
$$

if apart from the earlier requirements, we further assume

$$
\ln\theta > 2 \text{ and } K_1 \leq \frac{F_* C_1 \beta}{8}. \tag{5.22}
$$

For $|x| < \underline{h}(t) - (t+\theta)^\beta$, $\underline{u}(t,y) = K_2$ and

$$
\begin{aligned}
& d\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y - d\underline{u}(t,x) + f(\underline{u}(t,x)) \\
\geq\ & F_* \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\underline{u}(t,y)\mathrm{d}y \geq 0 = \underline{u}_t(t,x).
\end{aligned}
$$

Thus (5.17) holds. (Let us stress that it is possible to find K_1, K_2 and large θ such that (5.16), (5.19), (5.21) and (5.22) hold simultaneously.)

Step 3. We finally prove (5.13).

Clearly, $\underline{u}(t, \pm\underline{h}(t)) = 0$ for $t \geq 0$. Since spreading happens for (u, g, h) and $K_2 > 0$ is small, there is a large constant $t_0 > 0$ such that

$$
\begin{aligned}
& [-\underline{h}(0), \underline{h}(0)] \subset [g(t_0)/2, h(t_0)/2], \\
& \underline{u}(0,x) \leq K_2 \leq u(t_0, x) \quad \text{for} \quad x \in [-\underline{h}(0), \underline{h}(0)].
\end{aligned}
$$

By Remark 2.4 in [21], we see that the comparison principle still applies to our situation here, even though $\partial\underline{u}_t(t,x)$ has a jumping discontinuity at $|x| = \underline{h}(t) - (t+\theta)^\beta$. Therefore we have

$$
\begin{aligned}
& [-\underline{h}(t), \underline{h}(t)] \subset [g(t+t_0), h(t+t_0)], && t \geq 0, \\
& \underline{u}(t,x) \leq u(t+t_0, x), && t \geq 0,\ x \in [-\underline{h}(t), \underline{h}(t)].
\end{aligned}
$$

So (5.13) holds. This completes the proof of the lemma. □

Acknowledgment: This research was partially supported by the Australian Research Council.

References

[1] M. Alfaro and J. Coville, Propagation phenomena in monostable integro-differential equations: Acceleration or not? *J. Differ. Equ.* 263 (2017), 5727-5758.

[2] D. Aronson, H. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.* 30 (1978), 33-76.

[3] P. Bates, P. Fife, X. Ren, X. Wang, Traveling waves in a convolution model for phase transitions, *Arch. Rational Mech. Anal.* 138 (1997), 105-136.

[4] P. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, *J. Math. Anal. Appl.* 332 (2007), 428-440.

[5] H. Berestycki, J. Coville, H. Vo, Persistence criteria for populations with non-local dispersion, *J. Math. Biol.* 72 (2016), 1693-1745.

[6] M. D. Bramson, Maximal displacement of branching Brownian motion, *Comm. Pure Appl. Math.* 31 (1978), 531-581.

[7] E. Bouin, J. Garnier, C. Henderson, F. Patout, Thin front limit of an integro-differential Fisher–KPP equation with fat-tailed kernels, *SIAM J. Math. Anal.* 50 (2018), 3365-3394.

[8] E. Bouin, C. Henderson, L. Ryzhik, Super-linear spreading in local and nonlocal cane toads equations, *J. Math. Pures Appl.* 108 (2017), 724-750.

[9] G. Bunting, Y. Du, K. Krakowski, Spreading speed revisited: analysis of a free boundary model, *Netw. Heterog. Media* 7 (2012), 583-603.

[10] X. Cabré and J-M. Roquejoffre, The influence of fractional diffusion in Fisher-KPP equations, *Comm. Math. Phys.* 320 (2013), 679-722.
[11] J.-F. Cao, Y. Du, F. Li, W.-T. Li, The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, *J. Funct. Anal.* 277 (2019), 2772-2814.
[12] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, *Adv. Differential Equations* 2 (1997), 125-160.
[13] C. Cortázar, F. Quirós, N. Wolanski, A nonlocal diffusion problem with a sharp free boundary, *Interfaces Free Bound.*, 21 (2019), 441-462.
[14] J. Coville, J. Dávila and S. Martínez, Pulsating fronts for nonlocal dispersion and KPP nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 30 (2013), no. 2, 179-223.
[15] J. Coville, L. Dupaigne, On a non-local equation arising in population dynamics, *Proc. Roy. Soc. Edinburgh Sect. A* 137 (2007), 727-755.
[16] Y. Du and Z.M. Guo, The Stefan problem for the Fisher-KPP equation, *J. Diff. Eqns.* 253 (2012), 996-1035.
[17] Y. Du, F. Li and M. Zhou, Semi-wave and spreading speed of the nonlocal Fisher-KPP equation with free boundaries, *J. Math. Pure Appl.* to appear. (arXiv: 1909.03711).
[18] Y. Du, Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.* 42 (2010), 377-405.
[19] Y. Du, H. Matsuzawa, M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, *SIAM J. Math. Anal.* 46 (2014), 375-396.
[20] Y. Du and W. Ni, Analysis of a West Nile virus model with nonlocal diffusion and free boundaries, *Nonlinearity* 33(2020), 4407-4448.
[21] Y. Du and W. Ni, Spreading speed for some cooperative systems with nonlocal diffusion and free boundaries, part 1: Semi-wave and a threshold condition, preprint, 2021. (https://turing.une.edu.au/∼ydu/papers/SpreadingSpeed-part1-2021July.pdf)
[22] J. Fang, G. Faye, Monotone traveling waves for delayed neural field equations, *Math. Models Methods Appl. Sci.* 26 (2016), 1919-1954.
[23] J. Fang, X.Q. Zhao, Traveling waves for monotone semiflows with weak compactness, *SIAM J. Math. Anal.* 46 (2014), 3678-3704.
[24] D. Finkelshtein, P. Tkachov, Accelerated nonlocal nonsymmetric dispersion for monostable equations on the real line, *Appl. Anal.* 98 (2019), 756-780.
[25] R. Fisher, The wave of advance of advantage genes, *Ann. Eugenics* 7 (1937), 335-369.
[26] J. Garnier, Accelerating solutions in integro-differential equations, *SIAM J. Math. Anal.* 43 (2011), 1955-1974.
[27] C. Graham, Precise asymptotics for Fisher-KPP fronts, *Nonlinearity* 32 (2019), 1967-1998.
[28] F. Hamel, J. Nolen, J.-M. Roquejoffre and L. Ryzhik, A short proof of the logarithmic Bramson correction in Fisher-KPP equations, *Netw. Heterog. Media* 8 (2013), 275-289.
[29] F. Hamel, J. Nolen, J.-M. Roquejoffre and L. Ryzhik, The logarithmic delay of KPP fronts in a periodic medium, *J. Eur. Math. Soc.* 18 (2016), 465-505.
[30] F. Hamel and L. Roques, Fast propagation for KPP equations with slowly decaying initial conditions, *J. Differential Equations* 249 (2010), 1726-1745.
[31] V. Hutson, S. Martinez, K.Mischaikow, G. Vickers, The evolution of dispersal, *J. Math. Biol.* 47 (2003), 483-517.
[32] C. Kao, Y. Lou, W. Shen, Random dispersal vs. non-local dispersal, *Discrete Contin. Dyn. Syst.* 26 (2010), 551-596.
[33] A.N. Kolmogorov, I.G. Petrovski, N.S. Piskunov, A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem, *Bull. Moscow Univ. Math. Mech.* 1 (1937), 1-25.
[34] K.-S. Lau, On the nonlinear diffusion equation of Kolmogorov, Petrovskii and Piskunov, *J. Diff. Equ.* 59 (1985), 44-70.
[35] X. Liang and T. Zhou, Spreading speeds of nonlocal KPP equations in almost periodic media, *J. Funct. Anal.* 279 (2020), no. 9, 108723, 58 pp.
[36] J. Nolen, J.-M. Roquejoffre and L. Ryzhik, Convergence to a single wave in the Fisher-KPP equation, *Chinese Ann. Math. Ser. B* 38 (2017), 629-646.
[37] J. Nolen, J.-M. Roquejoffre and L. Ryzhik, Refined long time asymptotics for the Fisher-KPP fronts, *Comm. Contemp. Math.* 21 (2019), no. 7, 1850072, 25 pp.
[38] N. Rawal, W. Shen, A. Zhang, Spreading speeds and traveling waves of nonlocal monostable equations in time and space periodic habitats, *Discrete Contin. Dyn. Syst.* 35 (2015), 1609-1640.
[39] P.E. Souganidis and A. Tarfulea, Front propagation for nonlocal KPP reaction-diffusion equations in periodic media, *NoDEA. Nonlinear Differ. Equ. Appl.* 26 (2019), no. 4, Paper No. 29, 41 pp.
[40] Y.-J. Sun, W.-T. Li, Z.-C. Wang, Traveling waves for a nonlocal anisotropic dispersal equation with monostable nonlinearity, *Nonlinear Anal.* 74 (2011), 814-826.
[41] H. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.* 13 (1982), 353-396.
[42] W.B. Xu, W.T. Li, S. Ruan, Fast propagation for reaction-diffusion cooperative systems, *J. Differ. Equ.* 265 (2018), 645-670.
[43] H. Yagisita, Existence and nonexistence of traveling waves for a nonlocal monostable equation, *Publ. Res. Inst. Math. Sci.* 45 (2009), no. 4, 925-953.