# THE PERIODIC LOGISTIC EQUATION WITH SPATIAL AND TEMPORAL DEGENERACIES

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Abstract. In this article, we study the degenerate periodic logistic equation with homogeneous Neumann boundary conditions:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq, \not\equiv 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$, $a$ and $p > 1$ are constants. The function $b \in C^{\theta,\theta/2}(\overline{\Omega} \times \mathbb{R})$ ($0 < \theta < 1$) is T-periodic in $t$, nonnegative, and vanishes (i.e., has a degeneracy) in some subdomain of $\Omega \times \mathbb{R}$. We examine the effects of various natural spatial and temporal degeneracies of $b(x,t)$ on the long-time dynamical behavior of the positive solutions. Our analysis leads to a new eigenvalue problem for periodic-parabolic operators over a varying cylinder and certain parabolic boundary blow-up problems not known before. The investigation in this paper shows that the temporal degeneracy causes a fundamental change of the dynamical behavior of the equation only when spatial degeneracy also exists; but in sharp contrast, whether or not temporal degeneracy appears in the equation, the spatial degeneracy always induces fundamental changes of the behavior of the equation, though such changes differ significantly according to whether or not there is temporal degeneracy.

## 1. Introduction

One of the fundamental reaction-diffusion equations is the diffusive logistic equation, which is a basic model in population biology. In its simplest form, it can be written as

$$\begin{cases} \partial_t u - d\Delta u = au - bu^2 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

This equation describes the population density $u(x,t)$ of a species with initial density $u_0(x)$ and intrinsic growth rate $a$ in a habitat $\Omega$ that has carrying capacity $1/b$. The Neumann boundary condition means that the species is enclosed in $\Omega$ with no population flux across its boundary

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$\partial\Omega$. The coefficient $d$ stands for the dispersal (or diffusion) rate of the species. If the spatial and temporal variation of the environment is taken into account, the above equation should take the form

$$\begin{cases} \partial_t u - \operatorname{div}(d(x,t)\nabla u) = a(x,t)u - b(x,t)u^2 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Since the natural environment is typically periodic in time (for example, daily, seasonal or yearly), it is reasonable to assume that the coefficient functions $a(x,t), b(x,t)$ and $d(x,t)$ are periodic in $t$ of some given period $T > 0$.

In this article, we shall be concerned with such a periodic logistic equation. However, to emphasize our main points, we will only consider a simplified version with $d(x,t) \equiv 1$ and $a(x,t) \equiv a$, a constant. We will also replace $u^2$ by $u^p$ for some $p > 1$, since the treatment is the same. Thus the logistic equation we will consider in detail is given by

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq, \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

We assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with $C^{2+\theta}$ boundary $\partial\Omega$, $\nu$ is the outward unit normal vector on $\partial\Omega$, and $b(x,t)$ is a function in $C^{\theta,\theta/2}(\overline{\Omega} \times \mathbb{R})$ $(0 < \theta < 1)$, which is $T$-periodic in $t$ and satisfies $b(x,t) \geq, \not\equiv 0$ in $\Omega \times \mathbb{R}$. We remark that the techniques developed in this paper work as well if $a$ and $d$ are smooth positive functions that are $T$-periodic in $t$, but we choose to sacrifice such generality in order to keep the notations and presentation concise and transparent.

If $b(x,t) > 0$ in $\overline{\Omega} \times \mathbb{R}$, then a well-known result of Hess [7] states that

$$\lim_{n\to\infty} u(x, t + nT) = \begin{cases} 0 & \text{uniformly for } x \in \Omega \text{ and } t \in [0,T] \text{ if } a \leq 0, \\ u_a(x,t) & \text{uniformly for } x \in \Omega \text{ and } t \in [0,T] \text{ if } a > 0, \end{cases} \tag{1.2}$$

where $u_a$ is the unique positive $T$-periodic solution of

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega \times \mathbb{R}, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases} \tag{1.3}$$

which exists if and only if $a > 0$.

Our main interest in this paper is to examine the case that $b(x,t)$ vanishes in a proper subset of $\Omega \times \mathbb{R}$. We will call such a case a degeneracy in the logistic equation. The region where $b$ vanishes represents the extreme environmental situation that the species experiences no self-limitation for its growth there. A good understanding of such an extreme case is important in order to understand the scope of the possible behavior of the model as the environment varies heterogeneously. Indeed, it reveals how the dynamical behavior of the model makes fundamental changes.

When $b(x,t) \equiv b(x)$ is independent of $t$, it is well-known that such a degeneracy causes fundamental changes in the behavior of the logistic equation. Indeed, if $\{b(x) = 0\} := \{x \in \overline{\Omega} : b(x) = 0\}$ is a closed connected set $\overline{\Omega}_0$ contained in $\Omega$ with smooth boundary $\partial\Omega_0$, then it was shown in Du-Huang [3] and Du-Yamada [4] that, instead of the corresponding version of (1.2) (with $u_a$ now independent of $t$) the unique solution of (1.1) satisfies

$$\lim_{t\to\infty} u(x,t) = \begin{cases} 0 \quad \text{uniformly in } \overline{\Omega}, & \text{if } a \le 0; \\ u_a(x) \quad \text{uniformly in } \overline{\Omega}, & \text{if } 0 < a < \lambda_1^D(\Omega_0); \\ \begin{cases} U_a(x) & \text{locally uniformly in } \overline{\Omega}\setminus\overline{\Omega}_0, \\ \infty & \text{uniformly in } \overline{\Omega}_0, \end{cases} & \text{if } a \ge \lambda_1^D(\Omega_0). \end{cases} \tag{1.4}$$

Here we use $\lambda_1^D(\Omega_0)$ to denote the first eigenvalue of $-\Delta$ over $\Omega_0$ under Dirichlet boundary conditions; $u_a$ denotes the unique positive solution of

$$-\Delta u = au - b(x)u^p \text{ in } \Omega,\ \partial_\nu u = 0 \text{ on } \partial\Omega,$$

which exists if and only if $0 < a < \lambda_1^D(\Omega_0)$; $U_a$ denotes the minimal positive solution of the following boundary blow-up problem

$$-\Delta u = au - b(x)u^p \text{ in } \Omega\setminus\overline{\Omega}_0,\ \partial_\nu u|_{\partial\Omega} = 0,\ u|_{\partial\Omega_0} = \infty,$$

which exists for all $a \in \mathbb{R}$.

In this paper, we will examine the effect of a combination of spatial and temporal degeneracies on the behavior of (1.1), and reveal some new phenomena caused by the inclusion of temporal degeneracy in the model. Our results are best described in the special case that

$$b(x,t) = p(x)q(t),$$

where $p(x)$ and $q(t)$ are Hölder continuous nonnegative functions, and $q$ is $T$-periodic. We distinguish three different cases:

$$\begin{array}{lll} & \text{(i) No spatial degeneracy :} & p(x) > 0 \text{ in } \overline{\Omega} \text{ and } q(t) \ge, \not\equiv 0; \\ (1.5) & \text{(ii) No temporal degenecary :} & q(t) > 0 \text{ in } \mathbb{R}, \{p(x)=0\} = \overline{\Omega}_0 \subset \Omega; \\ & \text{(iii) Full degeneracy :} & \{p(x)=0\} \text{ is as in (ii)}, \{q(t)=0\}\cap[0,T] = [0,T^*]. \end{array}$$

Here $\Omega_0$ is a connected open set with $C^{2+\theta}$ boundary and $T^* \in (0,T)$.

We will show that in case (i), the long-time behavior of (1.1) is similar to (1.2), in case (ii) it is analogous to (1.4), but in case (iii) new behavior arises.

We now briefly describe the new behavior in case (iii). Firstly we show that there exists $a_* \in (0, \lambda_1^D(\Omega_0))$ such that (1.3) with $b(x,t) = p(x)q(t)$ has a unique positive periodic solution $u_a$ if $a \in (0, a_*)$ and it has no positive periodic solution otherwise. Moreover we show that $a_*$ is the principal eigenvalue of the following eigenvalue problem over a varying cylinder:

$$\begin{cases} \partial_t\varphi - \Delta\varphi = \lambda\varphi & \text{in } (\Omega\times(0,T^*])\cup(\Omega_0\times(T^*,T]), \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega\times(0,T^*], \\ \varphi(x,t) = 0 & \text{on } (\partial\Omega_0\times(T^*,T])\cup((\Omega\backslash\Omega_0)\times\{0\}), \\ \varphi(x,0) = \varphi(x,T) & \text{in } \Omega_0. \end{cases} \tag{1.6}$$

Secondly we show that the unique solution $u(x,t)$ of (1.1) with $b(x,t) = p(x)q(t)$ satisfies

(a) $\lim_{t\to\infty} u(x,t) = 0$ when $a \le 0$,

(b) $\lim_{n\to\infty} u(x, t+nT) = u_a(x,t)$ when $a \in (0, a_*)$,

(c) when $a \geq a_*$, $\lim_{n\to\infty} u(x,t+nT) = \infty$ locally uniformly on $(\overline{\Omega}\times(0,T^*])\cup(\overline{\Omega}_0\times[T^*,T])$, $\lim_{n\to\infty} u(x,t+nT) = U_a(x,t)$ uniformly on any compact subset of $(\overline{\Omega}\backslash\overline{\Omega}_0)\times(T^*,T)$,

where $U_a$ is the minimal positive solution of the following parabolic boundary blow-up problem

$$
\begin{cases}
\partial_t u - \Delta u = au - p(x)q(t)u^p & \text{in } (\Omega\setminus\overline{\Omega}_0)\times(T^*,T),\\
\partial_\nu u = 0 & \text{on } \partial\Omega\times(T^*,T),\\
u = \infty & \text{on } \partial\Omega_0\times(T^*,T),\\
u = \infty & \text{on } (\overline{\Omega}\setminus\Omega_0)\times\{T^*\}.
\end{cases}
\tag{1.7}
$$

Here by $u = \infty$ on $\partial\Omega_0\times(T^*,T)$, we mean that

$$u(x,t)\to\infty \text{ as } d(x,\Omega_0)\to 0 \text{ for each } t\in(T^*,T).$$

By $u=\infty$ on $(\overline{\Omega}\setminus\Omega_0)\times\{T^*\}$, we mean

$$u(x,t)\to\infty \text{ as } t \text{ decreases to } T^* \text{ for each } x\in\overline{\Omega}\setminus\Omega_0.$$

This paper seems to be the first to introduce and investigate an eigenvalue problem over a varying cylinder like (1.6) and to study a parabolic boundary blow-up problem of the form (1.7). Comparing case (i) with case (ii) in (1.5), we notice that the temporal degeneracy causes a fundamental change of the dynamical behavior of the equation only when spatial degeneracy also exists. In sharp contrast, by comparing all three cases in (1.5), one finds that whether or not temporal degeneracy appears in the equation, the spatial degeneracy always induces fundamental changes of the behavior of the equation, though such changes differ significantly according to whether there is temporal degeneracy or not.

The rest of the paper is organized as follows. In section 2, we present some preliminary results for later use. In section 3, we show how the eigenvalue problem (1.6) arises from the existence problem of positive periodic solutions of (1.3). In section 4, we examine the long-time behavior of (1.1) by making use of some parabolic boundary blow-up problems such as (1.7).

The techniques and ideas developed in this paper can be modified to treat a much more general version of (1.1). For example, the differential operator $\partial_t - \Delta$ can be replaced by one of the form $\partial_t + A(x,t,D)$ as given in section 2 below but with $A$ in divergence form, the nonlinear function $au - b(x,t)u^p$ can be replaced by a general function of the form $f(x,t,u)$ with the same key features, and the Neumann boundary operator can be replaced by a general boundary operator of the form $Bu$ given in section 2.

## 2. Preliminary results and existence of positive periodic solutions

In this section, for convenience, we recall some basic results on the initial boundary value problem of linear and semilinear parabolic equations. We also prove the existence and uniqueness of positive periodic solutions to (1.3).

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with $C^{2+\theta}$ boundary $\partial\Omega$, and let $A = A(x,t,D)$ given by

$$A(x,t,D)v = -\Sigma_{j,k=1}^N a_{ik}(x,t)\partial_j\partial_k v + \Sigma_{j=1}^N a_j(x,t)\partial_j v + a_0(x,t)v$$

be uniformly elliptic for each $t \in [0,T]$, where $T > 0$ is a given positive number. We assume that

$$a_{jk}, a_j, a_0 \in C^{\theta,\theta/2}(\overline{Q}_T),\ Q_T = \Omega \times [0,T].$$

Let $B = B(x,D)$ be given by

$$B(x,D)v = v \text{ or } B(x,D)v = \partial_\nu v + b_0(x)v,$$

where $\nu : \partial\Omega \to \mathbb{R}^N$ is an outward pointing, nowhere tangential vectorfield of class $C^{1+\theta}$, and $b_0 : \partial\Omega \to \mathbb{R}$ is of class $C^{1+\theta}$. We notice that $B = B(x,D)$ is independent of $t$.

Consider the initial-boundary value problem

$$\begin{cases} \partial_t u + A(x,t,D)u = f(x,t) & \text{in } \Omega \times (0,T], \\ Bu = 0 & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $f \in C^{\theta,\theta/2}(\overline{Q}_T)$, and $u_0 \in X_0 := L^p(\Omega)$ for some $p > 1$. Let $X_1 := W^{2,p}_B(\Omega) := \{v \in W^{2,p}(\Omega) : Bv = 0\}$. Then there exist a family of Banach spaces $X_\alpha$, $0 \le \alpha \le 1$, defined by the fractional power $A^\alpha$ of the differential operator $A$, with the properties:

(i) $0 \le \alpha < \beta \le 1$ implies that $X_\beta$ embeds into $X_\alpha$ compactly;

(ii) if $0 < \alpha < 1$, then for any given $\epsilon > 0$, there exists $C = C(\epsilon) > 0$ such that $\|v\|_{X_\alpha} \le \epsilon\|v\|_{X_1} + C\|v\|_0\ \forall v \in X_1$.

(iii) $X_\alpha$ compactly embeds into $C^{1+\lambda}_B(\overline{\Omega}) := \{v \in C^{1+\lambda}(\overline{\Omega}) : Bv = 0\}$ if $\frac{1}{2} + \frac{N}{2p} < \alpha \le 1$ and $0 \le \lambda < 2\alpha - 1 - \frac{N}{p}$.

By Theorem 1.2.1 on page 43 of Amann [1], problem (2.1) has a unique solution $u \in C^\theta((0,T], X_1) \cap C^{1+\theta}((0,T], X_0)$. Moreover, if $u_0 \in X_1$, then $u \in C^1([0,T], X_0)$.

Therefore, for $t > 0$, $u(\cdot,t) \in X_1 = W^{2,p}_B(\Omega) \hookrightarrow C^{1+\lambda}(\overline{\Omega})$ if $p > N$. One can actually use the Hölder theory to see that $u \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times (0,T])$.

The unique solution of (2.1) can be expressed by a constant of variation formula:

$$u(\cdot,t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\cdot,\tau)d\tau\ (0 \le t \le T), \tag{2.2}$$

where $U(t,0)u_0$ is the unique solution to (2.1) with $f \equiv 0$, and $U(t,\tau)$ satisfies:

(i) for each $v \in X_0$

$$U(\cdot)v : \Delta := \{(t,\tau) : 0 \le \tau \le t \le T\} \to X_0 \text{ is continuous},$$

(ii) $U(t,t) = I,\ \ U(s,t)U(t,\tau) = U(s,\tau)\ (0 \le \tau \le t \le s \le T)$,

(iii) $U(t,\tau) \in L(X_0, X_1)$ for $0 \le \tau < t \le T$,

(iv) for $0 \le \tau < t \le T$,

$$\begin{aligned} &\|U(t,\tau)\|_{L(X_\alpha,X_\beta)} \le C(\alpha,\beta) && \text{for } 0 \le \beta < \alpha \le 1, \\ &\|U(t,\tau)\|_{L(X_\alpha,X_\beta)} \le C(\alpha,\beta,\gamma)(t-\tau)^{-\gamma} && \text{for } 0 \le \alpha \le \beta, \beta - \alpha < \gamma < 1, \end{aligned}$$

(v) for $0 \le \alpha < \beta \le 1, 0 \le \gamma < \beta - \alpha$ and $(t,\tau),(s,\tau) \in \Delta$,

$$\|U(t,\tau) - U(s,\tau)\|_{L(X_\beta,X_\alpha)} \le C(\alpha,\beta,\gamma)|t-s|^\gamma,$$

(vi) for $0 \le \alpha < 1$, $g \in C([0,T], X_0)$ and $0 \le \gamma < 1-\alpha$,

$$\|\int_0^t U(t,\tau)g(\tau)d\tau - \int_0^s U(s,\tau)g(\tau)d\tau\|_{X_\alpha} \le C(\alpha,\gamma)|t-s|^\gamma \max_{0\le\tau\le T} \|g(\tau)\|_{X_0}.$$

Next we recall two definitions which will be used frequently later. The first one concerns the super- and sub-solutions to

$$\begin{cases} \partial_t u + A(x,t,D)u = f(x,t,u) & \text{in } \Omega\times[0,T], \\ Bu = 0 & \text{on } \partial\Omega\times[0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases} \tag{2.3}$$

where $f$ is continuous and $f(\cdot,\cdot,u)$ is of class $C^{\theta,\theta/2}(\overline{\Omega}\times[0,T])$ uniformly for $u$ in bounded subsets of $\mathbb{R}$, $\partial_u f$ is continuous on $\overline{\Omega}\times[0,T]\times\mathbb{R}$, and there exists a continuous function $c:(0,\infty)\to(0,\infty)$ such that

$$|f(x,t,u)| \le c(\rho)\ \forall \rho>0,\ \forall (x,t,u)\in\overline{\Omega}\times[0,T]\times[-\rho,\rho].$$

Following Hess [7], a function $\underline{u} \in C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega}\times[0,T])\cap C^{2,1}(\overline{\Omega}\times(0,T])$ is called a subsolution for the $T$-periodic problem (2.3) if

$$\begin{cases} \partial_t \underline{u} - \Delta\underline{u} \le f(x,t,\underline{u}) & \text{in } \Omega\times(0,T], \\ B\underline{u} \le 0 & \text{on } \partial\Omega\times(0,T], \\ \underline{u}(x,0) \le \underline{u}(x,T) & \text{in } \Omega. \end{cases}$$

A supersolution $\overline{u}$ is defined by reversing the inequality signs.

By Theorem 22.3 of [7], we know that if $\underline{u}\le\overline{u}$ is a pair of sub- and super-solutions to (2.3), then (2.3) has a solution $u$ satisfying $\underline{u}\le u\le\overline{u}$.

The above definition and existence result can be easily extended to the case that the boundary condition $Bu=0$ is replaced by $Bu=B\psi$, where $\psi\in C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[0,T])$. In such a case we simply let $v=u-\psi$ and the problem reduces to the standard case. A situation that arises frequently later in the paper is that $\partial\Omega$ has two components $\Gamma_1$ and $\Gamma_2$, and the boundary condition is given by $u|_{\Gamma_1}=\xi$, $\partial_\nu u|_{\Gamma_2}=0$, where $\xi\in C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[0,T])$. In such a case, we may choose a smooth function $\sigma(x)$ such that $\sigma=1$ near $\Gamma_1$ and $\sigma=0$ near $\Gamma_2$, and let $\psi=\sigma\xi$. Then it is easily seen that the given boundary condition is equivalent to $B_0u=B_0\psi$ on $\partial\Omega$, where $B_0u=u$ on $\Gamma_1$ and $B_0u=\partial_\nu u$ on $\Gamma_2$.

Let us also recall the theory of the principal eigenvalue for a linear periodic-parabolic eigenvalue problem. For any given $T$-periodic function $g\in C^{\theta,\theta/2}(\overline{\Omega}\times\mathbb{R})$, we consider the eigenvalue problem:

$$\begin{cases} \partial_t\varphi - \Delta\varphi + g(x,t)\varphi = \lambda\varphi & \text{in } \Omega\times\mathbb{R}, \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ \varphi(x,t) = \varphi(x,t+T) & \text{in } \Omega\times\mathbb{R}. \end{cases} \tag{2.4}$$

By Proposition 14.4 of [7], we know that (2.4) has a principal eigenvalue $\lambda=\lambda_1(g)$, which corresponds to a positive eigenfunction $\varphi\in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times\mathbb{R})$. Such a function $\varphi$ is usually called a principal eigenfunction.

With $b(x,t)$ as before, namely it belongs to $C^{\theta,\theta/2}(\overline{\Omega}\times\mathbb{R})$, is $T$-periodic in $t$ and $b \geq, \not\equiv 0$, for each $\mu\in\mathbb{R}$, by Lemmas 15.5 and 15.7 of [7], $\mu\mapsto\lambda_1(\mu b)$ is a strictly increasing continuous function with $\lambda_1(\mu b)>\lambda_1(0)=0$ when $\mu>0$. Therefore, we can define

$$\lambda_1(\infty) := \lim_{\mu\to\infty}\lambda_1(\mu b)\in(0,\infty]. \tag{2.5}$$

We are now ready to prove the basic existence and uniqueness result for the positive periodic solution to (1.3) and its global stability property as an element of the omega limit set of (1.1).

**Theorem 2.1.** *Problem* (1.3) *admits a unique positive $T$-periodic solution $u_a(x,t)$ if*

$$0<a<\lambda_1(\infty). \tag{2.6}$$

*It has no positive periodic solution otherwise. Moreover, if* (2.6) *holds, then the unique solution of* (1.1) *satisfies*

$$\lim_{n\to\infty}u(x,t+nT)=u_a(x,t) \text{ uniformly in } x\in\overline{\Omega} \text{ and } t\in[0,T].$$

*If $a\leq 0$,*

$$\lim_{t\to\infty}u(x,t)=0 \text{ uniformly in } \overline{\Omega}.$$

**Proof.** Assume that (1.3) has a positive $T$-periodic solution $u^*(x,t)$. Set $m=\max_{\overline{\Omega}\times[0,T]}u^*(x,t)$. Clearly $m>0$ and by the uniqueness and monotonicity properties of the principle eigenvalues we see that $a=\lambda_1(bu^{p-1})$, and

$$0=\lambda_1(0)<\lambda_1(bu^{p-1})\leq\lambda_1(bm^{p-1})<\lambda_1(\infty).$$

Hence $0<a<\lambda_1(\infty)$.

On the other hand, if (2.6) holds, we set $\overline{u}=M\varphi_\mu$, where $\varphi_\mu(x,t)$ is a positive principal eigenfunction corresponding to $\lambda_1(\mu b)$. We may fix $\mu>0$ sufficiently large such that $a<\lambda_1(\mu b)$. We then take $M$ so large that $(M\varphi_\mu)^{p-1}\geq\mu$ on $\overline{\Omega}\times[0,T]$. With such $\mu$ and $M$, it is easy to check that $\overline{u}:=M\varphi_\mu$ is a positive supersolution to (1.3). One also easily checks that any small positive constant $\underline{u}$ is a subsolution of (1.3). Thus (1.3) has a positive $T$-periodic solution.

Using the concavity of the nonlinearity in (1.3), one can follow a standard argument (see Theorem 27.1 in [7]) to show that the positive periodic solution of (1.3) is unique and attracts all the positive solutions of (1.1).

Finally suppose that $a\leq 0$. Then one can follow the argument in the proof of Theorem 28.1 in [7] to conclude that $\lim_{n\to\infty}u(x,t+nT)=0$ uniformly in $x\in\overline{\Omega}$ and $t\in[0,T]$. It follows that $\lim_{t\to\infty}u(x,t)=0$ uniformly in $x$. The proof is complete. □

In view of the above theorem, to obtain a full understanding of the long-time dynamical behavior of (1.1), we need to find a better description of $\lambda_1(\infty)$, and more importantly, we need to know the long-time behavior of the solution of (1.1) when $a\geq\lambda_1(\infty)$. The rest of the paper is devoted to answering these questions under suitable further conditions on $b(x,t)$.

## 3. Characterization of $\lambda_1(\infty)$ and an unconventional eigenvalue problem

In this section we characterize $\lambda_1(\infty)$ under suitable assumptions on $b(x,t)$, and show how this leads to a periodic-parabolic eigenvalue problem over a varying cylinder. Recall that $b \in C^{\theta,\theta/2}(\overline{\Omega}\times\mathbb{R})$ is $T$-periodic in $t$ and $b \geq, \not\equiv 0$. The simplest case is when $b(x,t)$ has no spatial degeneracy at some point in time, that is,

$$b(x,t_0) > 0 \quad \text{for all } x \in \overline{\Omega} \text{ and some } t_0 \in \mathbb{R}. \tag{3.1}$$

Without loss of generality, we may assume that $t_0 \in (0,T]$. Clearly case (i) in (1.5) belongs to this situation. We show that in this case $\lambda_1(\infty) = \infty$, and hence Theorem 2.1 gives a full description of the long-time dynamical behavior of (1.1), which is the same as in the classical case (i.e., the case that no degeneracy occurs in the logistic equation).

Indeed, when (3.1) holds, $\min_{\overline{\Omega}} b(x,t_0) > 0$ and hence, $\int_0^T \min_{\overline{\Omega}} b(x,t)dt > 0$. It follows that, for any given $M > 0$, there exists a large $\mu_0$ such that

$$\int_0^T \max_{\overline{\Omega}}(M - \mu b(x,t))\mathrm{d}t = \int_0^T [M - \mu \min_{\overline{\Omega}} b(x,t)]\mathrm{d}t < 0,$$

for all $\mu \geq \mu_0$. Hence, by Lemma 15.6 in [7],

$$\lambda_1(\mu b) - M = \lambda_1(\mu b - M) > 0,$$

for $\mu \geq \mu_0$, which implies $\lambda_1(\infty) = \infty$.

We thus have

**Theorem 3.1.** *Assume that* (3.1) *holds; then* $\lambda_1(\infty) = \infty$.

Next we consider a case that includes but generalizes case (ii) in (1.5), namely

$$c_1 p(x) \leq b(x,t) \leq c_2 p(x), \tag{3.2}$$

where $c_1$, $c_2$ are positive constants and $p(x)$ is as in case (ii) of (1.5).

We will show that in this case $\lambda_1(\infty) = \lambda_1^D(\Omega_0)$. Our argument is based on the properties of the first eigenvalues for elliptic operators. Let $O$ be a bounded domain, and $f(x)$ be an $L^\infty(O)$ function. We denote by $\lambda_1^D(f,O)$ and $\lambda_1^N(f,O)$ the first eigenvalue of the operator $-\Delta + f$ over $O$, with Dirichlet and Neumann boundary conditions, respectively. We also use the convention that $\lambda_1^D(0,O) = \lambda_1^D(O)$, $\lambda_1^N(0,O) = \lambda_1^N(O)$. For convenience of later use, we list some well known properties:

(1) $\lambda_1^D(f,O) > \lambda_1^N(f,O)$;

(2) $\lambda_1^B(f_1,O) > \lambda_1^B(f_2,O)$ if $f_1 \geq f_2$ and $f_1 \not\equiv f_2$, for $B = D$ or $B = N$;

(3) $\lambda_1^D(f,O_1) \geq \lambda_1^D(f,O_2)$ if $O_1 \subset O_2$.

Let us also note that by the uniqueness property of the principal eigenvalue of the periodic-parabolic operator $Lu = \partial_t u - \Delta u - g(x,t)u$, when $g(x,t) = g(x)$ is a function independent of the time variable $t$, then $\lambda_1(g) = \lambda_1^N(g,\Omega)$.

Let

$$\overline{b}(x) = \max_{[0,T]} b(x,t) \quad \text{and} \quad \underline{b}(x) = \min_{[0,T]} b(x,t).$$

Then, $\{\bar{b}(x) = 0\} = \{\underline{b}(x) = 0\} = \overline{\Omega}_0$.

By the monotonicity of the principal eigenvalues, we have

$$\lambda_1^N(\mu\underline{b}, \Omega) = \lambda_1(\mu\underline{b}) \le \lambda_1(\mu b) \le \lambda_1(\mu\bar{b}) = \lambda_1^N(\mu\bar{b}, \Omega).$$

By Theorem 2.4 of [5],

$$\lim_{\mu\to\infty} \lambda_1^N(\mu\underline{b}, \Omega) = \lim_{\mu\to\infty} \lambda_1^N(\mu\bar{b}, \Omega) = \lambda_1^D(\Omega_0).$$

Thus we have proved the following result:

**Theorem 3.2.** *Under the assumption* (3.2)*, we have* $\lambda_1(\infty) = \lambda_1^D(\Omega_0)$.

We now consider the third case, which includes but generalizes case (iii) in (1.5), namely

$$c_1 p(x)q(t) \le b(x,t) \le c_2 p(x)q(t) \text{ on } \overline{\Omega} \times \mathbb{R}, \tag{3.3}$$

where $p$ and $q$ are as in case (iii) of (1.5), and $c_1, c_2$ are positive constants. It turns out that this case is much more difficult to handle.

Our first main result on $\lambda_1(\infty)$ is the following:

**Theorem 3.3.** *When* (3.3) *holds, we have* $\lambda_1(\infty) < \lambda_1^D(\Omega_0)$*. Moreover, there exists a function* $\varphi(x,t)$ *which is continuous in* $\big(\overline{\Omega} \times [0,T]\big) \setminus \big[(\overline{\Omega} \setminus \Omega_0) \times \{T^*\}\big]$*, and satisfies*

$$\varphi > 0 \text{ in } \ (\overline{\Omega} \times (0,T^*]) \cup (\Omega_0 \times (T^*,T]), \ \ \varphi = 0 \text{ in } \ \big(\overline{\Omega} \setminus \Omega_0\big) \times (T^*,T], \tag{3.4}$$

$$\varphi \in C^{2+\theta, 1+\frac{\theta}{2}}\Big(\big[(\overline{\Omega} \times (0,T^*]) \cup (\overline{\Omega}_0 \times [T^*,T])\big] \setminus \big[\partial\Omega_0 \times \{T^*\}\big]\Big), \tag{3.5}$$

*and*

$$\begin{cases} \partial_t\varphi - \Delta\varphi = \lambda_1(\infty)\varphi & \text{in } (\Omega \times (0,T^*]) \cup (\Omega_0 \times (T^*,T]), \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega \times (0,T^*], \\ \varphi(x,t) = 0 & \text{on } (\partial\Omega_0 \times (T^*,T]) \cup (\Omega\backslash\Omega_0 \times \{0\}), \\ \varphi(x,0) = \varphi(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.6}$$

**Proof.** Let $\varphi = \varphi_\mu$ be a positive principal eigenfunction corresponding to $\lambda_1(\mu b)$. Then

$$\begin{cases} \partial_t\varphi - \Delta\varphi + \mu b(x,t)\varphi = \lambda_1(\mu b)\varphi & \text{in } \Omega \times (0,T), \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(x,0) = \varphi(x,T) & \text{in } \Omega. \end{cases} \tag{3.7}$$

By (3.3) and Theorem 2.4 of [5], we have

$$\lambda_1(\mu b) \le \lambda_1(\mu c_2 Q p) \to \lambda_1^D(\Omega_0), \quad \text{as } \mu \to \infty, \tag{3.8}$$

where $Q = \max_{[0,T]} q(t) > 0$. It follows that $\lambda_1(\infty) \le \lambda_1^D(\Omega_0)$.

Without loss of generality, we may assume that $\max_{\overline{\Omega}\times[0,T]} \varphi_\mu = 1$. Since $0 \le \varphi_\mu(x,t) \le 1$ on $\overline{\Omega} \times [0,T]$, we can find a sequence $\mu_n \to \infty$ as $n \to \infty$ such that

$$\varphi_{\mu_n}(x,t) \to \varphi^*(x,t) \ \text{ weakly in } \ L^2(\Omega \times (0,T)) \ \text{ as } n \to \infty,$$

and $0 \le \varphi^*(x,t) \le 1$ a.e. in $\Omega \times (0,T)$. For the sake of convenience, we will write $\varphi_n(x,t)$ instead of $\varphi_{\mu_n}(x,t)$.

In the following, we will investigate the properties of $\varphi^*$ through improved understanding of the convergence of $\varphi_n$. For clarity, the long discussions below are divided into several steps.

**Step 1:** $\varphi^*(x,t) \not\equiv 0$ in $\Omega \times (0,T)$.

We proceed by a contradiction argument. Suppose that

$$\varphi_n(x,t) \to 0 \ \text{ weakly in } L^2(\Omega \times (0,T)). \tag{3.9}$$

For any fixed $n \geq 1$, we consider the auxiliary problem:

$$\begin{cases} \partial_t \psi - \Delta\psi + \psi = [\lambda_1(\infty)+1]\varphi_n(x,t) & \text{in } \Omega \times (0,\infty), \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \psi(x,0) = 1 & \text{in } \Omega. \end{cases} \tag{3.10}$$

For any fixed $n$, (3.10) admits a unique solution $\psi_n(x,t) \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times (0,\infty))$. Furthermore, in view of $0 \leq \varphi_n(x,t) \leq 1$, the comparison principle for parabolic equations immediately infers

$$\varphi_n(x,t) \leq \psi_n(x,t) \leq \lambda_1(\infty)+1 \ \text{ in } \Omega \times (0,\infty) \quad \text{for each } n \geq 1. \tag{3.11}$$

Since the right side of the first equation in (3.10) has a bound in $L^\infty(\Omega \times [0,\infty))$ that is independent of $n$, by standard global parabolic $L^p$ estimates, we have, for any $p > 1$ and $\hat{T} > 0$,

$$\|\psi_n\|_{W_p^{2,1}(\Omega\times[0,\hat{T}])} \leq C_0$$

for some constant $C_0$ independent of $n$. Taking $p$ large enough and applying the Sobolev embedding result (see [8] Lemma II3.3), we obtain

$$\|\psi_n\|_{C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega}\times[0,\hat{T}])} \leq C = C_{\hat{T}}.$$

Therefore by passing to a subsequence we can assume that $\psi_n \to \psi^*$ in $C^{1,\frac{1}{2}}(\overline{\Omega} \times [0,\hat{T}])$. By this conclusion and a standard diagonal argument, we can pass to a further subsequence so that $\psi_n \to \psi^*$ in $C^{1,\frac{1}{2}}(\overline{\Omega} \times [0,\hat{T}])$ for any $\hat{T} \in (0,\infty)$.

We now use the weak formulation of (3.10) to show that $\psi^*(x,t)$ satisfies weakly (and then classically)

$$\begin{cases} \partial_t \psi - \Delta\psi + \psi = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \psi(x,0) = 1 & \text{in } \Omega, \end{cases} \tag{3.12}$$

which implies $\psi^*(x,t) = e^{-t}$ on $\overline{\Omega} \times [0,\infty)$.

For any given $\hat{T} > 0$, let $V$ be the space of all functions $u(x,t)$ in $L^2(\Omega \times [0,\hat{T}])$ such that $|\nabla_x u| \in L^2(\Omega \times [0,\hat{T}])$, $u(\cdot,t) \in L^2(\Omega)$ for all $t \in [0,\hat{T}]$, and the norm defined by

$$\|u\|_V^2 = \int_{\Omega\times[0,\hat{T}]} |\nabla_x u|^2 dxdt + \sup_{t\in[0,\hat{T}]} \int_\Omega u^2 dx$$

is finite. Following Lieberman [9] (page 136), $u \in V$ is called a weak solution of the initial boundary value problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0, \hat{T}], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, \hat{T}], \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.13}$$

with $f \in L^2(\Omega \times [0, \hat{T}])$ and $u_0 \in L^2(\Omega)$, if for all $v \in C^1(\overline{\Omega} \times [0, \hat{T}])$ satisfying $v(x, \hat{T}) = 0$, we have

$$\int_{\Omega\times[0,\hat{T}]} [-u\partial_t v + \nabla_x u \cdot \nabla_x v - fv] dxdt = \int_\Omega u_0(x)v(x, 0)dx.$$

The smoothness of $\psi^*$ implies that $\psi^* \in V$. Since $\varphi_n$ is $T$-periodic in $t$, $\varphi_n \to 0$ weakly in $L^2(\Omega \times (0, T))$ implies that it converges to 0 weakly in $L^2(\Omega \times (0, \hat{T}))$ for any $\hat{T} > 0$. We may now make use of the weak formulation of (3.10) and let $n \to \infty$ to see that $\psi^*$ is a weak solution of (3.12) for $t \in [0, \hat{T}]$. Since $\hat{T} > 0$ is arbitrary, it is a weak solution of (3.12). Thus $\psi^* \equiv e^{-t}$. Due to (3.11) we have, for any integer $k \geq 1$,

$$\max_{\overline{\Omega}\times[0,T]} \varphi_n(x, t) = \max_{\overline{\Omega}\times[0,T]} \varphi_n(x, t + kT) \leq \max_{\overline{\Omega}\times[0,T]} \psi_n(x, t + kT) \to e^{-kT}$$

as $n \to \infty$. It follows that

$$\limsup_{n\to\infty} \max_{\overline{\Omega}\times[0,T]} \varphi_n(x, t) \leq e^{-kT}.$$

Letting $k \to \infty$ we deduce

$$\limsup_{n\to\infty} \max_{\overline{\Omega}\times[0,T]} \varphi_n(x, t) \leq 0.$$

But this contradicts our assumption that $\max_{\overline{\Omega}\times[0,T]} \varphi_n = 1$. This contradiction proves that $\varphi^* \not\equiv 0$, and the proof of Step 1 is complete.

Next, we determine the differential equation satisfied by $\varphi^*(x, t)$. To this end, it is convenient to consider $\varphi^*$ over the regions $\overline{\Omega} \times (0, T^*]$ and $\overline{\Omega} \times (T^*, T]$ separately.

**Step 2:** $\varphi^*$ in the range $(x, t) \in \overline{\Omega} \times (0, T^*]$.

In this range, $\psi = \varphi_n$ is the unique solution of

$$\begin{cases} \partial_t \psi - \Delta\psi = \lambda_1(\mu_n b)\varphi_n(x, t) & \text{in } \Omega \times (0, T^*], \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega \times (0, T^*], \\ \psi(x, 0) = \varphi_n(x, 0) & \text{in } \Omega. \end{cases} \tag{3.14}$$

By the parabolic $L^p$ estimates, for any $\tau \in (0, T^*)$, there exists $C = C_\tau$ such that

$$\|\varphi_n\|_{C^{1+\theta, \frac{1+\theta}{2}}(\overline{\Omega}\times[\tau,T^*])} \leq C.$$

Therefore by passing to a subsequence and also using a diagonal argument, we can assume that $\varphi_n \to \varphi_*$ in $C^{1,\frac{1}{2}}(\overline{\Omega} \times [\tau, T^*])$ for any $\tau \in (0, T^*)$. We necessarily have $\varphi_* = \varphi^*$. Hence $\varphi^*$ satisfies weakly

$$\begin{cases} \partial_t \varphi^* - \Delta\varphi^* = \lambda_1(\infty)\varphi^*(x, t) & \text{in } \Omega \times (0, T^*], \\ \partial_\nu \varphi^* = 0 & \text{on } \partial\Omega \times (0, T^*]. \end{cases} \tag{3.15}$$

By standard parabolic regularity we know that $\varphi^* \in C^{2+\theta,1+\theta}(\overline{\Omega}\times(0,T^*])$ and satisfies the above equation in the classical sense.

**Step 3:** $\varphi^*$ in the range $(x,t) \in \overline{\Omega} \times (T^*, T]$.

This case turns out to be difficult to handle. We first prove that $\varphi^* = 0$ a.e. in $(\overline{\Omega} \setminus \Omega_0) \times (T^*, T]$. Take $v(x,t)$ to be a smooth $T$-periodic function on $\overline{\Omega} \times \mathbb{R}$ with $v = 0$ near $\partial\Omega \times \mathbb{R}$. Multiplying (3.7) by $v$ and then integrating over $\Omega \times (0,T)$, we derive

$$\int_0^T \int_\Omega \{-\varphi_n v_t - \varphi_n \Delta v + \mu_n b(x,t)\varphi_n v\} = \lambda_1(\mu_n b) \int_0^T \int_\Omega \varphi_n v.$$

Dividing the above identity by $\mu_n$ and then letting $n \to \infty$, we obtain

$$\int_0^T \int_\Omega b(x,t)\varphi^*(x,t)v(x,t) = 0.$$

Due to the arbitrariness of $v$, we necessarily have

$$b(x,t)\varphi^*(x,t) = 0 \quad \text{a.e in } \Omega \times (0,T). \tag{3.16}$$

Since $b(x,t) > 0$ in $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T)$, it follows that

$$\varphi^*(x,t) = 0 \quad \text{a.e. in } \Omega\backslash\Omega_0 \times (T^*, T). \tag{3.17}$$

Secondly we prove that restricted to $\Omega_0 \times \mathbb{R}$, $\varphi_n \to \varphi^*$ in $C^{2,1}_{\text{loc}}(\Omega_0 \times \mathbb{R})$. Indeed, in this range, $\varphi_n(x,t)$ satisfies

$$\begin{cases} \partial_t \varphi_n - \Delta\varphi_n = \lambda_1(\mu_n b)\varphi_n & \text{in } \Omega_0 \times \mathbb{R}, \\ \varphi_n(x,0) = \varphi_n(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.18}$$

Since $0 < \lambda_1(\mu_n b) \le \lambda_1(\infty)$ and $0 \le \varphi_n \le 1$, by standard interior estimates (see, e.g., [8] or [9]), for any compact subset $K \subset \Omega_0 \times \mathbb{R}$, there exists a positive constant $C = C_K$ independent of $n$ such that

$$\|\varphi_n(x,t)\|_{C^{2+\theta,1+\theta/2}(K)} \le C.$$

Therefore, by passing to a subsequence of $\{\varphi_n(x,t)\}$ and a diagonal argument, we may assume that

$$\varphi_n \to \varphi_* \quad \text{in } C^{2,1}_{\text{loc}}(\Omega_0 \times \mathbb{R}).$$

As before we necessarily have $\varphi_* = \varphi^*$. Clearly $\varphi^*$ satisfies

$$\begin{cases} \partial_t \varphi^* - \Delta\varphi^* = \lambda_1(\infty)\varphi^* & \text{in } \Omega_0 \times \mathbb{R}, \\ \varphi^*(x,0) = \varphi^*(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.19}$$

Thirdly we determine the boundary condition satisfied by $\varphi^*|_{\Omega_0 \times (T^*,T]}$ over $\partial\Omega_0 \times (T^*, T]$. Multiplying $\varphi_n$ to the equation satisfied by $\varphi_n$ and then integrating over $\Omega \times [0,T]$, we easily obtain

$$\int_0^T \int_\Omega |\nabla\varphi_n|^2 dxdt \le \lambda_1(\mu_n b) \int_0^T \int_\Omega \varphi_n^2 dxdt \le \lambda_1(\infty) T|\Omega|.$$

It follows that

$$\int_0^T \int_\Omega |\nabla \varphi_n|^2 dxdt + \int_0^T \int_\Omega \varphi_n^2 dxdt \le M_0 := [\lambda_1(\infty)+1]T|\Omega|. \tag{3.20}$$

That is, $\{\varphi_n\}$ is a bounded set in the Hilbert space $W_2^{1,0}(\Omega \times [0,T])$ with inner product

$$(u,v) = \int_0^T \int_\Omega \nabla u \cdot \nabla v dxdt + \int_0^T \int_\Omega uvdxdt.$$

Hence by passing to a subsequence $\varphi_n \to \varphi_*$ weakly in $W_2^{1,0}(\Omega \times [0,T])$. Necessarily $\varphi_* = \varphi^*$. Thus $\varphi^* \in W_2^{1,0}(\Omega \times [0,T])$, and hence for a.e. $t \in [0,T]$, $\varphi^*(\cdot,t) \in H^1(\Omega)$. By (3.17), for a.e. $t \in (T^*,T]$, $\varphi^*(\cdot,t) = 0$ over $\Omega \setminus \Omega_0$. Since $\partial\Omega_0$ is smooth (actually Lipschitz is enough here), it follows that $\varphi^*(\cdot,t)|_{\Omega_0} \in H_0^1(\Omega_0)$ for a.e. $t \in (T^*,T]$.

From (3.20) we deduce

$$\int_0^T \int_\Omega |\nabla \varphi^*|^2 dxdt + \int_0^T \int_\Omega (\varphi^*)^2 dxdt \le M_0.$$

As a consequence,

$$\int_{T^*}^T \int_{\Omega_0} |\nabla \varphi^*|^2 dxdt \le M_0.$$

Using this and $0 \le \varphi^* \le 1$, we obtain

$$\int_{T^*}^T \int_{\Omega_0} |\nabla \varphi^*|^2 dxdt + \sup_{t\in[T^*,T]} \int_{\Omega_0} (\varphi^*)^2 dxdt \le M_0 + |\Omega_0|.$$

By the above facts for $\varphi^*$ and the fact that $\varphi_n \to \varphi^*$ in $C^{2,1}_{\rm loc}(\Omega_0 \times \mathbb{R})$, we easily see that $\psi = \varphi^*$ is the unique weak solution of

$$\begin{cases} \partial_t \psi - \Delta\psi = \lambda_1(\infty)\varphi^* & \text{in } \Omega_0 \times (T^*,T], \\ \psi = 0 & \text{on } \partial\Omega_0 \times (T^*,T], \\ \psi(x,T^*) = \varphi^*(x,T^*) & \text{in } \Omega_0. \end{cases} \tag{3.21}$$

By standard regularity theory for weak solutions (see [9]) the weak solution of (3.21) belongs to $C^{\theta,\theta/2}(\overline{\Omega}_0 \times [\tau,T])$ for any $\tau \in (T^*,T)$. Hence $\varphi^* \in C^{\theta,\theta/2}(\overline{\Omega}_0 \times [\tau,T])$ and we can use the Hölder estimate to conclude that $\varphi^* \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}_0 \times (T^*,T])$.

To better understand the behavior of $\varphi^*$ near $\overline{\Omega} \setminus \overline{\Omega}_0 \times \{T\}$, we need

**Step 4:** $\varphi_n$ converges to 0 uniformly on any compact subset of $\overline{\Omega} \setminus \Omega_0 \times (T^*,T]$.

Since $\varphi_n \to \varphi^*$ weakly in $L^2(\Omega \times [0,T])$ and $\varphi^* = 0$ over $\Omega \setminus \Omega_0 \times (T^*,T]$, if we define

$$\xi_n(t) = \int_{\Omega\setminus\Omega_0} \varphi_n(x,t)dx,$$

then $\xi_n \to 0$ in $L^1([T^*,T])$. Hence $\xi_n \to 0$ a.e. in $[T^*,T]$. Thus we can find a sequence $t_k$ decreasing to $T^*$ such that $\xi_n(t_k) \to 0$ as $n \to \infty$ for each $k \ge 1$. It follows that

$$0 \le \int_{\Omega\setminus\Omega_0} \varphi_n(x,t_k)^2 dx \le \int_{\Omega\setminus\Omega_0} \varphi_n(x,t_k)dx \to 0$$

as $n \to \infty$ for each $k \ge 1$.

Due to the conclusions proved in the last part of Step 3, for any given small $\delta > 0$ and $k \ge 1$, we can find $\sigma > 0$ small such that $0 < \varphi^*(x,t) < \delta/2$ for $(x,t)$ satisfying $x \in \Omega_0 \setminus \Omega^\sigma$, $t \in [t_k,T]$, where $\Omega^\sigma = \{x \in \Omega_0 : d(x, \partial\Omega_0) > \sigma\}$. Since $\varphi_n \to \varphi^*$ in $C^{2,1}_{\rm loc}(\Omega_0 \times \mathbb{R})$, for all large $n$, $\varphi_n < \delta$ on $\partial\Omega^\sigma \times [t_k, T]$. We now consider the auxiliary problem

$$\begin{cases} \partial_t v - \Delta v = \lambda_1(\infty)\varphi_n & \text{in } \Omega \setminus \overline{\Omega^\sigma} \times (t_k, T], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (t_k, T], \\ v = \delta & \text{on } \partial\Omega^\sigma \times (t_k, T], \\ v(x, t_k) = \varphi_n(x, t_k) & \text{in } \Omega \setminus \overline{\Omega^\sigma}. \end{cases} \tag{3.22}$$

Let $v_n$ denote the unique solution of (3.22); a simple comparison consideration shows that for all large $n$, $\varphi_n \le v_n$ in $\Omega \setminus \overline{\Omega^\sigma} \times (t_k, T]$. Much as before, we can show that, by passing to a subsequence, $v_n \to v^*$ in $C^{1+\theta, \frac{1+\theta}{2}}(\overline{\Omega} \setminus \Omega^\sigma \times (\tau, T])$ ($\forall \tau \in (t_k, T)$) and $v = v^*$ is a weak solution of

$$\begin{cases} \partial_t v - \Delta v = \lambda_1(\infty)\varphi^* & \text{in } \Omega \setminus \overline{\Omega^\sigma} \times (t_k, T], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (t_k, T], \\ v = \delta & \text{on } \partial\Omega^\sigma \times (t_k, T], \\ v(x, t_k) = v_0(x) & \text{in } \Omega \setminus \overline{\Omega^\sigma}, \end{cases} \tag{3.23}$$

where $v_0(x) = 0$ in $\Omega \setminus \overline{\Omega}_0$, and $v_0(x) = \varphi^*(x, t_k)$ for $x \in \Omega_0 \setminus \Omega^\sigma$. Since $0 \le \varphi^* \le \delta$ in $\Omega \setminus \overline{\Omega^\sigma} \times (t_k, T]$, and $v_0 \le \delta$ in $\Omega \setminus \Omega^\sigma$, a direct calculation shows that $\tilde{v}(x,t) := [\lambda_1(\infty)t + 1]\delta$ is a supersolution of (3.23). Hence

$$v^* \le \tilde{v} \le [\lambda_1(\infty)T + 1]\delta \text{ in } \overline{\Omega} \setminus \Omega^\sigma \times (t_k, T].$$

It follows that, for all large $n$,

$$\varphi_n \le v_n \le v^* + \delta \le \tilde{v} + \delta \le [\lambda_1(\infty)T + 2]\delta$$

in $\overline{\Omega} \setminus \Omega^\sigma \times [t_{k-1}, T]$. This implies that $\varphi_n \to 0$ uniformly in $\overline{\Omega} \setminus \Omega_0 \times [t_{k-1}, T]$ as $n \to \infty$, for each $k \ge 2$. Since $t_k \to T^*$, this proves Step 4.

**Step 5:** Summary and positivity of $\varphi^*$.

To summarize, we have shown that, by passing to a subsequence,

- over $\overline{\Omega} \times (0, T^*]$, $\varphi_n \to \varphi^*$ locally in the $C^{2,1}$ norm,
- over $\Omega_0 \times \mathbb{R}$, $\varphi_n \to \varphi^*$ locally in the $C^{2,1}$ norm,
- over $\overline{\Omega} \setminus \Omega_0 \times (T^*, T]$, $\varphi_n \to 0 = \varphi^*$ locally uniformly,
- $\varphi^* \in C^{2,1}(\overline{\Omega}_0 \times (T^*, T])$ and $\varphi^* = 0$ on $\partial\Omega_0 \times (T^*, T]$.

These properties imply in particular that $\varphi_n(x, 0) = \varphi_n(x, T) \to \varphi^*(x, 0)$ in the $L^2(\Omega)$ norm (actually the convergence is in $C(\overline{\Omega})$), and we see from (3.14) that $v = \varphi^*$ is the unique weak solution of the problem

$$\begin{cases} \partial_t v - \Delta v = \lambda_1(\infty) v & \text{in } \Omega \times (0, T^*], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T^*], \\ v(x, 0) = \varphi^*(x, 0) & \text{in } \Omega. \end{cases} \tag{3.24}$$

As $\varphi^*(x, 0) = \varphi^*(x, T)$ is continuous over $\overline{\Omega}$ and equals 0 near $\partial\Omega$, and $\partial\Omega$ is smooth, by standard theory for parabolic equations (see Theorem 9 on page 69 of [6]) we know that $\varphi^* \in$

$C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times(0,T^*])\cap C(\overline{\Omega}\times[0,T^*])$. Hence $\varphi^*$ is a continuous function over $\overline{\Omega}\times[0,T]$ except a possible discontinuity along $\overline{\Omega}\setminus\Omega_0\times\{T^*\}$.

We now use the strong maximum principle to show that $\varphi^*>0$ in $\{\overline{\Omega}\times(0,T^*]\}\cup\{\Omega_0\times(T^*,T]\}$. Indeed we must have $\varphi^*(\cdot,0)\not\equiv 0$ in $\Omega_0$. Otherwise $\varphi^*(\cdot,0)\equiv 0$ and hence $v=0$ is the unique solution of (3.24). It follows that $\varphi^*=0$ over $\overline{\Omega}\times(0,T^*]$. Due to (3.21), $v=\varphi^*$ is the unique solution of

$$\begin{cases}\partial_t v-\Delta v=\lambda_1(\infty)v & \text{in } \Omega_0\times(T^*,T],\\ v=0 & \text{on } \partial\Omega_0\times(T^*,T],\\ v(x,T^*)=\varphi^*(x,T^*) & \text{in } \Omega_0.\end{cases}\tag{3.25}$$

Since now $\varphi^*(\cdot,T^*)\equiv 0$, clearly $v=0$ solves (3.25), and we deduce $\varphi^*=0$ over $\Omega_0\times(T^*,T]$. As we already know that $\varphi^*=0$ over $\Omega\setminus\Omega_0\times(T^*,T]$, we see that $\varphi^*\equiv 0$ over $\Omega\times(T^*,T]$. Hence $\varphi^*\equiv 0$ over $\Omega\times[0,T]$, contradicting our earlier conclusion that $\varphi^*\not\equiv 0$. This proves that $\varphi^*(\cdot,0)\geq,\not\equiv 0$ in $\Omega_0$. Thus we can apply the strong maximum principle to (3.24) to conclude that $\varphi^*(x,t)>0$ for $(x,t)\in\overline{\Omega}\times(0,T^*]$. We may then apply the strong maximum principle to (3.25) to see that $\varphi^*>0$ in $\Omega_0\times(T^*,T]$. Hence $\varphi^*(x,0)=\varphi^*(x,T)>0$ in $\Omega_0$. Let us note that the above conclusions show that $\varphi^*(x,t)$ does have a jumping discontinuity across $\overline{\Omega}\setminus\Omega_0\times\{T^*\}$.

Thus we find that

$$\begin{aligned}\varphi^*\in\ & C^{2+\theta,1+\frac{\theta}{2}}\Big(\big(\overline{\Omega}\times(0,T^*]\big)\cup\big(\overline{\Omega}_0\times[T^*,T]\big)\setminus\partial\Omega_0\times\{T^*\}\Big)\\ &\cap C^0\Big(\big(\overline{\Omega}\times[0,T]\big)\setminus\big[(\overline{\Omega}\setminus\Omega_0)\times\{T^*\}\big]\Big),\end{aligned}$$

$$\varphi^*>0 \text{ in } (\overline{\Omega}\times(0,T^*])\cup(\Omega_0\times(T^*,T]),\ \ \varphi=0 \text{ in } (\overline{\Omega}\setminus\Omega_0)\times(T^*,T],$$

and

$$\begin{cases}\partial_t\varphi^*-\Delta\varphi^*=\lambda_1(\infty)\varphi^* & \text{in } (\Omega\times(0,T^*])\cup(\Omega_0\times(T^*,T]),\\ \partial_\nu\varphi^*=0 & \text{on } \partial\Omega\times(0,T^*],\\ \varphi^*(x,t)=0 & \text{on } (\overline{\Omega}\backslash\Omega_0\times\{0\})\cup(\partial\Omega_0\times(T^*,T]),\\ \varphi^*(x,0)=\varphi^*(x,T) & \text{in } \Omega_0.\end{cases}\tag{3.26}$$

**Step 6:** $\lambda_1(\infty)<\lambda_1^D(\Omega_0)$.

Let $\varphi_*(x)$ be the corresponding eigenfunction of $\lambda_1^D(\Omega_0)$ with $\varphi_*(x)>0$, that is, $\varphi_*(x)$ satisfies:

$$-\Delta\varphi_*=\lambda_1^D(\Omega_0)\varphi_*,\ \varphi_*>0\ \text{ in } \Omega,\quad \varphi_*=0\ \text{ on } \partial\Omega_0.$$

Then, we multiply the equation in (3.26) by $\varphi_*(x)$ and integrate the resulting identity over $\Omega_0\times(0,T)$ to derive

$$\int_0^T\int_{\Omega_0}\partial_t\varphi^*\varphi_*-\int_0^T\int_{\Omega_0}\Delta\varphi^*\varphi_*=\lambda_1(\infty)\int_0^T\int_{\Omega_0}\varphi^*\varphi_*.\tag{3.27}$$

By the $T$-periodic property of $\varphi^*(x,t)$, it is clear that the first term in the left-hand side is zero. For the second term in the left-hand side, integrating by parts we have

$$\begin{aligned}
-\int_0^T\int_{\Omega_0}\Delta\varphi^*\varphi_* &= -\int_0^{T^*}\int_{\Omega_0}\Delta\varphi^*\varphi_* - \int_{T^*}^T\int_{\Omega_0}\Delta\varphi^*\varphi_* \\
&= -\int_0^{T^*}\int_{\Omega_0}\varphi^*\Delta\varphi_* + \int_0^{T^*}\int_{\partial\Omega_0}\varphi^*\partial_{\nu_0}\varphi_* - \int_{T^*}^T\int_{\Omega_0}\varphi^*\Delta\varphi_* \\
&= \lambda_1^D(\Omega_0)\int_0^T\int_{\Omega_0}\varphi^*\varphi_* + \int_0^{T^*}\int_{\partial\Omega_0}\varphi^*\partial_{\nu_0}\varphi_* \\
&< \lambda_1^D(\Omega_0)\int_0^T\int_{\Omega_0}\varphi^*\varphi_*,
\end{aligned}$$

where $\nu_0$ denotes the unit normal of $\partial\Omega_0$ pointing inward of $\Omega_0$. Hence, it follows from (3.27) that $\lambda_1(\infty) < \lambda_1^D(\Omega_0)$, which completes the proof of Step 6.

The theorem is now completely proved. □

Consider the eigenvalue problem

$$\begin{cases} \partial_t\varphi - \Delta\varphi = \lambda\varphi & \text{in } (\Omega\times(0,T^*])\cup(\Omega_0\times(T^*,T]), \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega\times(0,T^*], \\ \varphi(x,t) = 0 & \text{on } (\overline{\Omega}\backslash\Omega_0\times\{0,T\})\cup(\partial\Omega_0\times(T^*,T]), \\ \varphi(x,0) = \varphi(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.28}$$

**Theorem 3.4.** *The eigenvalue problem* (3.28) *admits a principal eigenvalue* $\lambda = \lambda_1 > 0$ *which corresponds to a positive eigenfunction* $\varphi_1(x,t)$ *satisfying* (3.5) *and* (3.4). *Conversely, if* (3.28) *has a solution* $\varphi$ *satisfying* (3.5) *and* (3.4), *then necessarily* $\lambda = \lambda_1$, *the principal eigenvalue of* (3.28), *and* $\varphi = c\varphi_1$ *for some constant* $c$.

**Proof.** For any given $u \in C_0^1(\overline{\Omega}_0)$, we extend it by 0 to $\overline{\Omega}$, and denote the resulting function by $\tilde{u}$. Clearly $\tilde{u} \in C(\overline{\Omega})$. Let $v(x,t)$ be the unique solution of the problem

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega\times(0,T^*], \\ \partial_\nu v = 0 & \text{on } \partial\Omega\times(0,T^*), \\ v(x,0) = \tilde{u}(x) & \text{in } \Omega. \end{cases} \tag{3.29}$$

By [6] we know that $v \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times(0,T^*])\cap C(\overline{\Omega}\times[0,T^*])$.

We then consider the problem

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega_0\times(T^*,T], \\ w = 0 & \text{on } \partial\Omega_0\times(T^*,T], \\ w(x,T^*) = v(x,T^*) & \text{in } \Omega_0. \end{cases} \tag{3.30}$$

By the existence result recalled in section 2 we know that this problem has a unique solution $w \in C^\theta((T^*,T],X_1)\cap C^{1+\theta}((T^*,T],X_0)$, where $X_0 = L^p(\Omega_0)$ and $X_1 = W_0^{2,p}(\Omega_0)$, $p > 1$. We may choose $p$ large enough such that $W_0^{2,p}(\Omega_0)$ embeds compactly into $E := C_0^1(\overline{\Omega}_0)$.

With $u$ and $w$ as above, we define the operator $K_0 : E \to E$ by

$$K_0 u = w(\cdot,T).$$

It is easily seen that $K_0$ is a linear operator. We show next that $K_0$ is compact. Suppose that $\{u_n\}$ is a bounded sequence in $E$. Then there exists $C > 0$ such that $-C \le \tilde{u}_n \le C$ in $\Omega$. If we denote by $v_n$ the unique solution of (3.29) with $\tilde{u}$ replaced by $\tilde{u}_n$, then a simple comparison consideration gives $-C \le v_n \le C$ in $\Omega \times (0, T^*]$. In particular, $-C \le v_n(x, T^*) \le C$ in $\Omega_0$. We may then apply the comparison principle to deduce that $-C \le w_n \le C$ in $\Omega_0 \times (T^*, T]$, where $w_n$ is the unique solution of (3.30) with $v(x, T^*)$ replaced by $v_n(x, T^*)$. We may now apply the standard $L^p$ estimates to the equation satisfied by $w_n$ to conclude that, for any $p > 1$ and $\tau \in (T^*, T)$, there exists $C_0 > 0$ such that

$$\|w_n\|_{W_p^{2,1}(\Omega_0 \times [\tau, T])} \le C_0 \text{ for all } n \ge 1.$$

By the Sobolev embedding result in [8] (Lemma II 3.3) we deduce

$$\|w_n\|_{C^{1+\theta, \frac{1+\theta}{2}}(\overline{\Omega}_0 \times [\tau, T])} \le C$$

for some constant $C$ and all $n \ge 1$. In particular, $\{w_n(\cdot, T)\}$ is bounded in $C^{1+\theta}(\overline{\Omega}_0)$. Hence it has a convergent subsequence in $E$. This proves the compactness of $K_0$.

Let $P$ denote the cone of nonnegative functions in $E$, and $P^o$ the interior of $P$. It is easily seen that $P$ is reproducing, namely, $E = P - P$. We show that $K_0$ is strongly positive, that is, $K_0(P \setminus \{0\}) \subset P^o$. Indeed, if $u \ge 0$ and $u \not\equiv 0$ in $E$, then by the strong maximum principle we know that the unique solution $v$ of (3.29) satisfies $v > 0$ in $\overline{\Omega} \times (0, T^*]$. It follows that the unique solution $w$ of (3.30) satisfies $w > 0$ in $\Omega_0 \times (T^*, T]$. By the Hopf boundary lemma we deduce $\partial_{\nu_0} w < 0$ on $\partial\Omega_0 \times (T^*, T]$, where $\nu_0$ denotes the unit outward normal of $\partial\Omega_0$. In particular we have $w(x, T) > 0$ in $\Omega_0$ and $\partial_{\nu_0} w(x, T) < 0$ on $\partial\Omega_0$. This implies that $w(\cdot, T) \in P^o$. Hence $K_0$ is strongly positive.

With the above properties for $K_0$, the Krein-Rutman theorem applies and hence the spectral radius $r(K_0)$ of $K_0$ is positive, it corresponds to an eigenvector $u_0 \in P^o$. Moreover, if $K_0 u_1 = r u_1$ for some $u_1 \in P^o$, then necessarily $r = r(K_0)$ and $u_1 = c u_0$ for some constant $c$.

Let us now see how $K_0$ and $r(K_0)$ are related to the eigenvalue problem (3.28). Let $u_0 \in P^o$ be an eigenvector of $K_0$ corresponding to $r(K_0)$: $K_0 u_0 = r(K_0) u_0$. Let $U_0(x, t)$ be defined by

$$U_0(x, t) = v_0(x, t) \text{ in } \overline{\Omega} \times [0, T^*],\ U_0(x, t) = w_0(x, t) \text{ in } \overline{\Omega}_0 \times (T^*, T],$$

where $v_0$ denotes the unique solution of (3.29) with $\tilde{u}_0$ in place of $\tilde{u}$, and $w_0$ is the unique solution of (3.30) with $v(x, T^*)$ replaced by $v_0(x, T^*)$.

By definition, $U_0(\cdot, T) = K_0 u_0 = r(K_0) u_0$ in $\overline{\Omega}_0$. We now define

$$\varphi_0(x, t) = e^{\lambda t} U_0(x, t) \text{ with } \lambda = -\frac{1}{T} \ln r(K_0).$$

Then clearly $\varphi_0$ satisfies (3.5) and (3.4). Moreover, a direct calculation shows that $\varphi_0$ satisfies (3.28).

Conversely, if (3.28) has a solution $\varphi$ satisfying (3.5) and (3.4), then let $r_0 = e^{-\lambda T}$ and $\psi(x, t) = e^{-\lambda t} \varphi(x, t)$. We easily see that $\psi$ satisfies (3.29) with $\tilde{u}$ replaced by $\varphi(x, 0)$ in $\overline{\Omega} \times [0, T^*]$, and it satisfies (3.30) with $v(x, T^*)$ replaced by $\psi(x, T^*)$. Moreover,

$$K_0 \psi(\cdot, 0) = \psi(\cdot, T) = e^{-\lambda T} \varphi(\cdot, T) = r_0 \varphi(\cdot, 0) = r_0 \psi(\cdot, 0).$$

Hence $u := \psi(\cdot, 0)|_{\overline{\Omega}_0} = \varphi(\cdot, T) \in P^o$ satisfies $K_0 u = r_0 u$. By the Krein-Rutman theorem, we necessarily have $r_0 = r(K_0)$ and $u = cu_0$ for some constant $c$. It follows that $\varphi = c\varphi_0$.

Our proof is complete. □

**Remark 3.5.** *By Theorem 3.4 we know that the limiting function $\varphi^*$ in Theorem 3.3 is uniquely determined by* (3.6)*. It follows that the limit* $\lim_{\mu\to\infty}\varphi_\mu$ *exists and equals* $\varphi^*$*.*

If we denote by $\lambda_1 = \lambda_1(\Omega, \Omega_0, T, T^*)$ the principal eigenvalue of (3.28), then it follows from Theorem 3.3 that $\lambda_1 < \lambda_1^D(\Omega_0)$. We now give a lower bound for $\lambda_1$, which will be used in the next section.

**Theorem 3.6.** $\lambda_1(\Omega, \Omega_0, T, T^*) \geq \left(1 - \frac{T^*}{T}\right)\lambda_1^D(\Omega_0)$.

**Proof.** Firstly we observe that the linear operator $K_0$ defined in the proof of Theorem 3.4 can be extended to a compact linear operator $\tilde{K}_0$ over $X_0 = L^2(\Omega_0)$. Indeed, for any $u \in L^2(\Omega_0)$ we define $\tilde{u}$ as the extension of $u$ by 0 to $\Omega$, and let $v$ be the unique solution of (3.29); then we have $v(\cdot, T^*) = U_1(T^*, 0)\tilde{u}$, where $U_1$ is the operator in (2.2) associated with (3.29). Similarly the unique solution $w$ of (3.30) is given by $w(\cdot, t) = U_2(t - T^*, 0)v(\cdot, T^*)|_{\Omega_0}$, where $U_2$ is the operator in (2.2) associated with (3.30). By the properties of $U_1$ and $U_2$, we know that $U_1(0, T^*)$ and $U_2(T - T^*, 0)$ are compact operators on $L^2(\Omega)$ and $L^2(\Omega_0)$, respectively. It follows easily that $\tilde{K}_0 = U_2(T - T^*, 0) \circ I \circ U_1(T^*, 0) \circ J$ is compact from $L^2(\Omega_0)$ to itself, where $Ju = \tilde{u}$ is the extension operator, and $Iv = v|_{\Omega_0}$ is the restriction operator. By the maximum principle we know that $\tilde{K}_0$ is also a positive operator: $\tilde{K}_0 u \geq 0$ if $u$ is a nonnegative function in $L^2(\Omega_0)$. Since the positive cone in $L^2(\Omega_0)$ is reproducing, we can apply the Krein-Rutman theorem to conclude that $r(\tilde{K}_0) \geq r(K_0)$ is an eigenvalue that corresponds to a positive eigenfunction: $\tilde{K}_0\phi = r(\tilde{K}_0)\phi$. Using the regularity of $\tilde{K}_0$ and the Sobolev embedding theorem we can easily deduce from an iteration argument that $\phi \in C_0^1(\overline{\Omega}_0)$ and hence $\tilde{K}_0\phi = K_0\phi$. It follows that $K_0\phi = r(\tilde{K}_0)\phi$. However, since $K_0$ is a strongly positive operator, the above equality implies that $r(\tilde{K}_0) = r(K_0)$. Clearly $r(\tilde{K}_0) \leq \|\tilde{K}_0\|$.

We now estimate $\|\tilde{K}_0\|$. Let $\lambda_k^N(\Omega)$ be the eigenvalues of $-\Delta$ over $\Omega$ with Neumann boundary conditions, with corresponding eigenfunctions $\phi_k$, $k \geq 1$; and let $\lambda_k^D(\Omega_0)$ denote the eigenvalues of $-\Delta$ over $\Omega_0$ with Dirichlet boundary conditions, with corresponding eigenfunctions $\psi_k$, $k \geq 1$. We may assume that the eigenfunctions are orthonormal:

$$\int_\Omega \phi_k\phi_j = \delta_{kj}, \int_{\Omega_0} \psi_k\psi_j = \delta_{kj}.$$

Then for any $u \in L^2(\Omega_0)$, we have

$$Ju = \tilde{u} = \Sigma_{k=1}^\infty a_k\phi_k,$$

and

$$\|u\|_{L^2(\Omega_0)} = \|Ju\|_{L^2(\Omega)} = \left(\Sigma_{k=1}^\infty a_k^2\right)^{1/2}.$$

It is easily seen that with $\tilde{u}$ expressed this way,

$$v(\cdot, t) = U_1(t, 0)\tilde{u} = \Sigma_{k=1}^\infty a_k e^{-\lambda_k^N(\Omega)t}\phi_k,$$

and hence

$$\|v(\cdot,T^*)\|_{L^2(\Omega)} = \left(\Sigma_{k=1}^{\infty} a_k^2 e^{-2\lambda_k^N(\Omega)T^*}\right)^{1/2} \leq \left(\Sigma_{k=1}^{\infty} a_k^2\right)^{1/2} = \|u\|_{L^2(\Omega_0)}.$$

Similarly, we can write

$$Iv(\cdot,T^*) = v(\cdot,T^*)|_{\Omega_0} = \Sigma_{k=1}^{\infty} b_k\psi_k,$$

and hence

$$\|Iv(\cdot,T^*)\|_{L^2(\Omega_0)} = \left(\Sigma_{k=1}^{\infty} b_k^2\right)^{1/2},$$

$$w(\cdot,t) = U_2(t-T^*,0)v(\cdot,T^*)|_{\Omega_0} = \Sigma_{k=1}^{\infty} b_k e^{-\lambda_k^D(\Omega_0)(t-T^*)}\psi_k(x).$$

It follows that

$$\|w(\cdot,T)\|_{L^2(\Omega_0)} = \left(\Sigma_{k=1}^{\infty} b_k^2 e^{-2\lambda_k^D(\Omega_0)(T-T^*)}\right)^{1/2} \leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|Iv(\cdot,T^*)\|_{L^2(\Omega_0)}.$$

We thus obtain

$$\begin{aligned}\|w(\cdot,T)\|_{L^2(\Omega_0)} &\leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|Iv(\cdot,T^*)\|_{L^2(\Omega_0)}\\ &\leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|v(\cdot,T^*)\|_{L^2(\Omega)} \leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|u\|_{L^2(\Omega_0)}.\end{aligned}$$

This implies that $\|\tilde{K}_0\| \leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}$ and hence

$$r(K_0) = r(\tilde{K}_0) \leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}.$$

From the proof of Theorem 3.4, we have

$$\lambda_1(\Omega,\Omega_0,T,T^*) = -\frac{1}{T}\ln r(K_0) \geq -\frac{1}{T}\ln e^{-\lambda_1^D(\Omega_0)(T-T^*)} = \left(1-\frac{T^*}{T}\right)\lambda_1^D(\Omega_0).$$

The proof is complete. □

## 4. Long-time dynamical behavior when $a \geq \lambda_1(\infty)$

Suppose that $\lambda_1(\infty) < \infty$, we now study the long-time behavior of the positive solution of (1.1). Recall that for $a < \lambda_1(\infty)$, the behavior of the solution is already given in Theorem 2.1.

We first consider the case that (3.3) holds, and then discuss the case (3.2). As we will see below, the limit $\lim_{a\to\lambda_1(\infty)} u_a$, where $u_a$ is the unique positive $T$-periodic solution of (1.3), which exists if and only if $a \in (0,\lambda_1(\infty))$ (see Theorem 2.1), will play a key role in our analysis. This limit turns out to be determined by certain boundary blow-up solutions, and the boundary blow-up problems are fundamentally different between the case (3.3) and the case (3.2).

4.1. **The case that** (3.3) **holds.** Throughout this subsection we assume that (3.3) holds. We first discuss the asymptotic behavior of $u_a(x,t)$ as $a \uparrow \lambda_1(\infty)$. For simplicity, we denote $a_\infty = \lambda_1(\infty)$. By a simple comparison and sub- and super-solution argument it is easily seen that $u_a(x,t)$ is strictly increasing in $a$ for $a \in (0,a_\infty)$. Hence, it suffices to consider a sequence $a_n$ with $a_n \to a_\infty$. In the discussions below, we also denote $u_n(x,t) = u_{a_n}(x,t)$ and $\Omega^* := \Omega \setminus \overline{\Omega}_0$ for simplicity.

**Theorem 4.1.** *$u_a(x,t) \to \infty$ uniformly on every compact subset of $(\overline{\Omega}\times(0,T^*]) \cup (\overline{\Omega}_0\times[0,T])$ as $a \to a_\infty$.*

The proof of this theorem requires the following result.

**Lemma 4.2.** *Let $m(x,t)$ be a given positive $T$-periodic function on $\overline{\Omega^*}\times[0,T]$ that belongs to the space $C^{2+\theta,1+\theta/2}(\overline{\Omega^*}\times[0,T])$. Then, for any $a\in(-\infty,\infty)$, the following periodic problem*

$$
(4.1)\qquad \begin{cases} \partial_t u-\Delta u=au-b(x,t)u^p & \text{in } \Omega^*\times[0,T],\\ \partial_\nu u=0 & \text{on } \partial\Omega\times[0,T],\\ u=m(x,t) & \text{on } \partial\Omega_0\times[0,T],\\ u(x,0)=u(x,T) & \text{in } \Omega^* \end{cases}
$$

*has a unique $T$-periodic solution $u_a^m\in C^{2,1}(\overline{\Omega^*}\times[0,T])$. Moreover $u_a^m(x,t)>0$ on $\overline{\Omega^*}\times[0,T]$, and $u_a^m(x,t)$ is a strict increasing function with respect to $m(x,t)$ and $a$ in the sense that $u_a^{m_1}>u_a^{m_2}$ in $\Omega^*\times[0,T]$ if $m_1(x,t)\geq,\not\equiv m_2(x,t)$ on $\partial\Omega_0\times[0,T]$, and $u_{a_1}^m>u_{a_2}^m$ if $a_1>a_2$.*

**Proof.** For small $\delta>0$ we define

$$\Omega_0^\delta:=\{x\in\Omega_0: d(x,\partial\Omega_0)<\delta\}.$$

We then choose a $C^\theta$ function $p_\delta(x)$ which is positive in $\Omega_0\setminus\overline{\Omega_0^\delta}$ and vanishes on $\partial\Omega_0^\delta\cap\Omega_0$. Then define

$$b_\delta(x,t)=p_\delta(x)q(t)\ \text{ for } (x,t)\in\big(\Omega_0\setminus\Omega_0^\delta\big)\times\mathbb{R},\ b_\delta(x,t)=b(x,t)\ \text{ elsewhere.}$$

It is clear that $b_\delta(x,t)$ satisfies a condition similar to (3.3) but with $\Omega_0$ replaced by $\Omega_0^\delta$. By Theorem 2.1 problem (1.3) with $b$ replaced by $b_\delta$ has a unique positive $T$-periodic solution $u_a^\delta$ if and only if $0<a<\lambda_1^\delta(\infty):=\lim_{\mu\to\infty}\lambda_1(\mu b_\delta)$. Moreover, by Theorems 3.3, 3.4 and 3.6,

$$\left(1-\frac{T^*}{T}\right)\lambda_1^D(\Omega_0^\delta)\leq\lambda_1^\delta(\infty)<\lambda_1^D(\Omega_0^\delta).$$

Since $\lambda_1^D(\Omega_0^\delta)\to\infty$ as $\delta\to 0$, for any given $a\in(0,\infty)$, we can find a $\delta>0$ such that $(1-T^*T^{-1})\lambda_1^D(\Omega_0^\delta)>a$ and hence $u_a^\delta$ exists. It is easily checked that for sufficiently large $M>1$, $\overline{u}:=Mu_a^\delta|_{\Omega^*\times[0,T]}$ is a super-solution to (4.1). On the other hand, clearly $\underline{u}:=0$ is a sub-solution. Hence (4.1) has a nonnegative $T$-periodic solution. The strong maximum principle then implies that the solution is positive.

If $a\leq 0$, then 0 is a sub-solution and any positive constant $M>\max m(x,t)$ is a super-solution. Hence (4.1) has a nonnegative $T$-periodic solution in the order interval $[0,M]$. Since $m>0$, by the strong maximum principle the solution is positive.

We now prove the uniqueness and monotonicity properties of the positive $T$-periodic solution. Suppose that (4.1) has two positive $T$-periodic solutions $u_1(x,t)$ and $u_2(x,t)$. We may choose $M_0>1$ such that $M_0^{-1}u_1(x,t)<u_i(x,t)<M_0u_1(x,t)$ for $i=1,2$. It is easily seen that $M_0u_1$ is a supersolution of (4.1) and $M_0^{-1}u_1$ is a sub-solution. Hence there exist a minimal and a maximal solution in the order interval $[M_0^{-1}u_1, M_0u_1]$, which we denote by $u_*(x,t)$ and $u^*(x,t)$, respectively. Thus $u_*(x,t)\leq u_i(x,t)\leq u^*(x,t)$ for $i=1,2$. Hence it suffices to show that $u_*(x,t)=u^*(x,t)$.

Define

$$\sigma_*:=\inf\{\sigma\in\mathbb{R}: u^*\leq\sigma u_*\}.$$

Clearly $\sigma_* \geq 1$ and $u^* \leq \sigma_* u_*$. To prove $u^* = u_*$, it is enough to show $\sigma_* = 1$. Suppose for contradiction that $\sigma_* > 1$. Then for $w(x,t) := \sigma_* u_*(x,t) - u^*(x,t)$ we have $w \geq 0$, $w(x,0) = w(x,T)$,

$$\begin{aligned}\partial_t w - \Delta w &= aw - b(x,t)[\sigma_*(u_*)^p - (u^*)^p] \\ &\geq aw - b(x,t)(u^*)^{p-1} w\end{aligned}$$

for $(x,t) \in \Omega^* \times [0,T]$, and $\partial_\nu w = 0$ on $\partial\Omega \times [0,T]$, $w = (\sigma_* - 1)m > 0$ on $\partial\Omega_0 \times [0,T]$. Hence we can use the strong maximum principle to deduce that $w(x,t) > 0$ on $\overline{\Omega^*} \times [0,T]$. It follows that $w(x,t) \geq \epsilon u^*(x,t)$ for some $\epsilon > 0$ small, and hence $u^* \leq (1+\epsilon)^{-1}\sigma_* u_*$, which contradicts the definition of $\sigma_*$. This contradiction shows that we must have $\sigma_* = 1$, and the uniqueness conclusion is proved.

We next show the monotonicity of $u^m = u_a^m$ with respect to $m$. Assume that $m_1(x,t) \geq, \not\equiv m_2(x,t)$ on $\partial\Omega_0 \times [0,T]$. Then, $u^{m_1}$ is a strict super-solution to the equation that $u^{m_2}$ satisfies, and so the super-sub solution argument and the above proved uniqueness result indicate $u^{m_1} \geq u^{m_2}$ in $\Omega^* \times [0,T]$. Consequently, combined with the $T$-periodicity, the well-known maximum principle for parabolic equations and the Hopf boundary lemma we deduce $u^{m_1} > u^{m_2}$ in $\Omega^* \times [0,T]$. The monotonicity of $u_a^m$ with respect to $a$ is proved similarly. The proof is now complete. □

**Proof of Theorem 4.1.** For fixed $\mu > 0$, as in the proof of Theorem 3.3, let $\varphi_\mu(x,t)$ be the eigenfunction corresponding to $\lambda_1(\mu b)$ with the properties $\varphi_\mu(x,t) > 0$ on $\overline{\Omega} \times [0,T]$ and $\max_{\overline{\Omega}\times[0,T]} \varphi_\mu = 1$.

By the monotonicity of $u_a$ with respect to $a$, we only need to prove the desired conclusion along a sequence $a_n \to a_\infty$. Since $\lambda_1(\mu b) \to a_\infty$ as $\mu \to \infty$, we take $a_n = \lambda_1(\mu_n b)$ with $\mu_n$ increasing to $\infty$ as $n \to \infty$. For simplicity, we denote $u_{a_n}$ by $u_n$ and $\varphi_{\mu_n}$ by $\varphi_n$.

A simple computation shows that

$$\underline{u}(x,t) = \mu_n^{\frac{1}{p-1}} \varphi_n(x,t) \quad \text{and} \quad \overline{u}(x,t) = M_n \varphi_n(x,t)$$

form a pair of sub and super solutions of (1.3), where $M_n$ satisfies

$$M_n^{p-1}[\varphi_n(x,t)]^{p-1} \geq \mu_n.$$

Then by the uniqueness of $u_n$ it immediately follows that

$$\mu_n^{\frac{1}{p-1}} \varphi_n(x,t) \leq u_n(x,t) \leq M_n \varphi_n(x,t) \quad \text{on } \overline{\Omega} \times [0,T].$$

On the other hand, by Remark 3.5 and Step 5 in the proof of Theorem 3.3, we see that for any compact subset $K \subset (\overline{\Omega} \times (0,T^*]) \cup (\Omega_0 \times \mathbb{R})$,

$$\varphi_n(x,t) \to \varphi^*(x,t) \ \text{ in } C^{2,1}(K) \quad \text{as } n \to \infty,$$

where $\varphi^*(x,t) > 0$ in $K$. Hence

$$u_n(x,t) \geq \mu_n^{\frac{1}{p-1}} \varphi_n(x,t) \to \infty \ \text{ uniformly in } K.$$

It remains to show that

$$u_n(x,t) \to \infty \text{ uniformly on } \overline{\Omega}_0 \times [0,T] \text{ as } n \to \infty.$$

We now follow an argument in the spirit of the proof of Lemma 3.3 and Lemma 3.4 of [3]. Note that $u_n(x,t)$ satisfies $\partial_t u_n - \Delta u_n = a_n u_n > 0$ in $\Omega_0 \times [0,T]$, $u_n(x,t) > 0$ on $\overline{\Omega}_0 \times [0,T]$, $u_n(x,t) \to \infty$ uniformly on any compact subset of $\Omega_0 \times \mathbb{R}$, and $u_n(x,T^*) \to \infty$ uniformly on $\overline{\Omega}_0$. By the maximum principle, it is sufficient to prove

$$u_n(x_n,t_n) = \min_{\partial\Omega_0 \times \mathbb{R}} u_n(x,t) \to \infty \quad \text{as} \quad n \to \infty, \tag{4.2}$$

where we may choose $(x_n,t_n) \in \partial\Omega_0 \times [T^*, T+T^*]$.

To verify (4.2), we shall use a contradiction argument. We suppose on the contrary that (4.2) is false. Then, we may assume that $u_n(x_n,t_n) \le C$ for all $n \ge 1$ and some positive constant $C$. We may now use the maximum principle and the fact that $u_n(x,T^*) \to \infty$ uniformly on $\overline{\Omega}_0$ to conclude that for all large $n$, $u_n(x_n,t_n) = \min_{\overline{\Omega}_0 \times [T^*,T+T^*]} u_n(x,t)$. Without loss of generality we assume that this holds for all $n \ge 1$.

Since $\partial\Omega_0$ is smooth, it enjoys the uniform interior ball property, that is, we can find a small $R > 0$ such that for any $x \in \partial\Omega_0$, there exists a ball $B_{x,R}$ of radius $R$ such that $B_{x,R} \subset \overline{\Omega}_0$ and $\overline{B_{x,R}} \cap \partial\Omega_0 = \{x\}$.

To produce a contradiction, we first claim that: there is a constant $\delta > 0$ and a sequence of constants $c_n$ satisfying $c_n \to \infty$, such that

$$u_n(x_n,t_n) + c_n\omega(x) \le u_n(x,t) \;\text{ if }\; \frac{R}{2} \le |x-y_n| \le R,\; T^* \le t \le T+T^*, \tag{4.3}$$

where $\omega(x) = e^{-\delta|x-y_n|^2} - e^{-\delta R^2}$, and $y_n$ is the center of the ball $B_{x_n,R}$.

A simple computation gives

$$\Delta\omega + a_n\omega = (4\delta^2|x-y_n|^2 - 2N\delta + a_n)e^{-\delta|x-y_n|^2} - a_n e^{-\delta R^2}.$$

Thus, we can take a large $\delta > 0$ such that

$$\Delta\omega + a_n\omega \ge 0 \quad \forall\, x \in B_{x_n,R} \backslash B_{R/2}(y_n),$$

where $B_{R/2}(y_n) = \{x \in R^N \,:\, |x-y_n| < R/2\}$.

We now choose a compact set $K \subset\subset \Omega_0$ such that $K \supset \cup_{n=1}^{\infty} B_{R/2}(y_n)$. By what has already been proved, $u_n(x,t) \to \infty$ uniformly in $K \times \mathbb{R}$, and hence there is a sequence $c_n$ with $c_n \to \infty$ such that

$$u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) \le u_n(x,t), \;\; \forall\, x \in \overline{B_{R/2}}(y_n) \subset K,\; t \in [T^*, T+T^*].$$

We may further require that

$$u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) \le u_n(x,T^*), \;\; \forall\, x \in \overline{\Omega}_0.$$

Then, as $u_n(x,t) \ge u_n(x_n,t_n)$ on $\overline{\Omega}_0 \times [T^*, T+T^*]$, we find that $u_n(x,t)$ is a super-solution of the problem

$$\begin{cases} \partial_t u - \Delta u = a_n u & \text{in } B_{x_n,R} \backslash \overline{B_{R/2}}(y_n) \times [T^*, T+T^*], \\ u = u_n(x_n,t_n) & \text{on } \partial B_{x_n,R} \times [T^*, T+T^*], \\ u = u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) & \text{on } \partial B_{R/2}(y_n) \times [T^*, T+T^*], \\ u(x,T^*) = u_n(x,T^*) & \text{in } \{R/2 < |x-y_n| < R\}. \end{cases} \tag{4.4}$$

One also sees that $u_n(x_n,t_n)+c_n\omega(x)$ is a subsolution to (4.4). The comparison principle for parabolic equations then yields (4.3). Consequently, as $n\to\infty$, we find

$$\partial_{\nu_n}u_n|_{(x_n,t_n)}\geq c_n\partial_{\nu_n}\omega|_{x_n}=2c_n\delta Re^{-\delta R^2}\to\infty, \tag{4.5}$$

where $\nu_n=(y_n-x_n)/|y_n-x_n|$.

On the other hand, for any $n\geq 1$, the following $T$-periodic problem

$$\begin{cases}\partial_t u-\Delta u=a_n u-b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_0\times[0,T],\\ \partial_\nu u=0 & \text{on } \partial\Omega\times[0,T],\\ u=u_n(x_n,t_n) & \text{on } \partial\Omega_0\times[0,T],\\ u(x,0)=u(x,T) & \text{in } \Omega\backslash\overline{\Omega}_0.\end{cases} \tag{4.6}$$

admits a unique positive solution $v_n(x,t)$ (see Lemma 4.2). Furthermore, $u_n(x,t)$ is a supersolution of (4.6). Due to Lemma 4.2, we have $v_n(x,t)\leq u_n(x,t)$ on $\overline{\Omega}\backslash\Omega_0\times[0,T]$. If we replace $a_n$ by $a_\infty$ and replace $u_n(x_n,t_n)$ by its upper bound $C$ in (4.6), we obtain a unique positive solution of (4.6), denoted by $U_0(x,t)$. By Lemma 4.2 again, $v_n(x,t)\leq U_0(x,t)$ on $\overline{\Omega}\backslash\Omega_0\times[0,T]$. In particular, $\|v_n\|_{L^\infty(\overline{\Omega}\backslash\Omega_0\times[0,T])}$ has a bound independent of $n$. Thus, the $L^p-$ estimates and Sobolev embedding theorem imply that $\{v_n\}$ is bounded in $C^{1+\theta,\theta/2}(\overline{\Omega}\backslash\Omega_0\times[0,T])$, and so $\|\nabla v_n(x_n,t_n)\|\leq C_0$ for some $C_0>0$. Since

$$v_n(x,t)\leq u_n(x,t)\ \forall(x,t)\in\overline{\Omega}\backslash\Omega_0\times[T^*,T+T^*]\ \text{ and }\ u_n(x_n,t_n)=v_n(x_n,t_n),$$

we conclude

$$\partial_{\nu_n}u_n|_{(x_n,t_n)}\leq\partial_{\nu_n}v_n|_{(x_n,t_n)}\leq C_0. \tag{4.7}$$

Clearly (4.5) and (4.7) contradict each other, which indicates that (4.2) is true. The proof of Theorem 4.1 is now complete. □

**Theorem 4.3.** *Let $a_\infty=\lambda_1(\infty)$. Then, as $a$ increases to $a_\infty$, $u_a(x,t)\to U_\infty(x,t)$ uniformly on any compact subset of $\overline{\Omega}\backslash\overline{\Omega}_0\times(T^*,T)$, where $U_\infty(x,t)$ is the minimal positive solution of*

$$\begin{cases}\partial_t u-\Delta u=a_\infty u-b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_0\times(T^*,T),\\ \partial_\nu u=0 & \text{on } \partial\Omega\times(T^*,T),\\ u=\infty & \text{on } \partial\Omega_0\times(T^*,T),\\ u(x,T^*)=\infty & \text{in } \overline{\Omega}\backslash\Omega_0.\end{cases} \tag{4.8}$$

**Proof.** As before, since $u_a$ is increasing in $a$, we only need to consider the limit of $u_n:=u_{a_n}$ along an increasing sequence $a_n$ which converges to $a_\infty$ as $n\to\infty$.

Assume that $\epsilon>0$ is small. Let $\Omega_\epsilon=\{x\in\Omega: d(x,\Omega_0)<\epsilon\}$. Since $b(x,t)>0$ in $\overline{\Omega}\backslash\overline{\Omega}_0\times(T^*,T)$, we may assume that $b(x,t)\geq M_\epsilon$ on $\overline{\Omega}\backslash\Omega_\epsilon\times[T^*+\epsilon,T-\epsilon]$ for some positive constant $M_\epsilon$.

Consider the problem:

$$\begin{cases} \partial_t u - \Delta u = a_\infty u - M_\epsilon u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon \times (T^*+\epsilon, T-\epsilon], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*+\epsilon, T-\epsilon), \\ u = u_n(x,t) & \text{on } \partial\Omega_\epsilon \times (T^*+\epsilon, T-\epsilon), \\ u(x, T^*+\epsilon) = u_n(x, T^*+\epsilon) & \text{in } \Omega\backslash\overline{\Omega}_\epsilon. \end{cases} \tag{4.9}$$

It is clear that $u_n(x,t)$ is a subsolution of (4.9).

In what follows, we find a supersolution of (4.9). For this purpose, we consider the following two auxiliary problems:

$$w_t = a_\infty w - M_\epsilon w^p, \quad t > T^* + \epsilon; \quad w(T^*+\epsilon) = \infty, \tag{4.10}$$

and

$$\begin{cases} -\Delta z = a_\infty z - M_\epsilon z^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon, \\ \partial_\nu z = 0 \ \text{ on } \partial\Omega, \quad z = \infty \ \text{ on } \partial\Omega_\epsilon. \end{cases} \tag{4.11}$$

The unique solution $w(t)$ of (4.10) can be explicitly written as

$$w(t) = \Big(\frac{a_\infty}{M_\epsilon}\Big)^{\frac{1}{p-1}} e^{a_\infty t}\Big[e^{a_\infty (p-1)t} - e^{a_\infty(p-1)(T^*+\epsilon)}\Big]^{\frac{1}{1-p}}, \quad t > T^*+\epsilon.$$

And by the result of [2, 3], we know that problem (4.11) also admits a unique positive solution, which we denote by $z(x)$.

For any fixed $n$, we have $w(t) + z(x) > u_n(x,t)$ in $\partial\Omega_\epsilon \times (T^*+\epsilon, T-\epsilon)$ and $w(T^*+\epsilon) > u_n(x, T^*+\epsilon)$ on $\overline{\Omega}\backslash\Omega_\epsilon$. We can also easily check that $w(t)+z(x)$ satisfies the required differential inequality for a supersolution of (4.9). Hence, for all $n \geq 1$, by the comparison principle for parabolic equations, we have $u_n(x,t) \leq w(t) + z(x)$ on $\overline{\Omega}\backslash\Omega_\epsilon \times [T^*+\epsilon, T-\epsilon]$. Observe that, for fixed small $\epsilon > 0$, $w(t)+z(x)$ is bounded on $\overline{\Omega}\backslash\Omega_{2\epsilon} \times [T^*+2\epsilon, T-\epsilon]$. As a result, by the standard regularity argument, it is clear that $u_n(x,t) \to U_\infty(x,t)$ uniformly on any compact subset of $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T)$ as $n \to \infty$, where $U_\infty(x,t)$ satisfies the first equation of (4.8), and $\partial_\nu U_\infty = 0$ on $\partial\Omega \times (T^*, T)$.

Next we show that

$$\lim_{t \downarrow T^*} U_\infty(x,t) = \infty \text{ uniformly for } x \in \overline{\Omega} \setminus \Omega_0, \tag{4.12}$$

$$\lim_{d(x,\Omega_0)\to 0} U_\infty(x,t) = \infty \text{ uniformly for } t \in [T^*, T). \tag{4.13}$$

Since $u_n$ increases to $U_\infty$ as $n \to \infty$, we have $U_\infty > u_k$ for all $k \geq 1$. Suppose for contradiction that there exist sequences $x_n \in \overline{\Omega} \setminus \Omega_0$ and $t_n$ decreasing to $T^*$ such that $U_\infty(x_n, t_n) \leq M$ for all $n \geq 1$ and some constant $M > 0$, then

$$u_k(x_n, t_n) \leq M \ \forall n \geq 1, \ \forall k \geq 1. \tag{4.14}$$

On the other hand, by Theorem 4.1 we know that $u_k(x_n, T^*) \to \infty$ as $k \to \infty$ uniformly in $n \geq 1$. Thus there exists $k_0$ large such that $u_{k_0}(x_n, T^*) \geq 3M$ for all $n \geq 1$. Since the function $u_{k_0}(x,t)$

is uniformly continuous in its variables, and $t_n \to T^*$, we deduce $|u_{k_0}(x_n,t_n) - u_{k_0}(x_n,T^*)| \to 0$ as $n \to \infty$. Thus for all large $n$,

$$u_{k_0}(x_n,t_n) \geq u_{k_0}(x_n,T^*) - M \geq 2M,$$

which is in contradiction to (4.14). This proves (4.12). The proof of (4.13) is similar, where we use $u_n \to \infty$ on $\partial\Omega_0 \times [0,T]$ (by Theorem 4.1), and $u_k < U_\infty$.

Thus $U_\infty$ is a solution to (4.8). It remains to show that $U_\infty$ is the minimal positive solution of (4.8). Let $U$ be any positive solution of (4.8). Then applying the parabolic comparison principle we easily see that $u_n < U$ in $\Omega \setminus \overline{\Omega}_0 \times (T^*,T)$. Letting $n \to \infty$ we deduce $U_\infty \leq U$. Hence $U_\infty$ is the minimal positive solution. □

For later use, we also need to consider the following more general version of (4.8), where $a_\infty$ is replaced by an arbitrary $a \in (-\infty,\infty)$:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_0 \times (T^*,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*,T), \\ u = \infty & \text{on } \partial\Omega_0 \times (T^*,T), \\ u(x,T^*) = \infty & \text{in } \overline{\Omega}\backslash\Omega_0. \end{cases} \tag{4.15}$$

**Theorem 4.4.** *For any $a \in (-\infty,\infty)$, (4.15) has a minimal positive solution $\underline{U}_a$ and a maximal positive solution $\overline{U}_a$, in the sense that if $U$ is any positive solution of (4.15), then $\underline{U}_a \leq U \leq \overline{U}_a$ in $\Omega \setminus \overline{\Omega}_0 \times (T^*,T)$.*

**Proof.** For $\epsilon \geq 0$ small, we define $\Omega_\epsilon$ as in the proof of Theorem 4.3 and then for each integer $n \geq 1$ consider the following initial boundary value problem:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon \times (T^*+\epsilon,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*+\epsilon,T), \\ u = n & \text{on } \partial\Omega_\epsilon \times (T^*+\epsilon,T), \\ u(x,T^*+\epsilon) = n & \text{in } \overline{\Omega}\backslash\Omega_\epsilon. \end{cases} \tag{4.16}$$

Let $u_n$ denote the unique positive solution of (4.16). By the same argument used in the proof of Theorem 4.3 we find that $u_n$ increases to $U_\epsilon$ as $n \to \infty$, and $U_\epsilon$ is the minimal positive solution of

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon \times (T^*+\epsilon,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*+\epsilon,T), \\ u = \infty & \text{on } \partial\Omega_\epsilon \times (T^*+\epsilon,T), \\ u(x,T^*+\epsilon) = \infty & \text{in } \overline{\Omega}\backslash\Omega_\epsilon. \end{cases} \tag{4.17}$$

Taking $\epsilon = 0$ we obtain the minimal positive solution of (4.15).

Using the parabolic comparison principle we easily deduce that $U_{\epsilon_1} \geq U_{\epsilon_2} \geq U_0$ when $\epsilon_1 > \epsilon_2 > 0$. Hence there is a decreasing sequence $\epsilon_n$ converging to 0 such that $U_{\epsilon_n} \to \overline{U}$ as $\epsilon_n \to 0$ and $\overline{U}$ is a positive solution of (4.15). We show that $\overline{U}$ is the maximal positive solution of (4.15). Indeed, if $U$ is any positive solution of (4.15), then we can apply the parabolic comparison principle to deduce $U_{\epsilon_n} > U$ for each $n$. Letting $n \to \infty$ we obtain $\overline{U} \geq U$. Hence $\overline{U}$ is the maximal positive solution of (4.15). The proof is complete. $\square$

We are now ready to state and prove the long-time asymptotic behavior of the unique positive solution of (1.1) for $a \geq a_\infty$.

**Theorem 4.5.** *Assume that $a \geq a_\infty$, $u_0 \in C(\overline{\Omega})$ and $u_0 \geq, \not\equiv 0$. Then, the unique solution $u(x,t)$ of* (1.1) *satisfies*

$$\lim_{n\to\infty} u(x,t+nT) = \begin{cases} \infty & \text{locally uniformly on } (\overline{\Omega}\times(0,T^*]) \cup (\overline{\Omega}_0 \times [0,T]), \\ \underline{U}_a(x,t) & \text{locally uniformly on } \overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*,T). \end{cases}$$

**Proof.** For any given $\epsilon > 0$, let $u^\epsilon(x,t)$ denote the unique solution of the problem

$$(4.18) \qquad \begin{cases} \partial_t u - \Delta u = (a_\infty - \epsilon)u - b(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega \times (0,\infty). \end{cases}$$

Since $a > a_\infty - \epsilon$, it is obvious that $u(x,t)$ is a supersolution to (4.18) and thus

$$(4.19) \qquad u^\epsilon(x,t) \leq u(x,t) \quad \text{on } \overline{\Omega}\times[0,\infty).$$

Let $u_{a_\infty-\epsilon}(x,t)$ be the unique $T$-periodic positive solution to

$$\begin{cases} \partial_t u - \Delta u = (a_\infty - \epsilon)u - b(x,t)u^p & \text{in } \Omega \times (0,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases}$$

By Theorem 2.1, we have

$$(4.20) \qquad u^\epsilon(x,t+nT) \to u_{a_\infty-\epsilon}(x,t) \ \text{ uniformly on } \overline{\Omega}\times[0,T] \quad \text{as } n \to \infty.$$

Using (4.19) and (4.20) we obtain that

$$\liminf_{n\to\infty} u(x,t+nT) \geq u_{a_\infty-\epsilon}(x,t)$$

for all small $\epsilon > 0$, uniformly on $\overline{\Omega}\times[0,T]$. Letting $\epsilon \to 0$ in the above inequality and using Theorems 4.1 and 4.3, we deduce that

$$\lim_{n\to\infty} u(x,t+nT) = \infty \text{ locally uniformly on } (\overline{\Omega}\times(0,T^*]) \cup (\overline{\Omega}_0\times[0,T]),$$

and

$$(4.21) \qquad \liminf_{n\to\infty} u(x,t+nT) \geq \underline{U}_{a_\infty}(x,t) \text{ locally uniformly on } \overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*,T).$$

On the other hand, by the parabolic comparison principle, we easily see that for every $n \geq 1$, $u(x,t+nT) < \underline{U}_a(x,t)$ in $\Omega \setminus \overline{\Omega}_0 \times (T^*,T)$, and hence

$$(4.22) \qquad \limsup_{n\to\infty} u(x,t+nT) \leq \underline{U}_a(x,t) \text{ uniformly on } \overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*,T).$$

Using this upper bound for $\tilde{u}_n(x,t) := u(x,t+nT)$ and standard parabolic estimates and (4.21), we easily see that, by passing to a subsequence, $\tilde{u}_n(x,t) \to \tilde{U}_a(x,t)$ which is a positive solution of (4.15). Hence $\tilde{U}_a \geq \underline{U}_a$. Together with (4.22), this implies that $\tilde{U}_a = \underline{U}_a$. Hence the entire original sequence converges and

$$\lim_{n\to\infty} u(x, t + nT) = \underline{U}_a(x, t).$$

By standard parabolic estimates, the above convergence is locally uniform in $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T)$. The proof is thus complete. □

**Remark 4.6.** *The following questions arise naturally:*

- (Q1) *Does* (4.15) *have at most one positive solution?*
- (Q2) *If $U$ is a positive solution to* (4.15)*, what is the asymptotic behavior of $U(x, t)$ as $t$ increases to $T$?*

*We will address these and related questions in a forthcoming paper.*

4.2. **The case that** (3.2) **holds.** In this subsection, we suppose that (3.2) holds, and show that the long-time dynamical behavior of the positive solution to (1.1) is analogous to (1.4). Let us recall that by Theorem 3.2, $\lambda_1(\infty) = \lambda_1^D(\Omega_0)$.

Our approach in this subsection follows the lines of the previous subsection. We start with the following result.

**Lemma 4.7.** *The conclusions in Lemma 4.2 remain valid under condition* (3.2)*.*

**Proof.** We only give the proof for existence; the other conclusions are proved in the same way as in Lemma 4.2.

We shall again use a super-sub solution argument. It is obvious that $\underline{u}(x, t) = 0$ is a sub-solution to (4.1). Next, we construct a super-solution. It is well-known that the following elliptic problem

$$\begin{cases} -\Delta u = au - \underline{b}(x)u^p & \text{in } \Omega^*, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \max\limits_{\partial\Omega_0\times[0,T]} m(x, t) & \text{on } \partial\Omega_0 \end{cases}$$

has a unique positive solution $\overline{u}(x) \in C^2(\overline{\Omega^*})$ (see, e.g., Lemma 2.3 in [3]), and we easily see that $\overline{u}(x)$ is a supersolution to (4.1). Thus by the standard super-sub solution iteration argument (4.1) admits a positive $T$-periodic solution. □

**Remark 4.8.** *By exactly the same proof, we see that when $b(x, t) > 0$ on $\overline{\Omega^*} \times [0, T]$,* Lemma 4.7 *remans valid.*

**Theorem 4.9.** *For any $a \in (-\infty, \infty)$, the following boundary blow-up problem*

$$\begin{cases} \partial_t v - \Delta v = av - b(x, t)v^p & \text{in } \Omega^* \times [0, T], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times [0, T], \\ v = \infty & \text{on } \partial\Omega_0 \times [0, T], \\ v(x, 0) = v(x, T) & \text{in } \Omega^* \end{cases} \tag{4.23}$$

*has a minimal positive solution $\underline{V}_a(x,t)$ and a maximal positive solution $\overline{V}_a(x,t)$ in the sense that any positive solution $V(x,t)$ of* (4.23) *satisfies $\underline{V}_a(x,t) \le V(x,t) \le \overline{V}_a(x,t)$ in $\Omega^* \times [0,T]$. Moreover, both the minimal and maximal solutions are nondecreasing in $a$.*

**Proof.** For small $\epsilon \ge 0$, we define $\Omega_\epsilon = \{x \in \Omega : \mathrm{d}(x,\Omega_0) < \epsilon\}$. Obviously, for small $\epsilon$, $\partial\Omega_\epsilon$ has the same smoothness as $\partial\Omega_0$. In (4.1), we take $m(x,t) = m$ and replace $\Omega_0$ by $\Omega_\epsilon$. By Lemma 4.7 and Remark 4.8, we know that the modified (4.1) has a unique positive $T$-periodic solution $u_a^m(x,t) = u_a^{m,\epsilon}(x,t)$. We claim that $\underline{V}_a^\epsilon(x,t) = \lim_{m\to\infty} u_a^m(x,t)$ is a minimal positive solution of (4.23) with $\Omega_0$ replaced by $\Omega_\epsilon$.

To prove this, we first show that for any fixed small $\delta > 0$, $u_a^m(x,t)$ is uniformly bounded on $\overline{\Omega}\backslash\Omega_{\epsilon+\delta} \times [0,T]$. To this end, we consider the elliptic problem

$$
\begin{cases}
-\Delta u = au - \underline{b}(x)u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon,\\
\partial_\nu u = 0 & \text{on } \partial\Omega,\\
u = m & \text{on } \partial\Omega_\epsilon.
\end{cases}
\tag{4.24}
$$

By Lemma 2.3 in [3], problem (4.24) has a unique positive solution, which we denote by $\underline{u}_a^m(x)$. Moreover $\underline{u}_a^m(x)$ is strictly increasing in $m$, and $U_a(x) := \lim_{m\to\infty} \underline{u}_a^m(x)$ exists and is the minimal positive solution of (4.24) with $m$ replaced by $\infty$. This implies in particular that $\underline{u}_a^m < U_a$ for all $m \ge 1$.

As in the proof of Lemma 4.2, we can use a super-sub solution argument, together with the uniqueness of $u_a^m(x,t)$, to show that $u_a^m(x,t) \le \underline{u}_a^m(x)$ on $\overline{\Omega}\backslash\Omega_\epsilon \times [0,T]$ for each $m \ge 1$. Therefore, we have $u_a^m(x,t) \le U_a(x)$ on $\overline{\Omega}\backslash\Omega_\epsilon \times [0,T]$ for all $m \ge 1$. This proves the required uniform boundedness of $u_a^m(x,t)$. Hence $\underline{V}_a^\epsilon(x,t) := \lim_{m\to\infty} u_a^m(x,t)$ exists. Moreover, using standard regularity theorem for parabolic equations and the embedding theorem, we can easily conclude that $\underline{V}_a^\epsilon = \lim_{m\to\infty} u_a^m$ holds in $C^{2,1}(K \times [0,T])$ for any compact subset $K$ of $\overline{\Omega^*}\backslash\overline{\Omega}_\epsilon$, and $V_a^\epsilon(x,t)$ satisfies (4.23) with $\Omega_0$ replaced by $\Omega_\epsilon$. Since each $u_a^m$ is increasing in $a$, $\underline{V}_a^\epsilon$ is nondecreasing in $a$.

We next show that $\underline{V}_a^\epsilon(x,t)$ obtained above is the minimal positive solution. Assume that $V(x,t)$ is an arbitrary positive solution of (4.23) with $\Omega_0$ replaced by $\Omega_\epsilon$. Since $\lim_{d(x,\Omega_\epsilon)\to 0} V(x,t) = \infty$ and $u_a^m = m$ on $\partial\Omega_\epsilon \times [0,T]$, there exists $\delta_m > 0$ sufficiently small such that $V > u_a^m$ on $\partial\Omega_{\epsilon+\delta} \times [0,T]$ for all $\delta \in (0,\delta_m]$. Hence for all such $\delta$, $V$ is a supersolution to the following problem:

$$
\begin{cases}
\partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_{\epsilon+\delta} \times [0,T],\\
\partial_\nu u = 0 & \text{on } \partial\Omega \times [0,T],\\
u = u_a^m(x,t) & \text{on } \partial\Omega_{\epsilon+\delta} \times [0,T],\\
u(x,0) = u(x,T) & \text{in } \Omega\backslash\overline{\Omega}_{\epsilon+\delta}.
\end{cases}
\tag{4.25}
$$

Since clearly 0 is a sub-solution to this problem, we conclude that (4.25) has a positive solution satisfying $u \le V$. Since clearly $u_a^m(x,t)$ solves (4.25), and by Remark 4.8 it is the unique positive solution, we deduce $u_a^m(x,t) \le V(x,t)$ for all $m \ge 1$ and $(x,t) \in \Omega \setminus \Omega_{\epsilon+\delta} \times [0,T]$. Since $\delta > 0$ can be arbitrarily small we deduce $u_a^m(x,t) \le V(x,t)$ for all $m \ge 1$ and $(x,t) \in \Omega \setminus \overline{\Omega}_\epsilon \times [0,T]$.

Letting $m \to \infty$ we obtain $\underline{V}_a^\epsilon \le V$. This proves that $\underline{V}_a^\epsilon$ is the minimal positive solution. Taking $\epsilon = 0$ we know that (4.23) has a minimal positive solution, and it is nondecreasing in $a$.

To show the existence of a maximal positive solution, we notice that for any small $\epsilon_1$, $\epsilon_2$ with $0 < \epsilon_1 < \epsilon_2$, we can use a comparison argument as in the last paragraph to deduce that, for any positive solution $V$ of (4.23),

$$\underline{V}_a^{\epsilon_2}(x,t) \ge \underline{V}_a^{\epsilon_1}(x,t) \ge V(x,t) \ \text{ in } \Omega\backslash\overline{\Omega}_{\epsilon_2} \times [0,T].$$

It follows that

$$\overline{V}_a(x,t) := \lim_{\epsilon \to 0} \underline{V}_a^\epsilon(x,t) \ge V(x,t),$$

exists, and moreover, $\overline{V}_a(x,t)$ is a positive solution of (4.23). Since $V(x,t) \le \overline{V}_a(x,t)$, we conclude that $\overline{V}_a(x,t)$ is the desired maximal positive solution. Since each $\underline{V}_a^\epsilon$ is nondecreasing in $a$, so is $\overline{V}_a$. □

**Remark 4.10.** *For the boundary blow-up problem* (4.23)*, if we assume that*

$$\sigma_1 d(x,\Omega_0)^\alpha \le b(x,t) \le \sigma_2 d(x,\Omega_0)^\alpha$$

*for some constants $\sigma_1 > 0$, $\sigma_2 > 0$, $\alpha > -2$, and for all $x$ close to $\partial\Omega_0$ and $t \in [0,T]$, then one can make use of Corollary 6.17 in* [2]*, and a convex function trick due to Marcus and Véron* [10, 11] *as in the proof of Theorem 6.18 of* [2] *to show that* (4.23) *has a unique positive solution. Some details of this idea are given in the proof of Theorem 4.12 below.*

**Theorem 4.11.** *Let $a_\infty = \lambda_1(\infty)$. Then, as $a$ increases to $a_\infty$, the unique positive $T$-periodic solution of* (1.3) *satisfies*

(i) $u_a(x,t) \to \infty$ *uniformly on* $\overline{\Omega}_0$*;*

(ii) $u_a(x,t) \to \underline{V}_{a_\infty}(x,t)$ *in* $C^{2,1}(K \times [0,T])$ *for any compact set* $K \subset \overline{\Omega}\backslash\overline{\Omega}_0$*.*

**Proof.** As before, by a simple super-sub solution argument, we find that $u_a(x,t)$ is strictly increasing in $a$ for $a \in (0, a_\infty)$.

From [3], we know that the following problem

$$\begin{cases} -\Delta u = au - \overline{b}(x)u^p & \text{in } \ \Omega, \\ \partial_\nu u = 0 & \text{on } \ \partial\Omega \end{cases}$$

has a unique positive solution if and only if $a \in (0, a_\infty)$; we denote it by $\underline{u}_a(x)$. Moreover, Theorem 1.2 in [3] tells us that $\underline{u}_a(x) \to \infty$ uniformly on $\overline{\Omega}_0$ as $a \to a_\infty$. On the other hand, by a simple sub-super solution argument we deduce that $\underline{u}_a(x) \le u_a(x,t)$. As a result, we can use Theorem 3.6 of [3] to obtain

$$u_a(x,t) \to \infty \ \text{ uniformly on } \overline{\Omega}_0, \ \text{ as } a \to a_\infty.$$

Furthermore, by the comparison argument we used in the proof of Theorem 4.9 to deduce $u_a^m \le V$ through (4.25), we can easily show that $u_a(x,t) \le \underline{V}_a(x,t) \le \underline{V}_{a_\infty}(x,t)$ in $\Omega^* \times [0,T]$. Since $u_a(x,t)$ is increasing in $a$ for $a \in (0, a_\infty)$, $u_*(x,t) := \lim_{a \to a_\infty} u_a(x,t)$ exists. Moreover, $u_*(x,t) \le \underline{V}_{a_\infty}(x,t)$ and it satisfies (4.23) with $a = a_\infty$. Hence, by Theorem 4.9, it is necessary that $u_*(x,t) = \underline{V}_{a_\infty}(x,t)$. Using the Sobolev embedding theorems and the interior estimates (see, e.g., [8, 9]), we easily see that, as $a \to a_\infty$, $u_a(x,t) \to \underline{V}_{a_\infty}(x,t)$ in $C^{2,1}(K \times [0,T])$ for any compact set $K \subset \overline{\Omega} \backslash \overline{\Omega}_0$. This completes the proof. □

**Theorem 4.12.** *Assume that $a \geq a_\infty$, $u_0 \in C(\overline{\Omega})$ and $u_0 \geq, \not\equiv 0$. Then, the unique solution $u(x,t)$ of (1.1) satisfies that*

(i) *$u(x,t) \to \infty$ uniformly on $\overline{\Omega}_0$ as $t \to \infty$;*

(ii) *$u(x,t+nT) \to \underline{V}_a(x,t)$ in $C^{2,1}(K \times [0,T])$ as $n \to \infty$ for any compact set $K \subset \overline{\Omega} \backslash \overline{\Omega}_0$.*

**Proof.** By the same argument as in the proof of Theorem 4.5, we can use Theorem 4.11 to obtain

$$\lim_{n\to\infty} u(x,t+nT) = \infty \text{ uniformly on } (\overline{\Omega}_0 \times [0,T]),$$

and

$$\liminf_{n\to\infty} u(x,t+nT) \geq \underline{V}_{a_\infty}(x,t) \text{ in } \overline{\Omega} \setminus \overline{\Omega}_0 \times [0,T].$$

Conclusion (i) in the theorem is thus proved.

We next prove (ii). Let us denote by $u_a^m(x,t)$ the unique positive solution of (4.1) with $m(x,t) = m > 0$ being a constant. We first verify that for any $v_0 \in C(\overline{\Omega}^*)$, $v_0 \geq 0$, the solution $v^m(x,t)$ of

$$\begin{cases} \partial_t v - \Delta v = av - b(x,t)v^p & \text{in } \Omega^* \times (0,\infty), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v = m & \text{on } \partial\Omega_0 \times (0,\infty), \\ v(x,0) = v_0(x) & \text{in } \Omega^* \end{cases} \tag{4.26}$$

satisfies

$$v^m(x,t+nT) \to u_a^m(x,t) \ \text{ uniformly for } (x,t) \in \overline{\Omega^*} \times [0,T], \quad \text{as } n \to \infty. \tag{4.27}$$

In fact, for any constant $M > 1$, $Mu_a^m(x,t)$ is a supersolution of (4.1), while 0 is a subsolution. We may choose $M > 1$ large enough so that $Mu_a^m(x,0) > v_0(x)$ on $\overline{\Omega^*}$. Let $\underline{v}^m(x,t)$ denote the unique solution of (4.26) with $v(x,0) \equiv 0$, and let $\overline{v}^m(x,t)$ be the unique solution of (4.26) with $v(x,0) = Mu_a^m(x,0)$. Then, the well-known comparison principle for parabolic equations infers that

$$\underline{v}^m(x,t) \leq v^m(x,t) \leq \overline{v}^m(x,t).$$

Moreover, by standard iteration procedure from sub- and super-solutions for periodic-parabolic problems (as in [7]), we see that, as $n \to \infty$, $\underline{v}^m(x,t+nT)$ increases to a positive $T$-periodic solution of (4.1) with $m(x,t) \equiv m$, and $\overline{v}^m(x,t+nT)$ decreases to such a solution. Since $u_a^m(x,t)$ is the unique positive $T$-periodic solution of (4.1) with $m(x,t) \equiv m$ by Lemma 4.2, we must have

$$\underline{v}^m(x,t+nT),\ \overline{v}^m(x,t+nT) \to u_a^m(x,t) \ \text{ uniformly for } (x,t) \in \overline{\Omega^*} \times [0,T], \quad \text{as } n \to \infty.$$

Clearly (4.27) is a consequence of this fact.

As before we know that as $m \to \infty$, $u_a^m(x,t)$ converges to $\underline{V}_a(x,t)$. Moreover, this convergence is uniform on $K \times [0,T]$ for any compact subset $K \subset \overline{\Omega} \backslash \overline{\Omega}_0$. Hence, for any given $\epsilon > 0$, we can find $m_\epsilon > 0$ large such that

$$u_a^{m_\epsilon}(x,t) \geq \underline{V}_a(x,t) - \epsilon/2 \quad \forall (x,t) \in K \times [0,T]. \tag{4.28}$$

In view of conclusion (i) proved above, one can find a large integer $N_\epsilon > 0$ such that $u(x,t) > m_\epsilon$ for $t \geq N_\epsilon T$ and $x \in \partial\Omega_0$. Consequently, $u(x, t + N_\epsilon T)$ is a supersolution of (4.26) with $m = m_\epsilon$ and $v_0(x) = 0$. It follows that

$$u(x, t + nT) \geq \underline{v}^{m_\epsilon}(x, t + (n - N_\epsilon)T) \geq u^{m_\epsilon}(x,t) - \epsilon/2$$

uniformly on $K \times [0,T]$ for all large $n \geq N_\epsilon$. Applying (4.28), it follows that for all large $n \geq N_\epsilon$,

$$u(x, t + nT) \geq \underline{V}_a(x,t) - \epsilon \ \text{ uniformly on } K \times [0,T],$$

from which we derive

$$\liminf_{n\to\infty} u(x, t + nT) \geq \underline{V}_a(x,t) \ \text{ uniformly on } K \times [0,T]. \tag{4.29}$$

We show next that

$$\lim_{n\to\infty} u(x, t + nT) = \underline{V}_a(x,t) \ \text{ uniformly on } K \times [0,T]. \tag{4.30}$$

To this end we consider the auxiliary problem

$$\begin{cases} \partial_t w - \Delta w = aw - \underline{b}(x) v^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu w = 0 & \text{on } \partial\Omega \times (0,\infty), \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{4.31}$$

Since $\underline{b} \leq b$, by the parabolic comparison principle we deduce

$$u(x,t) \leq w(x,t) \text{ in } \Omega \times (0,\infty).$$

By the main result in [4], for $a \geq a_\infty$ and $x \in \overline{\Omega} \setminus \overline{\Omega}_0$,

$$\lim_{t\to\infty} w(x,t) = \tilde{W}_a(x),$$

where the limits are locally uniform in $\overline{\Omega} \setminus \overline{\Omega}_0$, and $\tilde{W}_a$ is the minimal positive solution of

$$-\Delta W = aW - \underline{b}(x) W^p \text{ in } \Omega \setminus \overline{\Omega}_0, \ \partial_\nu W|_{\partial\Omega} = 0, \ W|_{\partial\Omega_0} = \infty.$$

Since $u(x,t) \leq w(x,t)$, we necessarily have

$$\underline{V}_a(x,t) \leq \liminf_{n\to\infty} u(x, t + nT) \leq \limsup_{n\to\infty} u(x, t + nT) \leq \tilde{W}_a(x) \tag{4.32}$$

locally uniformly for $x \in \overline{\Omega} \setminus \overline{\Omega}_0$.

Using the above bounds for $\tilde{u}_n(x,t) := u(x, t + nT)$ and standard parabolic estimates, we easily see that by passing to a subsequence $\tilde{u}_n(x,t) \to \hat{V}_a(x,t)$, and $\hat{V}_a$ satisfies

$$\begin{cases} \partial_t v - \Delta v = av - b(x,t) v^p & \text{in } \Omega^* \times \mathbb{R}, \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ v = \infty & \text{on } \partial\Omega_0 \times \mathbb{R}. \end{cases} \tag{4.33}$$

By (4.32), we deduce

$$\underline{V}_a \le \hat{V}_a.$$

On the other hand, if we choose $k > 1$ large enough such that $k\underline{V}_a(x,0) \ge u_0(x)$ in $\overline{\Omega} \setminus \overline{\Omega}_0$, then we can apply the parabolic comparison principle to deduce that $u(x,t) \le k\underline{V}_a(x,t)$ in $(\overline{\Omega} \setminus \overline{\Omega}_0) \times (0,\infty)$. It follows that

$$\hat{V}_a \le k\underline{V}_a.$$

To prove (4.30) it suffices to verify that $\underline{V}_a = \hat{V}_a$. Arguing by contradiction, we assume that $\underline{V}_a(x,t) \le, \not\equiv \hat{V}_a(x,t)$ in $\Omega^* \times \mathbb{R}$. Then, by the well-known strong maximum principle for parabolic equations, it is easily seen that $\underline{V}_a(x,t) < \hat{V}_a(x,t)$ in $\Omega^* \times \mathbb{R}$.

We now define

$$U(x,t) = \underline{V}_a(x,t) - (2k)^{-1}(\hat{V}_a(x,t) - \underline{V}_a(x,t)),$$

and use a convex function trick introduced by Marcus and Véron [10, 11] as in Theorem 6.18 of [2]. Simple direct computations show that

$$\underline{V}_a > U \ge \frac{k+1}{2k}\underline{V}_a \quad \text{in } \Omega^* \times \mathbb{R}, \tag{4.34}$$

and

$$\frac{2k}{2k+1}U + \frac{1}{2k+1}\hat{V}_a(x,t) = \underline{V}_a(x,t). \tag{4.35}$$

It is clear that $f(x,t,v) = -av + b(x,t)v^p$ is convex with respect to $v$ in $(0,\infty)$. Hence, by virtue of (4.35), we obtain

$$f(x,t,\underline{V}_a(x,t)) \le \frac{2k}{2k+1}f(x,t,U) + \frac{1}{2k+1}f(x,t,\hat{V}_a(x,t)).$$

It follows that

$$\partial_t U - \Delta U = -\frac{2k+1}{2k}f(x,t,\underline{V}_a(x,t)) + \frac{1}{2k}f(x,t,\hat{V}_a(x,t)) \ge -f(x,t,U),$$

from which and (4.34), we deduce

$$\begin{cases} \partial_t U - \Delta U \ge aU - b(x,t)U^p & \text{in } \Omega^* \times \mathbb{R}, \\ \partial_\nu U = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ U = \infty & \text{on } \partial\Omega_0 \times \mathbb{R}. \end{cases}$$

Note that due to the periodicity of $b(x,t)$ in $t$, for each $n \ge 1$, $U(x,t-nT)$ also satisfies the above system. Hence we may use the parabolic comparison principle to deduce that $U(x,t-nT) \ge u_m(x,t)$ in $\Omega^* \times (0,\infty)$ for all $m$, $n \ge 1$, where $u_m$ is the unique solution to

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega^* \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u = m & \text{on } \partial\Omega_0 \times (0,\infty), \\ u(x,0) = m_* & \text{in } \Omega^*, \end{cases} \tag{4.36}$$

with $m_* = \inf U \ge \frac{k+1}{2k}\min \underline{V}_a > 0$. As before we know that $\lim_{n\to\infty} u_m(x,t+nT) = u_a^m(x,t)$, which is the unique positive $T$-periodic solution of (4.1) with $m(x,t) \equiv m$. Thus $u_a^m(x,t) =$

$\lim_{n\to\infty} u_m(x, t + nT) \le U(x, t)$. Letting $m \to \infty$ we deduce $\underline{V}_a(x, t) \le U(x, t)$. But this is a contradiction with (4.34). This proves (4.30).

Using standard parabolic regularity theory and embedding theorems, we easily see that the convergence in (4.30) holds in $C^{2,1}(K \times [0, T])$ for any compact subset $K$ of $\overline{\Omega} \setminus \overline{\Omega}_0$. This finishes the proof of the theorem. □

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