A DIFFUSIVE LOGISTIC MODEL WITH A FREE BOUNDARY IN TIME-PERIODIC ENVIRONMENT

YIHONG DU[†], ZONGMING GUO[‡] AND RUI PENG[♯]

Abstract. We study the diffusive logistic equation with a free boundary in time-periodic environment. Such a model may be used to describe the spreading of a new or invasive species, with the free boundary representing the expanding front. For time independent environment, in the cases of one space dimension, and higher space dimensions with radial symmetry, this free boundary problem has been studied in [12, 9]. In both cases, a spreading-vanishing dichotomy was established, and when spreading occurs, the asymptotic spreading speed was determined. In this paper, we show that the spreading-vanishing dichotomy is retained in time-periodic environment, and we also determine the spreading speed. The former is achieved by further developing the earlier techniques, and the latter is proved by introducing new ideas and methods.

1. Introduction

We study the evolution of the positive solution $u(t,r)$ ($r=|x|$, $x\in\mathbb{R}^N,\ N\geq 2$), governed by the following diffusive logistic equation with a free boundary:

$$(1.1)\qquad \begin{cases} u_t - d\Delta u = u(\alpha(t,r)-\beta(t,r)u), & t>0,\ 0<r<h(t),\\ u_r(t,0)=0,\ u(t,h(t))=0, & t>0,\\ h'(t)=-\mu u_r(t,h(t)), & t>0,\\ h(0)=h_0, u(0,r)=u_0(r), & 0\leq r\leq h_0, \end{cases}$$

where $\Delta u = u_{rr}+\frac{N-1}{r}u_r$; $r=h(t)$ is the free boundary to be determined; h_0, μ and d are given positive constants; $u_0\in C^2([0,h_0])$ is positive in $[0,h_0)$ and $u_0'(0)=u_0(h_0)=0$; the functions $\alpha(t,r)$ and $\beta(t,r)$ satisfy the following conditions:

$$(1.2)\qquad \begin{cases} \text{(i)} & \alpha,\beta\in C^{\nu_0/2,\nu_0}(\mathbb{R}\times[0,\infty)) \text{ for some } \nu_0\in(0,1),\\ & \text{and are } T\text{-periodic in } t \text{ for some } T>0;\\ \text{(ii)} & \text{there are positive constants } \kappa_1,\ \kappa_2 \text{ such that}\\ & \kappa_1\leq\alpha(t,r)\leq\kappa_2,\ \kappa_1\leq\beta(t,r)\leq\kappa_2,\ \forall r\in[0,\infty),\ \forall t\in[0,T]. \end{cases}$$

Problem (1.1) may be viewed as describing the spreading of a new or invasive species with population density $u(t,|x|)$ over an N-dimensional habitat, which is radially symmetric but heterogeneous. The initial function $u_0(|x|)$ stands for the population in its early stage of introduction. Its spreading front is represented by the free boundary $|x|=h(t)$, which is a sphere

1991 *Mathematics Subject Classification.* 35K20, 35R35, 35J60, 92B05.

Key words and phrases. Diffusive logistic equation, free boundary, spreading-vanishing dichotomy, periodic environment, spreading speed.

This work was supported by the Australian Research Council and by NSFC of China (10871060, 10801090, 10871185, 10771032, 11271167, 11171319). R. Peng was also partially supported by the Program for New Century Excellent Talents in University and the Priority Academic Program Development of Jiangsu Higher Education Institutions. Y. Du thanks Prof. Xing Liang for useful discussions.

[†] School of Science and Technology, University of New England, Armidale, NSW 2351, Australia.

[‡] Department of Mathematics, Henan Normal University, Xinxiang, 453007, China.

[♯] Department of Mathematics, Jiangsu Normal University, Xuzhou, 221116, China.

Emails: ydu@turing.une.edu.au (Y. Du), gzm@htu.cn(Z. Guo), pengrui_seu@163.com(R. Peng).

$\partial B_{h(t)}$ with radius $h(t)$ growing at a speed proportional to the gradient of the population density at the front: $h'(t) = -\mu u_r(t, h(t))$. The coefficient functions $\alpha(t, |x|)$ and $\beta(t, |x|)$ represent the intrinsic growth rate of the species and its intra-specific competition respectively, and d is the random diffusion rate.

By restricting to the radially symmetric setting, we are able to avoid the difficult mathematical problem of regularity of the free boundary, and focus on the new phenomena exhibited by the free boundary model. The general (non-radial) case in several space dimensions was treated in [10] and [16] by completely different techniques.

In the special case that the functions α and β are independent of time t, problem (1.1) was studied recently in [9], and when α, β are positive constants and the space dimension is one, this problem was considered earlier in [12]. (Actually more general situations were investigated in [12], e.g., u_0 needs not be symmetric.) In both cases, it was shown that a unique solution pair (u, h) exists, with $u(t, r) > 0$ and $h'(t) > 0$ for $t > 0$ and $0 \le r < h(t)$, and a spreading-vanishing dichotomy holds, namely, a spatial barrier $r = R^*$ exists, such that either

- **Spreading:** the free boundary breaks the barrier at some finite time (i.e., $h(t_0) \ge R^*$ for some $t_0 \ge 0$), and then the free boundary goes to infinity as $t \to \infty$ (i.e., $\lim_{t\to\infty} h(t) = \infty$), and the population spreads to the entire space and stabilizes at its positive steady-state, or
- **Vanishing:** the free boundary never breaks the barrier ($h(t) < R^*$ for all $t > 0$), and the population vanishes ($\lim_{t\to\infty} u(t, r) = 0$).

Moreover, when spreading occurs, the asymptotic spreading speed can be determined (namely $\lim_{t\to\infty} h(t)/t$ exists and is uniquely determined).

The purpose of this paper is to examine (1.1) in the time-periodic case, a situation that more closely reflects the periodic variation of the natural environment, such as daily or seasonal changes. We will show that the above spreading-vanishing dichotomy is retained in the time-periodic setting, and by introducing new ideas and techniques we also determine the spreading speed.

In most spreading processes in the natural world, a spreading front can be observed. Under the assumption of radial symmetry and logistic growth law, if a new or invasive species initially occupies a spherical region $\{|x| < h_0\}$ with density $u_0(|x|)$, then as time t increases from 0, it is natural to expect that the boundary of the initial region evolves into an invading front, which encloses an expanding ball $\{|x| < h(t)\}$ inside which the initial function $u_0(|x|)$ evolves into a positive function governed by the logistic equation $u_t - d\Delta u = u(\alpha - \beta u)$, with u vanishing on $\{|x| = h(t)\}$. To determine the evolution of the front $\{|x| = h(t)\}$ with time, we assume as in [12, 9] that the front invades at a speed that is proportional to the spatial gradient of the density function u there, which gives rise to the free boundary condition in (1.1). A deduction of this free boundary condition based on ecological assumptions can be found in [7].

The investigation of front propagation has a long history. A considerable amount of work is based on the following diffusive logistic equation over the entire space $\mathbb{R}^N$:

$$u_t - d\Delta u = u(a - bu),\ t > 0,\ x \in \mathbb{R}^N, \tag{1.3}$$

with d, a and b positive constants. In the pioneering works of Fisher [18] and Kolmogorov et al [23], for space dimension $N = 1$, traveling wave solutions have been found for (1.3). For any $c \ge c^* := 2\sqrt{ad}$, there exists a solution $u(t, x) := W(x - ct)$ with the property that

$$W'(y) < 0 \text{ for } y \in \mathbb{R}^1,\ W(-\infty) = a/b,\ \ W(+\infty) = 0;$$

no such solution exists if $c < c^*$. The number c^* is called the minimal speed of the traveling waves. Fisher [18] claims that c^* is the spreading speed for the advantageous gene in his research, and used a probabilistic argument to support his claim. The first well known ecological example exhibiting a linear spreading rate in time is due to Skellam [31]. He considered the case of spreading of muskrat in Europe in the early 1900s: he calculated the area of the muskrat range from a map obtained from field data, took the square root and plotted it against years, and found that the data points lay on a straight line. (Further ecological examples obeying this linear spreading rule may be found in [30].) Skellam [31] used a linear model (i.e., (1.3) with $b = 0$) and a probabilistic consideration to argue that c^* should be the speed of spreading. A clear description and rigorous proof of this fact were given by Aronson and Weinberger (see Section 4 in [1]), who showed that for a new population $u(t, x)$ (governed by the above logistic equation) with initial distribution $u(0, x)$ confined to a compact set of x (i.e., $u(0, x) = 0$ outside a compact set), one has

$$\lim_{t\to\infty,\ |x|\le(c^*-\epsilon)t} u(t,x) = a/b, \qquad \lim_{t\to\infty,\ |x|\ge(c^*+\epsilon)t} u(t,x) = 0$$

for any small $\epsilon > 0$, where the convergence is uniform in the indicated range of x. These results have been extended to higher dimensions in [2], and extensive further development on traveling wave solutions and the spreading speed has been achieved in several directions, in particular, to situations of various heterogeneous environments; see, for example, [3, 4, 5, 6, 25, 32, 33] and the references therein for more details.

Generally speaking, the Cauchy problem is the limiting problem of the corresponding free boundary model as $\mu \to \infty$. This was shown in [10] in a very general setting. If $\mu = 0$, clearly the free boundary problem reduces to a fixed boundary problem with Dirichlet boundary conditions.

The enormous success of (1.3) nevertheless carries a shortcoming. The above conclusion for (1.3) predicts successful spreading and establishment of the new species with any nontrivial initial population $u(0, x)$, regardless of its initial size and supporting area. However, this is not supported by empirical evidences, which suggest, in the contrary, that success of spreading is dependent on the initial size of the population; for example, the introduction of several bird species from Europe to North America in the 1900s was successful only after many initial attempts (cf. [30] and [26], where more examples can be found).

This defect of (1.3) can be removed if the logistic nonlinear term in the equation is replaced by a bistable one (see, e.g., [24]), to represent an Allee effect on the growth rate of the species. A typical bistable $f(u)$ is $u(u - \theta)(1 - u)$ with $\theta \in (0, 1/2)$. It is well known that (1.3) with a bistable nonlinearity has traveling wave solutions only for one wave speed c_*, and the unique solution of the Cauchy problem with large nonnegative initial u_0 converges to 1 with spreading speed c_* as $t \to \infty$, and for small u_0 the solution converges to 0; see [2]. In one space dimension, it was shown in [15] that as the nonnegative initial function u_0 (with compact support) is varied, exactly three types of behavior can be observed for the unique solution u of the Cauchy problem: $\lim_{t\to\infty} u(t, x) = 0$, $\lim_{t\to\infty} u(t, x) = 1$ or $\lim_{t\to\infty} u(t, x) = v(x)$, where $v(x)$ is a ground state solution of $-dv_{xx} = f(v)$ in $\mathbb{R}^1$, namely it is positive and decays to 0 at $\pm\infty$. Moreover, the third type of behavior occurs as an exceptional case; roughly speaking, if the initial function u_0 is properly parameterized by a parameter λ, then this type of behavior only occurs at a threshold value λ^* of the parameter.

Our results in [12, 9] and in this paper indicate that the above mentioned shortcoming of (1.3) does not appear even with the original logistic nonlinearity, if instead of the Cauchy problem, one uses the corresponding free boundary model to describe the spreading process. However, in contrast to the above mentioned trichotomy of [15] in one space dimension with a bistable nonlinearity associated with the Cauchy problem, the free boundary model with a logistic type nonlinearity exhibits a spreading-vanishing dichotomy. Furthermore, unlike the Cauchy problem model in which the spreading front is represented by an unspecified level set of the solution, the free boundary model gives a precise location of the spreading front for any given time. We note that the important feature of (1.3), namely the spreading front invades at a linear rate in time, is retained by the free boundary model.

We now describe the main results of this paper.

Theorem 1.1. (Existence and uniqueness) *Problem* (1.1) *admits a unique solution* $(u(t,r),h(t))$, *which is defined for all* $t>0$. *Moreover,* $h\in C^1([0,\infty))$, $u\in C^{1,2}(D)$ *with* $D=\{(t,r):t>0, 0\le r\le h(t)\}$, *and* $u(t,r)>0$ *for* $t>0$ *and* $0\le r<h(t)$, $h'(t)>0$ *for* $t>0$.

Theorem 1.2. (Spreading-vanishing dichotomy) *Let* $(u(t,r),h(t))$ *be the solution of* (1.1). *Then the following alternative holds:*

Either

(i) Spreading: $\lim_{t\to\infty} h(t)=+\infty$ *and*

$$\lim_{t\to\infty}|u(t,r)-\hat{U}(t,r)|=0 \ \ \text{locally uniformly for } r\in[0,\infty),$$

where $\hat{U}(t,|x|)$ *is the unique positive* T*-periodic solution of*

$$U_t-d\Delta U=U[\alpha(t,|x|)-\beta(t,|x|)U],\quad (t,x)\in\mathbb{R}^1\times\mathbb{R}^N,$$

or

(ii) Vanishing: $\lim_{t\to\infty} h(t)\le R^*$ *and* $\lim_{t\to+\infty}\ ||u(t,\cdot)||_{C([0,h(t)])}=0$, *where* $R^*>0$ *is the unique value such that the following linear problem has a positive* T*-periodic solution when* $R=R^*$*:*

$$\begin{cases}\phi_t-d\Delta\phi &= \alpha(t,|x|)\phi \quad \text{for } t\in\mathbb{R}^1 \text{ and } |x|<R,\\ \phi &= 0 \quad\quad\quad\quad\ \text{for } t\in\mathbb{R}^1 \text{ and } |x|=R.\end{cases}$$

Theorem 1.3. (Spreading-vanishing criteria)

(a) *If* $h_0\ge R^*$, *then spreading always occurs.*

(b) *If* $h_0<R^*$, *then there exists a unique* $\mu^*>0$ *depending on* u_0 *such that vanishing occurs if* $0<\mu\le\mu^*$, *and spreading happens if* $\mu>\mu^*$.

Let us note that, when $h_0<R^*$, since μ^* varies with u_0, for fixed μ, whether spreading or vanishing happens depends on the size of u_0.

Theorem 1.4. (Spreading speed and profile) *Suppose that*

$$\lim_{r\to\infty}\alpha(t,r)=\alpha_*(t),\ \lim_{r\to\infty}\beta(t,r)=\beta_*(t)$$

uniformly for $t\in[0,T]$. *Then in the case of spreading, there exists a positive* T*-periodic function* $k_0(t)$ *such that*

$$\lim_{t\to\infty}\frac{h(t)}{t}=\overline{k}_0:=\frac{1}{T}\int_0^T k_0(t)dt.$$

Moreover, for any $c\in(0,\overline{k}_0)$, *we have*

$$\lim_{t\to\infty}\max_{0\le r\le ct}|u(t,r)-\hat{U}(t,r)|=0.$$

Remark 1.5. *Clearly* $\overline{k}_0$ *depends on* μ. *If we denote* $\overline{k}_0=\overline{k}_0(\mu)$ *to stress this dependence, we will show that* $\overline{k}_0(\mu)<2\sqrt{\overline{\alpha}_*d}$ *and* $\lim_{\mu\to+\infty}\overline{k}_0(\mu)=2\sqrt{\overline{\alpha}_*d}$, *where* $\overline{\alpha}_*=T^{-1}\int_0^T\alpha_*(t)dt$.

The periodic function $k_0(t)$ is uniquely determined. This will be shown by a new approach. First, the existence of $k_0(t)$ is proved by the Schauder fixed point theorem, for a nonlinear operator arising in the following way. Given a nonnegative T-periodic Hölder continuous function $k(t)$, find a positive solution $U(t,r)$ to

$$\begin{cases} U_t - dU_{rr} + k(t)U_r = U[\alpha_*(t) - \beta_*(t)U], & (t,r) \in \mathbb{R}^1 \times (0,\infty), \\ U(t,0) = 0,\ U(t,r) = U(t+T,r), & t \in \mathbb{R}^1,\ r > 0. \end{cases} \tag{1.4}$$

If we denote by U^k such a positive solution (when exists), and define an operator A acting on nonnegative T-periodic functions by

$$Ak(t) = \mu U_r^k(t,0),$$

then k_0 will be a fixed point of A, and hence satisfies

$$k_0(t) = \mu U_r^{k_0}(t,0)\ \forall t \in \mathbb{R}^1. \tag{1.5}$$

Second, the uniqueness of such $k_0(t)$ and its dependence on the parameter μ are established by a new device, which turns out to be useful also for the study of the space-periodic case of the free boundary model [11], and for the study of a seasonal succession model [29]. We believe that these ideas may have further applications in related problems.

Let us note that if a T-periodic function $k(t)$ gives rise to a positive solution U^k of (1.4), then U^k can be used to generate a family of one dimensional "semi-waves" in the following way. Define, for each constant $c \in \mathbb{R}^1$,

$$K(t) = \int_0^t k(s)ds + c,\ \ V(t,x_1) = U^k(t, K(t) - x_1).$$

Then V satisfies, for $t \in \mathbb{R}^1$,

$$V_t - dV_{x_1x_1} = V[\alpha_*(t) - \beta_*(t)V], V > 0 \text{ for } x_1 < K(t),\ V(t,K(t)) = 0.$$

Thus as t increases, V behaves like a wave traveling to the positive direction of x_1, with the front at $x_1 = K(t)$ moving at the T-periodic speed $k(t)$. Since for fixed t, $V(t,x_1)$ is defined only on the half-line $x_1 \le K(t)$, it makes sense to call it a semi-wave. We will also call the profile of V, $U^k(t,r)$, a semi-wave. U^k clearly also generates a family of semi-waves traveling to the negative direction of x_1:

$$\tilde{V}(t,x_1) = U^k(t, x_1 + K(t)).$$

Note that the semi-wave V generated by U^{k_0} has the extra property that, at the front $x_1 = K_0(t) := \int_0^t k_0(s)ds + c$,

$$K_0'(t) = -\mu V_{x_1}(t, K_0(t)),$$

that is, the movement of the front of this particular semi-wave satisfies the 1-d free boundary condition. This property of $k_0(t)$ will allow us to construct upper and lower solutions to (1.1) based on suitable variations of U^{k_0} to determine the spreading speed.

Most of the innovations of this paper are contained in sections 2 and 4, with section 3 consisting of extensions of earlier techniques (except Lemma 3.10), where the proofs are often brief or omitted whenever possible. More specifically, in section 2, by introducing a completely new approach we study the semi-waves determined by (1.4): we prove the existence and uniqueness of $k_0(t)$, and investigate the dependence of $k_0(t)$ on $a(t), b(t)$ and μ. Section 3 is devoted to the proof of Theorems 1.2 and 1.3, by establishing a more general version of Theorem 1.1, and by extending many techniques in [9]. In section 4, we prove Theorem 1.4, based on our results established in section 2, and some new techniques.

We end this section by mentioning some recent related research. In [21], some of the results of [12] were extended to the case that the solution satisfies a Dirichlet boundary condition at $x = 0$

and a free boundary condition at $x = h(t)$, covering monostable and bistable nonlinearities. In [19], a competition system with free boundary conditions was investigated. In [13], the results of [12] were extended to one-dimensional free boundary problems with general nonlinear terms, including nonlinearities of monostable, bistable and combustion types. Sharper estimate of the spreading speed was obtained in [17] under the general setting of [13]. In [16], a general nonlinear Stefan problem in high space dimension without any symmetric assumption on the initial function or the free boundary was considered, and the regularity of the free boundary and the long-time behavior was investigated; in particular, it was shown that the spreading-vanishing dichotomy of [12] in one-space dimension remains valid in this general high dimension setting.

2. Semi-waves

The main purpose of this section is to prove the existence and uniqueness of a positive T-periodic function $k_0(t)$ so that (1.4) has a positive solution when $k = k_0$, and it satisfies (1.5). We also study how $k_0(t)$ varies as a, b and μ change.

So we consider the following problem

$$\begin{cases} U_t - dU_{rr} + k(t)U_r = U[a(t) - b(t)U], & (t, r) \in [0, T] \times (0, \infty), \\ U(t, 0) = 0, & t \in [0, T], \\ U(0, r) = U(T, r), & r \in (0, \infty), \end{cases} \tag{2.1}$$

where $d > 0$ is a given constant, and k, a, b are given T-periodic Hölder continuous functions with a, b positive and k nonnegative. Our first result on (2.1) is the following.

Proposition 2.1. *For any given positive T-periodic functions a, $b \in C^{\nu_0/2}([0, T])$ and any nonnegative continuous T-periodic function $k(t)$ in $C^{\nu_0/2}([0, T])$, problem* (2.1) *admits a maximal nonnegative T-periodic solution $U^k(t, r)$. Moreover, either $U^k \equiv 0$ or $U^k > 0$ in $[0, T] \times (0, \infty)$. Furthermore, if $U^k > 0$, then it is the only positive solution of* (2.1)*, $U^k_r(t, r) > 0$ in $[0, T] \times [0, \infty)$ and $U^k(t, r) \to V(t)$ uniformly for $t \in [0, T]$ as $r \to +\infty$, where $V(t)$ is the unique positive solution of the problem*

$$\frac{dV}{dt} = V[a(t) - b(t)V] \ \ \text{in } [0, T], \quad V(0) = V(T); \tag{2.2}$$

in addition, for any given nonnegative T-periodic function k_1 in $C^{\nu_0/2}([0, T])$, the assumption $k_1 \leq, \not\equiv k$ implies

$$U^{k_1}_r(t, 0) > U^k_r(t, 0), \quad U^{k_1}(t, r) > U^k(t, r) \quad \text{for } t \in [0, T] \text{ and } r \in (0, \infty).$$

Proof. We divide the proof of this proposition into several steps.

Step 1. Problem (2.1) always has a maximal nonnegative solution $\overline{U}$, and it satisfies

$$\overline{U}(t, r) \leq \frac{\max_{t \in [0,T]} a(t)}{\min_{t \in [0,T]} b(t)} := C_0, \forall (t, r) \in [0, T] \times [0, \infty). \tag{2.3}$$

Clearly 0 is always a nonnegative solution of (2.1). We show next that it has a maximal nonnegative solution and (2.3) holds.

To this end, for any given constant $\ell > 0$, we consider the following boundary blow-up problem:

$$\begin{cases} -du_{rr} = u\{[\max_{t \in [0,T]} a(t)] - [\min_{t \in [0,T]} b(t)]u\} & \text{in } (0, \ell), \\ u(0) = 0, \quad u(\ell) = \infty. \end{cases} \tag{2.4}$$

It is easily seen from arguments similar to those in the proof of Lemma 2.3 of [14] that, for any $\ell > 0$, problem (2.4) admits a unique positive solution u^ℓ and u^ℓ decreases to u^∞ uniformly over any bounded interval $[0, R]$ as ℓ increases to ∞. Moreover, u^∞ is the unique positive solution of

(2.4) with $\ell = \infty$, and $u^\infty < C_0$ in $[0,\infty)$. By a simple moving plane consideration we also see that $u^\infty_r(r) > 0$ for $r \in [0,\infty)$. Clearly u^∞ is an upper solution to the problem

$$
(2.5)\qquad \begin{cases} w_t - dw_{rr} + k(t)w_r = w[a(t) - b(t)w], & (t,r) \in [0,T] \times (0,\ell),\\ w(t,0) = 0, & t \in (0,T),\\ w(t,\ell) = u^\infty(\ell), & t \in [0,T],\\ w(t,r) \text{ is } T\text{-periodic in } t. \end{cases}
$$

Since 0 is a lower solution to (2.5), we find by standard upper and lower solution argument that (2.5) has at least one positive solution U^ℓ and $U^\ell \le u^\infty$. Since the right hand side of the first equation in (2.5) is concave in w, by a standard argument (along the lines of Step 4 below) we find that U^ℓ is the unique positive solution of (2.5). One may then use a simple upper and lower solution argument to (2.5) and the monotonicity of $u^\infty(r)$ to deduce that U^ℓ is decreasing in ℓ and $U^\ell \to \overline{U}$ as $\ell \to \infty$, where $\overline{U}$ is a nonnegative solution of (2.1). Clearly $\overline{U} \le u^\infty < C_0$.

It remains to show that $\overline{U}$ is the maximal nonnegative solution of (2.1). Let U be an arbitrary nonnegative solution of (2.1). If $U \equiv 0$ then clearly $U \le \overline{U}$. Suppose now $U \ge, \not\equiv 0$. Then $U(t,r) > 0$ in $[0,T] \times (0,\infty)$ due to the strong maximum principle of parabolic equations. We show next that $U(t,r) \le \overline{U}(t,r)$ for $(t,r) \in [0,T] \times [0,\infty)$.

Firstly for fixed $\ell > 0$ we can find $M > 0$ large such that $Mu^\ell(r) \ge U(t,r)$ for $(t,r) \in [0,T] \times [0,\ell)$. We claim that the above inequality also holds for $M = 1$. Otherwise let M_0 be the infimum of the set of M for which this inequality holds, then $M_0 > 1$. Since $M_0 u^\ell \ge, \not\equiv U$, we can apply the strong maximum principle to deduce that $M_0 u^\ell(r) > U(t,r)$ and $M_0 u^\ell_r(0) > U_r(t,0)$ for $r \in (0,\ell)$ and $t > 0$. Since U is periodic in t, this implies that there exists $M_1 < M_0$ such that $Mu^\ell \ge U$ in $[0,T] \times [0,\ell)$ for all $M \ge M_1$, which contradicts the definition of M_0. Thus we have proved that $u^\ell \ge U$ in $[0,T] \times [0,\ell)$. Letting $\ell \to \infty$ we deduce $u^\infty \ge U$ in $[0,T] \times [0,\infty)$. It follows that U is always a lower solution to (2.5), which implies $U \le U^\ell$ in $[0,T] \times [0,\ell]$, due to the uniqueness of U^ℓ. Letting $\ell \to \infty$, we deduce $U \le \overline{U}$, as we wanted. This completes the proof of step 1.

Step 2. For any given nonnegative T-periodic and continuous function $k(t)$, we claim that $U_r(t,r) > 0$ in $[0,T] \times [0,\infty)$ whenever U is a positive solution of (2.1).

We use the moving plane argument to prove the conclusion here. It follows from the Hopf boundary lemma for parabolic equations that $U_r(t,0) > 0$ for $t \in [0,T]$. Thus, setting

$$
\Lambda = \Big\{\lambda > 0 :\ U(t, 2\lambda - r) > U(t,r) \text{ for } (t,r) \in [0,T] \times [0,\lambda) \text{ and } U_r(t,r) > 0 \text{ for } (t,r) \in [0,T] \times [0,\lambda]\Big\}
$$

we see that Λ contains all sufficiently small $\lambda > 0$. Let $\lambda^* := \sup \Lambda$. We show that $\lambda^* = \infty$, which would imply our claim in this step.

Suppose by way of contradiction that $\lambda^* \in (0,\infty)$. Then

$$
U(t, 2\lambda^* - r) \ge U(t,r) \text{ and } U_r(t,r) \ge 0 \text{ for } (t,r) \in [0,T] \times [0,\lambda^*].
$$

Define $\tilde{V}(t,r) = U(t, 2\lambda^* - r)$ for $r \in [\lambda^*, 2\lambda^*]$. Then

$$
\tilde{V}_t - d\tilde{V}_{rr} + k(t)\tilde{V}_r = \tilde{V}[a(t) - b(t)\tilde{V}] - 2k(t)U_\xi, \ \ \xi = 2\lambda^* - r \in [0,\lambda^*].
$$

Now we set

$$
W(t,r;\lambda^*) = W(t,r) = \tilde{V}(t,r) - U(t,r) = U(t,\xi) - U(t, 2\lambda^* - \xi).
$$

Then $W \le 0$ for $(t,r) \in [0,T] \times [\lambda^*, 2\lambda^*]$, and it satisfies

$$\begin{cases} W_t - dW_{rr} + k(t)W_r + c(t,r)W = -2k(t)U_\xi \le 0, & (t,r) \in [0,T] \times [\lambda^*, 2\lambda^*], \\ W(t,\lambda^*) = 0, & t \in [0,T], \\ W(t,2\lambda^*) = -U(t,2\lambda^*) < 0, & t \in [0,T], \\ W(0,r) = W(T,r), & r \in [\lambda^*, 2\lambda^*], \end{cases} \tag{2.6}$$

where $c(t,r) := -[a(t) - b(t)(\tilde{V}(t,r) + U(t,r))]$ is a bounded and T-periodic function on $[0,T] \times [\lambda^*, 2\lambda^*]$ due to the assertion of step 1. Since W is periodic in t, the strong maximum principle and Hopf boundary lemma then imply that

$$W(t,r) < 0 \ \ \forall (t,r) \in [0,T] \times (\lambda^*, 2\lambda^*], \tag{2.7}$$

and

$$W_r(t,\lambda^*) < 0 \ \text{ for } t \in [0,T]. \tag{2.8}$$

By continuity, for all small $\epsilon \ge 0$,

$$W_r(t, \lambda^* + \epsilon; \lambda^* + \epsilon) < 0 \ \text{ for } t \in [0,T] \tag{2.9}$$

and

$$W(t,r;\lambda^* + \epsilon) < 0 \ \ \forall (t,r) \in [0,T] \times (\lambda^* + \epsilon, 2\lambda^* + 2\epsilon]. \tag{2.10}$$

It follows that

$$U(t, 2\lambda^* + 2\epsilon - \xi) > U(t,\xi) \ \text{ for } (t,\xi) \in [0,T] \times [0, \lambda^* + \epsilon),$$

and since $W_r(t, \lambda^* + \epsilon; \lambda^* + \epsilon) = -2U_r(t, \lambda^* + \epsilon)$, we see from (2.9) that

$$U_r(t, \lambda^* + \epsilon) > 0 \ \ \forall t \in [0,T].$$

But these facts contradict the definition of λ^*. This completes the proof of step 2.

Step 3. We obtain the asymptotic behavior of positive solution U of (2.1) as $r \to \infty$.

In view of steps 1 and 2, there exists $V(t)$ such that

$$\lim_{r \to +\infty} U(t,r) = V(t) \ \ \forall t \in [0,T].$$

Moreover, $V(t)$ is a positive T-periodic function. For any sequence $\{r_n\}$ with $r_n \to +\infty$ as $n \to \infty$, we define $U_n(t,r) = U(t, r_n + r)$. Then U_n solves the same equation as U but over $(0,T) \times (-r_n, \infty)$. Since $U_n \le C_0$, the standard regularity argument allows us to conclude that we can extract a subsequence of $\{U_n\}$ (still denoted by $\{U_n\}$) such that

$$U_n \to \tilde{U} \ \text{ locally in } C^{1,2}([0,T] \times (-\infty, \infty)) \text{ as } n \to \infty$$

and $\tilde{U}$ is a positive solution of

$$\begin{cases} w_t - dw_{rr} + k(t)w_r = w[a(t) - b(t)w], & (t,r) \in (0,T) \times (-\infty, \infty), \\ w(0,r) = w(T,r), & r \in (-\infty, \infty). \end{cases}$$

On the other hand, it follows from $\lim_{r \to +\infty} U(t,r) = V(t)$ that

$$\lim_{n \to \infty} U_n(t,r) = \lim_{n \to \infty} U(t, r_n + r) = V(t).$$

This implies that

$$\tilde{U} \equiv V.$$

Therefore, $V > 0$ satisfies

$$\frac{dV}{dt} = V[a(t) - b(t)V] \ \text{ in } [0,T], \quad V(0) = V(T). \tag{2.11}$$

It is well known that (2.11) has a unique positive solution.

Step 4. We show that (2.1) has at most one positive solution.

Suppose that (2.1) admits two positive T-periodic solutions U_1 and U_2. By the conclusions of step 3 and the Hopf boundary lemma, we can choose $M > 1$ such that $M^{-1}U_1 < U_i < MU_1$ in $[0, T] \times (0, \infty)$ for $i = 1, 2$. Note that $(M^{-1}U_1, MU_1)$ is a pair of sub- and supersolutions of (2.1). By the sub- and supersolution argument similar to [22] on unbounded spatial domains, (2.1) possesses a minimal and a maximal solution in the order interval $[M^{-1}U_1, MU_1]$, which are denoted by U_* and U^* respectively. Therefore,

$$U_* \le U_i \le U^* \text{ in } [0, T] \times (0, \infty),\ i = 1, 2.$$

To obtain $U_1 \equiv U_2$, we only need to show

$$U_* \equiv U^*.$$

Define

$$\sigma_* := \inf\{\sigma \in \mathbb{R} : U^* \le \sigma U_* \text{ in } [0, T] \times (0, \infty)\}.$$

Clearly $\sigma_* \ge 1$ and $U^* \le \sigma_* U_*$. To prove $U^* = U_*$, it suffices to show $\sigma_* = 1$. Suppose for contradiction that $\sigma_* > 1$. Then for $W(t, r) := \sigma_* U_* - U^*$ we have $W \ge 0$, $W(0, r) = W(T, r)$, and

$$W_t - dW_{rr} + k(t)W_r = a(t)W - b(t)[\sigma_*(U_*)^2 - (U^*)^2] \ge (a(t) - b(t)U^*)W$$

for $(t, r) \in [0, T] \times (0, \infty)$, and $W(t, 0) = 0$, $W(t, \infty) = (\sigma_* - 1)V(t) > 0$ on $[0, T]$. Thus, we can use the strong maximum principle and Hopf boundary lemma to deduce that $W \ge \epsilon U^*$ in $[0, T] \times [0, \infty)$ for some $\epsilon > 0$ small, and this implies that

$$U^* \le (1 + \epsilon)^{-1} \sigma_* U_* \ \ \forall (t, r) \in [0, T] \times (0, \infty).$$

This contradicts the definition of σ_*. Thus, it is necessary that $\sigma_* = 1$, and the uniqueness is established.

Step 5. Monotonicity in k. Assume that U^k is a positive solution of (2.1) and k_1 is a T-periodic continuous function satisfying $0 \le k_1 \le, \not\equiv k$. Let U^{k_1} be the maximal nonnegative solution of (2.1) with $k = k_1$. The conclusion of step 1 implies that any large constant $C > C_0$ is a supersolution for (2.1) with $k = k_1$, and by step 2 we see that U^k is a subsolution to this equation and $U^k \le C_0 < C$. Consequently, together with the uniqueness of positive solution to (2.1) with $k = k_1$, we see that

$$U^{k_1} \ge U^k \ \ \forall (t, r) \in [0, T] \times (0, \infty).$$

The strong maximum principle implies that

$$U^{k_1}(t, r) > U^k(t, r) \ \ \forall (t, r) \in [0, T] \times (0, \infty).$$

Moreover, the Hopf boundary lemma yields

$$U_r^{k_1}(t, 0) > U_r^k(t, 0) \ \text{ for } t \in [0, T].$$

This completes the proof of the proposition. □

Next we give a necessary and sufficient condition for the existence of a positive solution to (2.1). We will need some results of Nadin in [27] on the principal eigenvalue of linear periodic-parabolic operators.

For any given T-periodic functions $p, q \in C^{\nu_0/2}([0, T])$, we consider the linear periodic-parabolic operator

$$\mathcal{L}\phi = \phi_t - d\phi_{rr} + q(t)\phi_r + p(t)\phi \ \text{ for } \phi \in C^{1,2}([0, T] \times \mathbb{R}),$$

and the corresponding generalized principal eigenvalue $\lambda_1(\mathcal{L})$ defined by

$$\lambda_1(\mathcal{L}) = \sup\{\lambda \in \mathbb{R} : \ \exists \phi \in C^{1,2}([0,T]\times\mathbb{R}),\ \phi \text{ is } T\text{-periodic}, \ \phi > 0 \ \text{ and } \mathcal{L}\phi \geq \lambda\phi \text{ in } [0,T]\times\mathbb{R}\}. \tag{2.12}$$

By Propositions 2.3 and 2.4 of [27], if we denote by $\lambda_1^n(\mathcal{L})$ the principal eigenvalue of

$$\begin{cases} \mathcal{L}\phi = \lambda\phi, & (t,r)\in[0,T]\times(-n,n),\\ \phi(t,\pm n) = 0, & t\in[0,T],\\ \phi(0,r)=\phi(T,r), & r\in(-n,n), \end{cases} \tag{2.13}$$

then

$$\lambda_1^n(\mathcal{L}) \to \lambda_1(\mathcal{L}) \text{ as } n\to\infty, \tag{2.14}$$

and moreover, $\lambda_1(\mathcal{L})$ corresponds to a generalized principal eigenfunction $\phi_1 \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi_1 > 0$ in $[0,T]\times\mathbb{R}$ and

$$\begin{cases} \mathcal{L}\phi_1 = \lambda_1(\mathcal{L})\phi_1, & (t,r)\in[0,T]\times\mathbb{R},\\ \phi_1(0,r)=\phi_1(T,r), & r\in\mathbb{R}. \end{cases} \tag{2.15}$$

If further we assume

$$\overline{q} := \frac{1}{T}\int_0^T q(t)dt = 0, \tag{2.16}$$

then we can apply Theorems 2.7, 2.13 and Proposition 2.14[1] of [27] to obtain the following result.

Proposition 2.2. *If* (2.16) *holds, then the principal eigenfunction of* (2.15) *can be chosen to be a positive function which is T-periodic in t and independent of r.*

Thus when (2.16) holds, we can choose $\phi_1 = \phi_1(t)$, and obtain from (2.15)

$$\phi_1' + p(t)\phi_1 = \lambda_1(\mathcal{L})\phi_1,\ \phi_1(0)=\phi_1(T).$$

It follows easily that

$$\lambda_1(\mathcal{L}) = \overline{p},\ \phi_1(t) = \phi_1(0)e^{\int_0^t[\overline{p}-p(s)]ds}. \tag{2.17}$$

We are now ready to obtain a necessary and sufficient condition for (2.1) to have a positive solution.

Proposition 2.3. *Under the assumptions of Proposition 2.1, problem* (2.1) *admits a positive solution $U \in C^{1,2}([0,T]\times[0,\infty))$ if and only if $\overline{a} > \overline{k}^2/(4d)$.*

Proof. Write

$$k(t) = \overline{k} + \tilde{k}(t), \ \text{ so } \int_0^T \tilde{k}(t)dt = 0.$$

Then take

$$q(t) = \tilde{k}(t),\ p(t) = \frac{\overline{k}^2}{4d} + \frac{\overline{k}}{2d}\tilde{k}(t) - a(t)$$

in the operator $\mathcal{L}$. Clearly Proposition 2.2 applies and thus

$$\lambda_1(\mathcal{L}) = \overline{p} = \frac{\overline{k}^2}{4d} - \overline{a}.$$

[1]The proof of Proposition 2.14 in [27] contains a gap (the function $Q^i(t,x)$ defined at the end of page 286 in [27] may not be a constant in general), and it is unclear whether all the conclusions in this proposition hold true as stated. However, it is easy to use the ideas in [27] to show that the conclusions hold for the special case used in this paper here.

Firstly we show that when $\overline{a} > \overline{k}^2/(4d)$, (2.1) has a positive solution. In this case, we have $\lambda_1(\mathcal{L}) < 0$. By (2.14), for all large n, $\lambda_1^n(\mathcal{L}) < 0$. Let $\phi^n(t,r)$ be the corresponding positive eigenfunction of $\lambda_1^n(\mathcal{L})$, and define $\psi^n(t,r) = e^{\frac{\overline{k}}{2d}r}\phi^n(t, r-n)$. Then a simple calculation shows

$$\psi_t^n(t,r) - d\psi_{rr}^n(t,r) + k(t)\psi_r^n(t,r) - a(t)\psi^n(t,r) = e^{\frac{\overline{k}}{2d}r}\mathcal{L}\phi^n(t,r-n),$$

and thus

$$\begin{cases} \psi_t^n - d\psi_{rr}^n + k(t)\psi_r^n - a(t)\psi^n = \lambda_1^n(\mathcal{L})\psi^n, & (t,r) \in [0,T]\times(0,2n), \\ \psi^n(t,0) = \psi^n(t,2n) = 0, & t \in [0,T], \\ \psi^n(0,r) = \psi^n(T,r), & r \in (0,2n). \end{cases} \tag{2.18}$$

We now fix n such that $\lambda_1^n(\mathcal{L}) < 0$, and then choose $\epsilon_0 > 0$ sufficiently small so that $\epsilon_0\psi^n < C_0$ on $[0,T]\times[0,2n]$. Denote

$$\underline{U} = \begin{cases} \epsilon\psi^n, & (t,r) \in [0,T]\times[0,2n], \\ 0, & (t,r) \in [0,T]\times(2n,\infty). \end{cases}$$

Then $\underline{U}$ is a subsolution of (2.1) for all sufficiently small $\epsilon \in (0,\epsilon_0]$. For every $C > C_0$ clearly $\overline{U} \equiv C$ is a supersolution to (2.1). Evidently

$$\underline{U} < C \ \ \forall (t,r) \in [0,T]\times[0,\infty).$$

Therefore, it follows from the sub- and supersolution argument (see, e.g. [22]) that (2.1) admits at least one nontrivial nonnegative solution, which is the unique positive solution of (2.1) due to Proposition 2.1.

Next we show that (2.1) does not admit a positive solution when $\overline{a} \le \overline{k}^2/(4d)$. In this case, we have $\lambda_1(\mathcal{L}) \ge 0$, and by Proposition 2.2, there is a positive T-periodic function $\phi_1(t)$ satisfying

$$\mathcal{L}\phi_1 = \lambda_1(\mathcal{L})\phi_1.$$

Define

$$\psi_1(t,r) = e^{\frac{\overline{k}}{2d}r}\phi_1(t);$$

then one easily checks that

$$(\psi_1)_t - d(\psi_1)_{rr} + k(t)(\psi_1)_r - a(t)\psi_1 = \lambda_1(\mathcal{L})\psi_1.$$

Since $\overline{k} \ge 2\sqrt{\overline{a}d} > 0$, we have

$$\psi_1(t,r) \to +\infty \ \text{ uniformly on } [0,T] \text{ as } r \to +\infty,$$

and

$$\psi_1(t,r) \ge \min_{t\in[0,T]} \phi_1(t) > 0 \text{ in } [0,T]\times[0,\infty).$$

Suppose by way of contradiction that (2.1) admits a positive solution U. Then we know from step 1 in the proof of Proposition 2.1 that

$$U \le C_0 \ \text{ for } (t,r) \in [0,T]\times[0,\infty).$$

Define the set

$$\Sigma = \{\tau \in (0,\infty) : \ \tau\psi_1(t,r) \ge U(t,r) \ \text{ for } (t,r) \in [0,T]\times[0,\infty)\}.$$

Clearly $\Sigma \ne \emptyset$ and it is relatively closed in $(0,\infty)$. We are going to show that Σ is also open. Assume that $\tau_0 \in \Sigma$. Then, $\tau_0\psi_1 \ge U$. The fact that $\psi_1(t,r) \to +\infty$ as $r \to \infty$ uniformly for $t \in [0,T]$ enables us to find a large $\kappa_0 > 0$ and a small $\epsilon_0 > 0$ such that $(\tau_0 - \epsilon)\psi_1 > U$ for

$(t,r) \in [0,T] \times [\kappa_0, \infty)$ for all $\epsilon \in (0, \epsilon_0]$. Let $Z = \tau_0\psi_1 - U$. Since $\lambda_1(\mathcal{L}) \geq 0$, we see that for $(t,r) \in [0,T] \times (0, \kappa_0)$ and $\tau > 0$,

$$(\tau\psi_1)_t - d(\tau\psi_1)_{rr} + k(t)(\tau\psi_1)_r - a(t)(\tau\psi_1) + b(t)(\tau\psi_1)^2 \geq \lambda_1(\mathcal{L})(\tau\psi_1) \geq 0.$$

Moreover, simple calculations yield

$$\begin{cases} Z_t - dZ_{rr} + k(t)Z_r + \tilde{c}(r,t)Z \geq 0, & (t,r) \in (0,T) \times (0,\kappa_0), \\ Z(t,0) \geq 0, & t \in (0,T), \\ Z(t,\kappa_0) > 0, & t \in (0,T), \\ Z(0,r) = Z(T,r), & r \in (0,\kappa_0) \end{cases}$$

where $\tilde{c} = -[a(t) - b(t)(\tau_0\psi_1 + U)]$ is a bounded T-periodic function on $[0,T] \times [0,\kappa_0]$. Hence, the strong maximum principle concludes that there is a positive constant $\epsilon_1 \leq \epsilon_0$ such that

$$Z \geq \epsilon_1\psi_1 \ \ \forall (t,r) \in [0,T] \times [0,\kappa_0].$$

As a consequence, we have that

$$(\tau_0 - \epsilon)\psi_1 \geq U \ \ \forall (t,r) \in [0,T] \times [0,\infty)$$

for all $\epsilon \in (0, \epsilon_1]$. This clearly indicates that Σ is an open subset of $(0,\infty)$.

The above arguments imply $\Sigma = (0,\infty)$ and hence the inequality

$$\tau\psi_1(t,r) \geq U(t,r) \ \text{ for } (t,r) \in [0,T] \times [0,\infty)$$

holds for $\tau \in (0,\infty)$. This contradicts the fact that U is positive. The proof of Proposition 2.3 is complete. □

We are now in a position to state and prove the first main result of this section.

Theorem 2.4. *Under the assumptions of Proposition 2.1, for each $\mu > 0$, there exists a positive continuous T-periodic function $k_0(t) = k_0(\mu,a,b)(t) > 0$ such that $\mu U_r^{k_0}(t,0) = k_0(t)$ on $[0,T]$. Moreover, $0 < \overline{k}_0(\mu,a,b) < 2\sqrt{\overline{a}d}$ for every $\mu > 0$.*

Proof. Set

$$E = \{k \in C^{\nu_0/2}([0,T]) : \ k \geq 0, \ k(0) = k(T)\}.$$

Define the operator A by

$$Ak = \mu U_r^k(\cdot, 0), \ \ k \in E,$$

where U^k is the unique maximal nonnegative solution of (2.1) proved in Proposition 2.1.

We first observe that if $k \equiv 0$, then U^0 is a positive solution to (2.1). Indeed, when $k \equiv 0$, $\overline{a} > 0 = \overline{k}^2/(4d)$ and Proposition 2.3 infers that $U^0 > 0$. Thus, by the Hopf lemma,

$$A(0)(t) = \mu U_r^0(t,0) > 0 \ \text{ for } t \in [0,T].$$

We now denote $k^* = A(0)$ and $E_0 = \{k \in E : \ 0 \leq k(t) \leq k^*(t) \text{ for } t \in [0,T]\}$. Then, by Proposition 2.1, it is obvious that A maps E_0 to itself. In the following, we are going to show that A is a continuous operator on E_0 and maps E_0 into a precompact set. This will enable us to reach a setting for applying the Schauder fixed point theorem to obtain a fixed point of A.

We first prove that A is continuous. Assume that $k_n \in E_0$ and $k_n \to k$ in $C^{\nu_0/2}([0,T])$ as $n \to \infty$. Clearly $k \in E_0$. We need to prove

$$A(k_n) \to A(k) \ \text{ in } C^{\nu_0/2}([0,T]) \text{ as } n \to \infty.$$

To obtain this, we first prove

$$U^{k_n} \to U^k \ \text{ locally in } C^{1,2}([0,T] \times [0,\infty)) \text{ as } n \to \infty.$$

Since U^{k_n} satisfies (2.1) with k replaced by k_n and $0 \le U^{k_n} \le C_0$, using standard regularity theory for parabolic equations (up to the boundary), we see that there is a subsequence of U^{k_n} (still denoted by $\{U^{k_n}\}$) such that

$$U^{k_n} \to Z \text{ locally in } C^{1,2}([0,T] \times [0,\infty)) \text{ as } n \to \infty,$$

where Z is a nonnegative solution to (2.1).

We now claim that $Z \equiv U^k$. If $Z > 0$, then our claim is true by the uniqueness of positive solution to (2.1). If $Z \equiv 0$, we will show $U^k \equiv 0$ and so $Z \equiv U^k$. We will use an indirect argument and suppose that U^k is a positive solution of (2.1). Then, by Proposition 2.3, $\overline{k}^2 < 4\overline{a}d$ and so we can find a small $\epsilon^* = \epsilon^*(k,a) > 0$ such that

$$(\overline{k} + \epsilon^*)^2 < 4\overline{a}d.$$

Therefore, we can apply Proposition 2.3 to conclude that (2.1) with k replaced by $k + \epsilon^*$ admits a unique positive solution $U^{k+\epsilon^*}$. On the other hand, since $k_n \to k$ uniformly on $[0,T]$, we can find a large $n^* = n^*(k,a,\epsilon^*) > 0$ such that $k_n \le k + \epsilon^*$ for all $n \ge n^*$. It follows from Proposition 2.1 that for each $n \ge n^*$, (2.1) with $k = k_n$ has a unique positive solution U^{k_n}, and

$$U^{k_n} \ge U^{k+\epsilon^*} \text{ for all } n \ge n^*.$$

This contradicts the fact that $U^{k_n} \to 0$ as $n \to \infty$ locally in $C^{1,2}([0,T] \times [0,\infty))$.

We have thus proved that

$$U^{k_n} \to U^k \text{ locally in } C^{1,2}([0,T] \times [0,\infty)) \text{ as } n \to \infty.$$

The regularity of parabolic equations then implies

$$U^{k_n} \to U^k \text{ locally in } C^{1+\nu_0/2, 2+\nu_0}([0,T] \times [0,\infty)) \text{ as } n \to \infty.$$

This implies that

$$A(k_n) \to A(k) \text{ in } C^{1+\nu_0/2}([0,T]) \text{ as } n \to \infty,$$

and so A is continuous.

We show next that $A(E_0)$ is precompact. Let $\{k_n\}$ be a sequence in E_0. Denote $U^n(t,r) = U^{k_n}(t,r)$. Then

$$A(k_n)(t) = \mu U_r^n(t,0) \ \forall t.$$

Since

$$0 \le k_n(t) \le k^*(t), \ 0 \le U^n(t,r) \le U^0(t,r),$$

we find that k_n and U^n both have an L^∞ bound that is independent of n. Thus we can apply the L^p theory to the equation of U^n to conclude that for any $p > 1$, $\{U^n\}$ is a bounded set in $W_p^{1,2}(K)$ for any compact subset K of $[0,T] \times [0,\infty)$. Hence by passing to a subsequence we may assume that $U^n \to U^*$ locally in $C^{(1+\nu_0)/2, 1+\nu_0}([0,T] \times [0,\infty))$. In particular, $A(k_n) \to \mu U_r^*(t,0)$ in $C^{\nu_0}([0,T])$. This shows that $A(E_0)$ is precompact in $C^{\nu_0/2}([0,T])$.

Let E_1 denote the closed convex hull of $A(E_0)$. Then E_1 is compact and convex. Since E_0 is convex and closed, and $A(E_0) \subset E_0$, we have $E_1 \subset E_0$ and hence $A(E_1) \subset A(E_0) \subset E_1$.

Now E_1 is a compact convex subset of $C^{\nu_0/2}([0,T])$, A maps E_1 into itself and is continuous. Thus we can apply the Schauder fixed point theorem to conclude that there exists $k_0 \in E_1$ such that $A(k_0) = k_0$. Since $A(0) = \mu U_r^0(t,0) > 0$, we find that $k_0 \ne 0$. Thus U^{k_0} must be a positive solution and by the Hopf boundary lemma, $k_0(t) = U_r^{k_0}(t,0) > 0$ on $[0,T]$. By Proposition 2.3, we deduce $\overline{k}_0 < 2\sqrt{\overline{a}d}$. This completes the proof. □

Theorem 2.5. *The T-periodic function $k_0(t)$ given in Theorem 2.4 is uniquely determined by $a(t)$, $b(t)$ and μ.*

Proof. Using the notations of the proof of Theorem 2.4, for fixed a, b and $\mu > 0$, we assume that

$$k_0^i(t) = \mu U_r^{k_0^i}(t,0) \text{ in } [0,T] \text{ for } i = 1,2.$$

We will show that $\overline{k_0^1} = \overline{k_0^2}$ implies $k_0^1 \equiv k_0^2$, and $\overline{k_0^1} \neq \overline{k_0^2}$ leads to a contradiction. This will be done in three steps below, with the above two facts proved in steps 1 and 2 respectively, under the assumption of a fact to be proved in step 3.

Step 1. $\overline{k_0^1} = \overline{k_0^2}$ implies $k_0^1 \equiv k_0^2$.

Denote $K^i(t) = \int_0^t k_0^i(s)ds$ and suppose that $\overline{k_0^1} = \overline{k_0^2}$. Then $\tilde{K}(t) := K^1(t) - K^2(t)$ is a T-periodic function satisfying $\tilde{K}(0) = \tilde{K}(T) = 0$. If $\tilde{K} \equiv 0$, then clearly $k_0^1 \equiv k_0^2$, as we wanted. If $\tilde{K} \not\equiv 0$, we are going to derive a contradiction. In such a case, clearly there exists $t_0 \in (0,T)$ such that $C_0 := \max_{t\in\mathbb{R}} \tilde{K}(t) = \tilde{K}(t_0 + nT) > 0$, where $n = 0, \pm1, \pm2, ...$ It follows that $K^1(t) \le K^2(t) + C_0$ for all $t \in \mathbb{R}$ and $K^1(t_0 + nT) = K^2(t_0 + nT) + C_0$. Hence

$$k_0^1(t_0 + nT) = k_0^2(t_0 + nT). \tag{2.19}$$

To derive a contradiction, we consider the functions

$$V^1(t,r) := U^{k_0^1}(t, K^1(t) - r) \text{ and } V^2(t,r) := U^{k_0^2}(t, K^2(t) + C_0 - r).$$

One easily checks that

$$\begin{cases} V_t^1 - dV_{rr}^1 = a(t)V^1 - b(t)(V^1)^2, \ V^1 > 0, & t \in \mathbb{R}, \ r \in (-\infty, K^1(t)), \\ V^1(t, K^1(t)) = 0, \ k_0^1(t) = -\mu V_r^1(t, K^1(t)), & t \in \mathbb{R}, \end{cases} \tag{2.20}$$

and

$$\begin{cases} V_t^2 - dV_{rr}^2 = a(t)V^2 - b(t)(V^2)^2, \ V^2 > 0, & t \in \mathbb{R}, \ r \in (-\infty, K^2(t) + C_0), \\ V^2(t, K^2(t) + C_0) = 0, & t \in \mathbb{R}, \\ k_0^2(t) = -\mu V_r^2(t, K^2(t) + C_0), & t \in \mathbb{R}. \end{cases} \tag{2.21}$$

Set $W(t,r) := V^2(t,r) - V^1(t,r)$ for $(t,r) \in D := \{(t,r) \in \mathbb{R}^2 : t \in \mathbb{R}, \ r \in (-\infty, K^1(t))\}$. Then clearly $W(t,r) \ge, \not\equiv 0$ on ∂D with $W(t_0, K^1(t_0)) = 0$. We will show in Step 3 that this implies $W > 0$ in D, and $W_r(t_0, K^1(t_0)) < 0$, that is

$$V_r^1(t_0, K^1(t_0)) > V_r^2(t_0, K^2(t_0) + C_0), \text{ which leads to } k_0^1(t_0) < k_0^2(t_0),$$

a contradiction to (2.19). Thus the conclusion of Step 1 will follow if we can show that $W_r(t_0, K^1(t_0)) < 0$. This will be done in Step 3 below.

Step 2. $\overline{k_0^1} \neq \overline{k_0^2}$ leads to a contradiction.

Without loss of generality, we assume that $\overline{k_0^1} > \overline{k_0^2}$. Then

$$\lim_{t\to\pm\infty} \frac{K^1(t)}{t} = \overline{k_0^1} > \lim_{t\to\pm\infty} \frac{K^2(t)}{t} = \overline{k_0^2}. \tag{2.22}$$

This, together with $K^1(0) = K^2(0) = 0$, implies that the curves $r = K^1(t)$ and $r = K^2(t)$ has an intersection point (t_0, r_0) with the smallest t_0 value, namely

$$K^1(t) < K^2(t) \text{ for } t < t_0, \ K^1(t_0) = K^2(t_0).$$

It follows that

$$(K^1)'(t_0) \ge (K^2)'(t_0). \tag{2.23}$$

Define V^1 as in Step 1, and let $V^2(t,r) := U^{k_0^2}(t, K^2(t) - r)$, $W(t,r) := V^2(t,r) - V^1(t,r)$ for $(t,r) \in D_0 := \{(t,r) \in \mathbb{R}^2 : t \in (-\infty, t_0], r \in (-\infty, K^1(t))\}$. Then it is easily seen that V^1 and V^2 satisfies (2.20) and (2.21) respectively, except that in (2.21), C_0 should be replaced by

0. Moreover, $W > 0$ on $\partial D_0 \setminus \{(t,r) : t = t_0\}$ and $W(t_0, r_0) = W(t_0, K^1(t_0)) = 0$. By Step 3 below, we have $W > 0$ in D_0 and $W_r(t_0, r_0) < 0$, which implies that

$$(K^1)'(t_0) < (K^2)'(t_0).$$

But this is in contradiction to (2.23). This completes the proof of the conclusion in Step 2, except that it remains to prove Step 3.

Step 3. Let W be as in Steps 1 and 2, then $W > 0$ in D and D_0 respectively, and $W_r(t_0, K^1(t_0)) < 0$.

We consider the case in Step 1 first. To simplify notations we write $U_1 = U^{k_0^1}$ and $U_2 = U^{k_0^2}$. Let us recall that, by Proposition 2.1, for $i = 1, 2$, $(U_i)_r(t, 0) > 0$ in $[0, T]$, $U_i(t, r)$ increases in r and $U_i(t, r) \to V(t)$ uniformly in $t \in [0, T]$ as $r \to \infty$, where $V(t)$ is a positive T-periodic function. Hence we can find a positive constant $c_0 > 0$ such that $U_2(t, r) \geq c_0 U_1(t, r)$. From $K^2(t) + C_0 \geq K^1(t)$ and the monotonicity of U_2 in r, we thus deduce

$$V^2(t, r) \geq c_0 V^1(t, r) \text{ for all } t \in \mathbb{R}, r \leq K^1(t).$$

Define

$$c^* := \sup\{c > 0 : V^2(t, r) \geq c V^1(t, r)\ \forall t \in \mathbb{R},\ \forall r < K^1(t)\}.$$

We clearly have $c^* \geq c_0 > 0$. Since $V^i(t, r) \to V(t)$ as $r \to -\infty$, we also have $c^* \leq 1$. Thus $0 < c^* \leq 1$ and

$$W^*(t, r) := V^2(t, r) - c^* V^1(t, r) \geq 0\ \forall t \in \mathbb{R},\ \forall r \leq K^1(t).$$

Using $0 < c^* \leq 1$, we easily deduce from (2.20) and (2.21) that

$$\begin{cases} W_t^* - dW_{rr}^* + c(t, r) W^* \geq 0, & t \in \mathbb{R},\ r < K^1(t), \\ W^*(t, K^1(t)) \geq, \not\equiv 0, & t \in \mathbb{R}, \end{cases} \tag{2.24}$$

where $c(t, r) = -a(t) + b(t)[V^2(t, r) + c^* V^1(t, r)]$. Thus we can apply the strong maximum principle to (2.24) to conclude that the nonnegative function W^* is positive in $D = \{(t, r) : t \in \mathbb{R},\ r < K^1(t)\}$. Since $W^*(t_0, K^1(t_0)) = W(t_0, K^1(t_0)) = 0$, the Hopf lemma infers that $W_r^*(t_0, K^1(t_0)) < 0$. If we can show that $c^* = 1$, then $W^* = W$ and the required fact is proved.

We use an indirect argument to show that $c^* = 1$. Suppose by way of contradiction that $0 < c^* < 1$. Then by the definition of c^*, for any sequence of positive numbers $\epsilon_n \to 0$, there exists $(t_n, r_n) \in D$ such that

$$V^2(t_n, r_n) \leq (c^* + \epsilon_n) V^1(t_n, r_n)\ \forall n \geq 1. \tag{2.25}$$

We may write $t_n = m_n T + \tilde{t}_n$ with m_n an integer and $\tilde{t}_n \in [0, T]$. By passing to a subsequence, we may assume that $\tilde{t}_n \to \tilde{t} \in [0, T]$. We claim that $K^1(t_n) - r_n$ has an upper bound independent of n. Otherwise by passing to a subsequence we may assume that $K^1(t_n) - r_n \to \infty$ as $n \to \infty$. Then

$$K^2(t_n) + C_0 - r_n \geq K^1(t_n) - r_n \to \infty$$

and hence, for $i = 1, 2$,

$$V^i(t_n, r_n) \to V(\tilde{t})$$

as $n \to \infty$. It follows that

$$V(\tilde{t}) \leq c^* V(\tilde{t}) < V(\tilde{t}).$$

This contradiction proves our claim. Thus by passing to a subsequence we may assume that $K^1(t_n) - r_n \to \tilde{r} \in [0, \infty)$. Then

$$K^2(t_n) + C_0 - r_n = K^1(t_n) - r_n - \tilde{K}(\tilde{t}_n) + C_0 \to \tilde{r} - \tilde{K}(\tilde{t}) + C_0$$

and from (2.25), we deduce

$$U_2(\tilde{t}, \tilde{r} - \tilde{K}(\tilde{t}) + C_0) \le c^* U_1(\tilde{t}, \tilde{r}),$$

that is, $W^*(\tilde{t}, K^1(\tilde{t}) - \tilde{r}) \le 0$. Since $W^* > 0$ in D, we necessarily have $\tilde{r} = 0$, $W^*(\tilde{t}, K^1(\tilde{t})) = 0$ and $W_r^*(\tilde{t}, K^1(\tilde{t})) < 0$. By continuity we can find positive constants ϵ_0 and δ_0 such that

$$W_r^*(t, K^1(t)) < -2\delta_0 \ \forall t \in [\tilde{t} - \epsilon_0, \ \tilde{t} + \epsilon_0].$$

This implies that

$$W^*(t_n, r_n) = W^*(\tilde{t}_n, \tilde{r}_n) \ge \delta_0[K^1(\tilde{t}_n) - \tilde{r}_n] \text{ for all large } n,$$

where

$$\tilde{r}_n := K^1(\tilde{t}_n) + r_n - K^1(t_n) \to K^1(\tilde{t}) \text{ as } n \to \infty,$$

due to $\tilde{r} = 0$ and $\tilde{t}_n \to \tilde{t}$. On the other hand, from $V^1(t, K^1(t)) = 0$ and $V_r^1(t, K_1(t)) = -(U_1)_r(t, 0) < 0$ we find that

$$V^1(t_n, r_n) = V^1(\tilde{t}_n, \tilde{r}_n) \le M_0[K^1(\tilde{t}_n) - \tilde{r}_n] \text{ for all large } n,$$

where $M_0 = \max_{t\in[0,T]}(U_1)_r(t, 0)$. Thus for all large n,

$$V^2(t_n, r_n) \ge c^* V^1(t_n, r_n) + \delta_0[K^1(\tilde{t}_n) - \tilde{r}_n] \ge \left(c^* + \frac{\delta_0}{M_0}\right) V^1(t_n, r_n).$$

But this is in contradiction to (2.25). This proves $c^* = 1$ and thus $W > 0$ in D and $W_r(t_0, K^1(t_0)) < 0$, as required in Step 1.

The proof of the conclusion required in Step 2 follows a similar consideration. This time we define

$$c^* := \sup\{c > 0 : V^2(t, r) \ge cV^1(t, r) \ \forall t \le t_0, \ \forall r < K^1(t)\}.$$

We similarly have $c^* \ge c_0 > 0$ and $c^* \le 1$. Thus

$$W^*(t, r) := V^2(t, r) - c^* V^1(t, r) \ge 0 \ \forall t \le t_0, \ \forall r \le K^1(t).$$

Using $0 < c^* \le 1$ and $K^2(t) > K^1(t)$ for $t < t_0$, we easily deduce

$$\begin{cases} W_t^* - dW_{rr}^* + c(t, r)W^* \ge 0, & t \le t_0, \ r < K^1(t), \\ W^*(t, K^1(t)) > 0, & t < t_0, \end{cases} \tag{2.26}$$

where $c(t, r) = -a(t) + b(t)[V^2(t, r) + c^* V^1(t, r)]$. Thus we can apply the strong maximum principle to (2.26) to conclude that the nonnegative function W^* is positive in $D_0 = \{(t, r) : t \le t_0, \ r < K^1(t)\}$. Since $W^*(t_0, K^1(t_0)) = W(t_0, K^1(t_0)) = 0$, the Hopf lemma infers that $W_r^*(t_0, K^1(t_0)) < 0$. If we can show that $c^* = 1$, then $W^* = W$ and the required fact is proved.

Suppose by way of contradiction that $0 < c^* < 1$. Then by the definition of c^*, for any sequence of positive numbers $\epsilon_n \to 0$, there exists $(t_n, r_n) \in D_0$ such that (2.25) holds. We may write $t_n = m_n T + \tilde{t}_n$ with m_n an integer and $\tilde{t}_n \in [0, T]$. By passing to a subsequence, we may assume that $\tilde{t}_n \to \tilde{t} \in [0, T]$. We claim that t_n has a lower bound that is independent of n. Otherwise, by passing to a subsequence we may assume that $t_n \to -\infty$ as $n \to \infty$. Then from (2.22) we deduce $K^2(t_n) - K^1(t_n) \to +\infty$ as $n \to \infty$. It follows that

$$V^2(t_n, r_n) = U_2(\tilde{t}_n, K^2(t_n) - r_n) \ge U_2(\tilde{t}_n, K^2(t_n) - K^1(t_n)) \to V(\tilde{t})$$

as $n \to \infty$. On the other hand

$$V^1(t_n, r_n) = U_1(\tilde{t}_n, K^1(t_n) - r_n) \le V(\tilde{t}_n) \to V(\tilde{t})$$

as $n \to \infty$. Thus letting $n \to \infty$ in (2.25) we deduce $V(\tilde{t}) \le c^* V(\tilde{t}) < V(\tilde{t})$. This contradiction proves our claim. Hence we may assume, by passing to a subsequence, that $t_n \to \hat{t} \in (-\infty, t_0]$.

We can now easily see that r_n has a lower bound independent of n, for otherwise we may assume that $r_n \to -\infty$, which leads to

$$V^i(t_n, r_n) \to V(\hat{t}),\ V(\hat{t}) \le c^* V(\hat{t}) < V(\hat{t}).$$

Thus we may assume that $r_n \to \hat{r} \in (-\infty, K^1(\hat{t})]$ as $n \to \infty$. Letting $n \to \infty$ in (2.25), we deduce $V^2(\hat{t}, \hat{r}) \le c^* V^1(\hat{t}, \hat{r})$, that is $W^*(\hat{t}, \hat{r}) \le 0$. Since $W^* > 0$ in $D \cup \{(t, r) : r = K^1(t), t < t_0\}$, we necessarily have $(\hat{t}, \hat{r}) = (t_0, K^1(t_0))$ and $W^*(t_0, K^1(t_0)) = 0$. By the Hopf lemma we have $W_r^*(t_0, K^1(t_0)) < 0$, and we can then derive a contradiction to (2.25) as before.

The proof is now complete. □

Theorem 2.6. *The unique positive T-periodic function $k_0(t) = k_0(\mu, a, b)(t)$ in Theorem 2.5 depends continuously on $a(t), b(t)$ and μ; namely, if $a_n \to a$ in $C^{\nu_0/2}([0, T])$, $b_n \to b$ in $C^{\nu_0/2}([0, T])$ and $\mu_n \to \mu$ as $n \to \infty$, with a, b, μ as in Theorem 2.4, then $k_0(\mu_n, a_n, b_n)(t) \to k_0(\mu, a, b)(t)$ in $C^{\nu_0/2}([0, T])$.*

Proof. Let $a_n \to a$, $b_n \to b$ and $\mu_n \to \mu$ be as given in the theorem. To simplify notations we write $k_n(t) = k_0(\mu_n, a_n, b_n)(t)$, $U^n(t, r) = U^{k_n}(t, r)$. Thus

$$k_n(t) = \mu_n U_r^n(t, 0)\ \forall t.$$

From the proof of Theorem 2.4 we know that

$$k_n(t) \le \mu_n U_r^0(t, 0),\ U^n(t, r) \le U^0(t, r).$$

It follows that k_n and U^n both have an L^∞ bound that is independent of n. Thus we can apply the L^p theory to the equation of U^n to conclude that for any $p > 1$, $\{U^n\}$ is a bounded set in $W_p^{1,2}(K)$ for any compact subset K of $[0, T] \times [0, \infty)$. Hence by passing to a subsequence we may assume that $U^n \to U^*$ locally in $C^{(1+\nu_0)/2, 1+\nu_0}([0, T] \times [0, \infty))$. We may also assume $k_n \to k_*$ weakly in $L^2([0, T])$. Then it is easily seen that U^* is a weak solution of (2.1) with k replaced by k_*. Clearly U^* is nonnegative, and by the strong maximum principle it is either identically 0 or is a positive solution. We also have

$$k_n(t) = \mu_n U_r^n(t, 0) \to \mu U_r^*(t, 0) \text{ in } C^{\nu_0/2}([0, T]).$$

Thus $k_*(t) = \mu U_r^*(t, 0)$ and $k_n \to k_*$ in $C^{\nu_0/2}([0, T])$. If $U^* = 0$, then $k_* = 0$ and hence $k_n \to 0$ in $C^{\nu_0/2}([0, T])$. Hence for all large n, $k_n \le \epsilon$, where $\epsilon > 0$ satisfies $\epsilon < 2\sqrt{(\overline{a}/2)d}$. We may also assume that $a_n(t) > a(t)/2$ and $b_n(t) < 2b(t)$ for all such n. It then follows from the comparison principle that for all large n, $U^n \ge U_*$, where U_* denotes the unique positive solution of (2.1) with (k, a, b) replaced by $(\epsilon, a/2, 2b)$. It follows that

$$k_n(t) = \mu_n U_r^n(t, 0) \ge \mu_n (U_*)_r(t, 0) \to \mu (U_*)_r(t, 0) > 0.$$

This contradiction shows that $U^* = 0$ cannot happen. Therefore $U^* > 0$ and $k_* = \mu U_r^*(t, 0)$. By Theorem 2.5, we necessarily have $k_*(t) = k_0(\mu, a, b)(t)$. Thus $k_n(t) \to k_0(\mu, a, b)(t)$ in $C^{\nu_0/2}([0, T])$. This implies the continuity of $k_0(\mu, a, b)$ on (μ, a, b). □

Finally in this section, we study the dependence of $k_0(\mu, a, b)(t)$ on μ. Since $a(t)$ and $b(t)$ are fixed, we write $k_0(\mu) = k_0(\mu, a, b)$. We will show that $\overline{k_0(\mu)}$ is increasing in μ and

$$\lim_{\mu \to \infty} \overline{k_0(\mu)} = 2\sqrt{\overline{a} d}. \tag{2.27}$$

In order to prove (2.27), we consider the following variant of (2.1):

$$\begin{cases} U_t - dU_{rr} + \min\{k(t), M\} U_r = U[a(t) - b(t)U], & (t, r) \in [0, T] \times (0, \infty), \\ U(t, 0) = 0, & t \in [0, T], \\ U(0, r) = U(T, r), & r \in (0, \infty), \end{cases} \tag{2.28}$$

where d and M are given positive constants, and k, a, b are given T-periodic Hölder continuous functions with a, b positive and k nonnegative. For convenience, in the following we will use the notation

$$k^M(t) := \min\{k(t), M\}.$$

Recall that (2.28) has a unique maximal nonnegative solution U^{k^M}, and it is positive if and only if $\overline{k^M} < 2\sqrt{\overline{a}d}$.

We have the following result.

Proposition 2.7. *Under the above assumptions for* (2.28)*, for each $\mu > 0$ and $M > 0$, there exists a positive continuous T-periodic function $\hat{k}(t) = \hat{k}_\mu(t)$ such that $\mu U_r^{\hat{k}^M}(t,0) = \hat{k}(t)$ on $[0,T]$. Moreover,*

$$0 < \overline{\hat{k}_\mu^M} < 2\sqrt{\overline{a}d}, \text{ and when } M > 2\sqrt{\overline{a}d}, \lim_{\mu\to\infty} \overline{\hat{k}_\mu^M} = 2\sqrt{\overline{a}d}.$$

Proof. The existence of $\hat{k}$ can be proved by a simple variation of the proof of Theorem 2.4. We define the set E as before but replace the operator A there by

$$A_M k = \mu U_r^{k^M}(\cdot, 0),\ k \in E.$$

Since $k_n \to k$ in $C^{\nu_0/2}([0,T])$ implies $k_n^M \to k^M$ in $C^{\nu_0/2}([0,T])$, one easily sees that all the arguments in the proof of Theorem 2.4 for the operator A carry over to A_M. Thus A_M has a fixed point $\hat{k}(t)$. By Proposition 2.3 we know that $\overline{\hat{k}^M} < 2\sqrt{\overline{a}d}$.

To find the limit of $\overline{\hat{k}^M}$ as $\mu \to \infty$, we let μ_n be a sequence of positive numbers increasing to ∞ as $n \to \infty$, and $\hat{k}_n$ a corresponding T-periodic function satisfying $\hat{k}_n(t) = \mu_n U_r^{\hat{k}_n^M}(t,0)$. To simplify notations, we will write

$$k_n = \hat{k}_n \text{ and } U^n = U^{\hat{k}_n^M}.$$

Since k_n^M and U^n both have an L^∞ bound that is independent of n, we can apply the L^p theory to conclude that for any $p > 1$, $\{U^n\}$ is a bounded set in $W_p^{1,2}(K)$ for any compact subset K of $[0,T] \times [0,\infty)$. Hence by passing to a subsequence we may assume that $U^n \to U^*$ locally in $C^{(1+\nu_0)/2, 1+\nu_0}([0,T] \times [0,\infty))$. We may also assume $k_n^M \to k_*$ weakly in $L^2([0,T])$. Then it is easily seen that U^* is a weak solution of (2.28) with k^M replaced by k_*. Clearly U^* is nonnegative, and by the strong maximum principle it is either identically 0 or is a positive solution. If U^* is a positive solution, then we can use the Hopf lemma again to deduce $U_r^*(t,0) > 0$, and hence $k_n(t)/\mu_n = U_r^n(t,0) \to U_r^*(t,0)$ uniformly in $[0,T]$, which implies that $k_n^M \equiv M$ for all large n. Since $M > 2\sqrt{\overline{a}d}$, we can apply Proposition 2.3 to conclude that $U^n \equiv 0$ for all such n. This contradiction shows that we necessarily have $U^* \equiv 0$. Thus $U^n \to 0$ locally in $C^{(1+\nu_0)/2, 1+\nu_0}([0,T] \times [0,\infty))$. We will show in the following that this fact implies that $\overline{k_*} = 2\sqrt{\overline{a}d}$, and it is clear that the required limit for $\overline{k_n^M}$ is an easy consequence of this fact.

Set

$$V^n(t,r) = U^n(t,r)/U^n(T/2,1).$$

Clearly $V^n(T/2,1) = 1$, $V^n(t,r) > 0$ in $[0,T] \times (0,\infty)$, and

$$\begin{cases} V_t^n - dV_{rr}^n + k_n^M V_r^n = (a - bU^n)V^n, & (t,r) \in [0,T] \times (0,\infty), \\ V^n(t,0) = 0, & t \in [0,T], \\ V^n(0,r) = V^n(T,r), & r \in (0,\infty). \end{cases}$$

Since k_n^M and $(a - bU^n)$ both have an L^∞ bound that are independent of n, we can apply the Harnack inequality to V^n, noting that it is T-periodic in t, to obtain

$$V^n(t,r) \le C_R \text{ for } (t,r) \in [0,T] \times [1/2, 1+R],$$

where C_R is a constant depending on R but independent of n. By Proposition 2.3 we know that U^n and hence V^n is monotone increasing in r, and hence the above estimate also hold in $[0,T] \times [0,1/2]$. Thus we may apply the L^p estimates to the above equation for V^n and the embedding theorem to conclude that, by passing to a subsequence, $V^n \to V$ locally in $C^{(1+\nu_0)/2, 1+\nu_0}([0,T] \times [0,\infty))$, and V satisfies $V \ge 0$, $V(T/2,1) = 1$ and

$$\begin{cases} V_t - dV_{rr} + k_* V_r = aV, & (t,r) \in [0,T] \times (0,\infty), \\ V(t,0) = 0, & t \in [0,T], \\ V(0,r) = V(T,r), & r \in (0,\infty). \end{cases}$$

By the strong maximum principle we must have $V > 0$ in $[0,T] \times (0,\infty)$, and hence $V_r(t,0)$ is a positive T-periodic function in $C^{\nu_0}([0,T])$.

Denote $\xi_n = [\mu_n U^n(T/2,1)]^{-1}$. Then

$$\xi_n k_n(t) = \xi_n \mu_n U_r^n(t,0) = V_r^n(t,0) \to V_r(t,0) \tag{2.29}$$

in $C^{\nu_0}([0,T])$. By passing to a further subsequence, we have three possibilities for the sequence $\{\xi_n\}$: (i) $\xi_n \to 0$, (ii) $\xi_n \to \infty$, (iii) $\xi_n \to c_0 > 0$.

If case (i) happens, by (2.29) we deduce

$$k_n(t) = \xi_n^{-1} V_r^n(t,0) \to +\infty \text{ uniformly in } [0,T],$$

which implies that $k_n^M \equiv M$ for all large n, and hence $U^n \equiv 0$ for such n, a contradiction. If (ii) happens we deduce $k_n \to 0$ in $C^{\nu_0}([0,T])$ and if (iii) happens we deduce $k_n(t) \to c_0^{-1} V_r(t,0)$ in $C^{\nu_0}([0,T])$. Thus in either case (ii) or (iii), we have $k_n^M \to k_*$ in $C^{\nu_0}([0,T])$. We may now use the argument leading to the continuity of the operator A in the proof of Theorem 2.4 to see that $U^n \to U^{k_*}$ locally in $C^{1,2}([0,T] \times [0,\infty))$. Since we already know that this limit is 0, we thus have $U^{k_*} \equiv 0$, and by Proposition 2.3, we deduce $\overline{k_*} \ge 2\sqrt{\overline{a}d}$. On the other hand, from $\overline{k_n^M} < 2\sqrt{\overline{a}d}$ we deduce $\overline{k_*} \le 2\sqrt{\overline{a}d}$. Thus we necessarily have $\overline{k_*} = 2\sqrt{\overline{a}d}$, as we wanted. The proof is now complete. ☐

We are now ready for the last main result of this section.

Theorem 2.8. *The unique positive T-periodic function $k_0(t) = k_0(\mu)(t)$ in Theorem 2.5 has the following properties:*

$$\overline{k_0(\mu)} \text{ is increasing in } \mu \text{ and } \lim_{\mu \to \infty} \overline{k_0(\mu)} = 2\sqrt{\overline{a}d}.$$

Proof. We first prove that $\mu_1 < \mu_2$ implies $\overline{k_0(\mu_1)} < \overline{k_0(\mu_2)}$.

Arguing indirectly we assume that $\overline{k_0(\mu_1)} \ge \overline{k_0(\mu_2)}$. To simplify notations, we write

$$k_0^1(t) = k_0(\mu_1)(t), \; k_0^2(t) = k_0(\mu_2)(t).$$

To derive a contradiction, we can now argue exactly as in the proof of Theorem 2.5, the only minor difference occurs in the reasoning from $W_r(t_0, K^1(t_0)) < 0$ to $k_0^1(t_0) < k_0^2(t_0)$. We now argue as follows.

From $W_r(t_0, K^1(t_0)) < 0$ we obtain

$$U_r^{k_0^2}(t_0, 0) > U_r^{k_0^1}(t_0, 0).$$

Hence, due to $\mu_1 < \mu_2$, we have

$$k_0^2(t_0) = \mu_2 U_r^{k_0^2}(t_0,0) > \mu_1 U_r^{k_0^2}(t_0,0) > \mu_1 U_r^{k_0^1}(t_0,0) = k_0^1(t_0).$$

Next we prove that for any fixed $\mu > 0$ and $M > 0$,

$$\overline{\hat{k}_\mu^M} \le \overline{k_0(\mu)}, \tag{2.30}$$

where $\hat{k}_\mu(t)$ is given by Proposition 2.7. Since we always have $\overline{k_0(\mu)} < 2\sqrt{\overline{a}d}$, the required limit for $\overline{k_0(\mu)}$ is a consequence of (2.30) and Proposition 2.7.

We again argue by contradiction. Suppose that $\overline{\hat{k}_\mu^M} > \overline{k_0(\mu)}$. For convenience we write

$$\hat{k}(t) = \hat{k}_\mu(t),\ k_0(t) = k_0(\mu)(t),\ \hat{U}(t,r) = U^{\hat{k}^M}(t,r),\ U_0(t,r) = U^{k_0}(t,r).$$

Define

$$\hat{K}(t) = \int_0^t \hat{k}^M(s)ds,\ K_0(t) = \int_0^t k_0(s)ds.$$

Then from $\overline{\hat{k}^M} > \overline{k_0}$ we find, as in Step 2 of the proof of Theorem 2.5, that there exists t_0 such that

$$\hat{K}(t) < K_0(t) \text{ for } t < t_0,\ \hat{K}(t_0) = K_0(t_0).$$

It follows that

$$\hat{K}'(t_0) \ge K_0'(t_0). \tag{2.31}$$

We now follow the proof of Theorem 2.5 again to derive a contradiction. Define

$$V^1(t,r) = \hat{U}(t, \hat{K}(t) - r),\ V^2(t,r) = U_0(t, K_0(t) - r),$$

and

$$W(t,r) = V^2(t,r) - V^1(t,r).$$

Then the same arguments used in Steps 2 and 3 of the proof of Theorem 2.5 yield

$$W_r(t_0, \hat{K}(t_0)) < 0,$$

that is

$$\hat{U}_r(t_0, 0) < (U_0)_r(t_0, 0).$$

Since

$$\hat{K}'(t_0) = \hat{k}^M(t_0) \le \hat{k}(t_0) = \mu \hat{U}_r(t_0, 0)$$

and

$$K_0'(t_0) = k_0(t_0) = \mu (U_0)_r(t_0, 0),$$

we immediately deduce

$$\hat{K}'(t_0) < K_0'(t_0).$$

But this contradicts (2.31). Therefore (2.30) holds, and the proof is complete. □

3. The spreading-vanishing dichotomy

In this section we establish the spreading-vanishing dichotomy. The approach follows [9]. So some of the proofs are sketchy or omitted. For future applications, sometimes we consider a more general class of problems by replacing the special nonlinear term in (1.1) by a function $g(t,r,u)$ with the following properties:

$$\begin{cases} \text{(i) } g(t,r,u) \text{ is Hölder continuous for } (t,r,u) \in [0,\infty) \times [0,\infty) \times [0,\infty), \\ \text{(ii) } g(t,r,u) \text{ is locally Lipschitz in } u \text{ uniformly for } (t,r) \in [0,T] \times [0,\infty), \\ \text{(iii) there exists } c^* > 0 \text{ such that} \\ \qquad g(t,r,u) \le c^* u \quad \forall (t,r,u) \in [0,T] \times [0,\infty) \times [0,\infty). \end{cases} \tag{3.1}$$

We consider the radially symmetric free boundary problem

$$
(3.2) \qquad \begin{cases} u_t - d\Delta u = g(t,r,u), & t>0, 0<r<h(t),\\ u_r(t,0)=0, \;\; u(t,h(t))=0, & t>0,\\ h'(t) = -\mu u_r(t,h(t)), & t>0,\\ h(0)=\underline{R}, \;\; u(0,r)=u_0(r), & 0\le r\le \underline{R}, \end{cases}
$$

where $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, $u_0 \in C^2([0,\underline{R}])$ and $u_0(r)>0$ in $[0,\underline{R})$, $u_0'(0)=u_0(\underline{R})=0$. We have the following theorem:

Theorem 3.1. *For any given g satisfying the conditions in* (3.1)*, problem* (3.2) *admits a unique solution (u,h) defined for all $t>0$. Moreover $h\in C^1([0,\infty))$, $u\in C^{1,2}(D)$ with $D=\{(t,r): t>0, 0\le r\le h(t)\}$, and $h'(t)>0$ for $t>0$, $u(t,r)>0$ for $t>0$ and $0\le r<h(t)$.*

Proof. The proof is a simple modification of that for Theorem 4.1 in [9]. So we only briefly describe the main steps.

Step 1. The local existence and uniqueness of positive solution of (3.2).

This step can be obtained by exactly the same argument used in the proof of Theorem 4.1 in [9], as the special nonlinearity in [9] was not needed in the proof there.

Step 2. The local solution can be extended to all $t>0$.

To show this conclusion, we need the following estimates: if (u,h) is a solution of (3.2) defined for $t\in(0,T_0)$ for some $T_0\in(0,\infty)$, then for any given $T>T_0$, there exist constants C_1 and C_2 depending on T but independent of $T_0\in(0,T)$ such that

$$
(3.3) \qquad 0<u(t,r)\le C_1, \;\; 0<h'(t)\le C_2 \text{ for } 0<t<T_0,\, 0\le r<h(t).
$$

To find C_1, we use $g(t,r,u)\le c^*u$ and the comparison principle to obtain

$$
(3.4) \qquad u(t,r) < \hat{u}(t) := \|u_0\|_\infty e^{c^*t},
$$

and hence we may take $C_1 := \|u_0\|_\infty e^{c^*T}$.

To find C_2, we use the same construction as in the proof of Lemma 4.2 in [9], with some obvious modifications.

The rest of the proof is the same as in [9].

Step 3. The solution of (3.2) exists and is unique for all $t>0$.

This conclusion can be proved by exactly the same argument used in the proof of Theorem 4.3 in [9]. □

Clearly, Theorem 3.1 implies that $r=h(t)$ is increasing in t, and thus there exists $h_\infty\in(0,+\infty]$ such that $\lim_{t\to+\infty}h(t)=h_\infty$.

For a given positive T-periodic function $\alpha\in C^{\nu_0/2,\nu_0}(\mathbb{R}\times[0,\infty))$, it is well-known that the eigenvalue problem

$$
(3.5) \qquad \begin{cases} \phi_t - d\Delta\phi = \lambda\alpha(t,|x|)\phi & \text{in } [0,T]\times B_R,\\ \phi = 0 & \text{on } [0,T]\times\partial B_R,\\ \phi(0,x)=\phi(T,x) & \text{for } x\in B_R \end{cases}
$$

possesses a unique positive principal eigenvalue $\lambda=\lambda_1(d,\alpha,R,T)$, which corresponds to a positive eigenfunction $\varphi\in C^{1,2}([0,T]\times\overline{B_R})$ (see, for example, Proposition 14.4 of [20]). Moreover, $\varphi(t,x)$ is radially symmetric in x and this fact is a consequence of the moving-plane argument in [8].

In what follows, we present some further properties of $\lambda=\lambda_1(d,\alpha,R,T)$.

Lemma 3.2. *Let $\alpha(t,r)$ be a function satisfying* (1.2)*. Then $\lambda_1(d,\alpha,\cdot,T)$ is a strictly decreasing continuous function in $(0,\infty)$ for fixed d,α,T, and $\lambda_1(d,\cdot,R,T)$ is a strictly decreasing function* *in the sense that,* $\lambda_1(d,\alpha_1,R,T) < \lambda_1(d,\alpha_2,R,T)$ *if the two positive* T*-periodic continuous functions* α_1, α_2 *satisfy* $\alpha_1 \geq, \not\equiv \alpha_2$ *on* $[0,T]\times B_R$. *Moreover,*

$$\lim_{R\to 0^+}\lambda_1(d,\alpha,R,T)=+\infty,\quad \lim_{R\to+\infty}\lambda_1(d,\alpha,R,T)=0. \tag{3.6}$$

Proof. The continuity of $\lambda_1(d,\alpha,\cdot,T)$ with fixed d,α and T can be obtained by using a simple re-scaling argument of the spatial variable r, which also gives the monotonicity of $\lambda_1(d,\alpha,\cdot,T)$. The proof of the monotonicity of $\lambda_1(d,\cdot,R,T)$ is folklore. We present a proof for completeness.

Assume that α_1, α_2 are given positive T-periodic continuous functions and satisfy $\alpha_1 \geq, \not\equiv \alpha_2$ on $[0,T]\times B_R$. Let ϕ_1 be the corresponding eigenfunction of $\lambda_1(d,\alpha_1,R,T)$. From [20], it follows that $\lambda_1(d,\alpha_2,R,T)$ is the principal eigenvalue of the adjoint problem

$$\begin{cases} -\psi_t - d\Delta\psi = \lambda\alpha_2(t,|x|)\psi & \text{in } [0,T]\times B_R,\\ \psi = 0 & \text{on } [0,T]\times\partial B_R,\\ \psi(0,x)=\psi(T,x) & \text{in } B_R. \end{cases} \tag{3.7}$$

Furthermore, (3.7) has a T-periodic positive eigenfunction ψ_2 corresponding to $\lambda_1(d,\alpha_2,R,T)$. Now, we multiply the equation of ϕ_1 by ψ_2 and the equation of ψ_2 by ϕ_1, integrate over $(0,T)\times B_R$ and then subtract the resulting identities to obtain that

$$\lambda_1(d,\alpha_1,R,T)\int_0^T\int_{B_R}\alpha_1\phi_1\psi_2 = \lambda_1(d,\alpha_2,R,T)\int_0^T\int_{B_R}\alpha_2\phi_1\psi_2,$$

which obviously implies

$$\lambda_1(d,\alpha_1,R,T) < \lambda_1(d,\alpha_2,R,T). \tag{3.8}$$

We now prove (3.6). It follows from condition (ii) in (1.2) and (3.8) that

$$\lambda_1(d,\max_{[0,T]}\kappa_2,R,T)\leq\lambda_1(d,\alpha,R,T)\leq\lambda_1(d,\min_{[0,T]}\kappa_1,R,T). \tag{3.9}$$

It is also easy to see that $\lambda_1(d,\min_{[0,T]}\kappa_1,R,T)$ and $\lambda_1(d,\max_{[0,T]}\kappa_2,R,T)$ are the principal eigenvalues of the elliptic problems

$$-d\Delta\psi=\lambda[\min_{[0,T]}\kappa_1]\psi \ \text{ in } B_R,\quad \psi=0 \ \text{ on } \partial B_R$$

and

$$-d\Delta\psi=\lambda[\max_{[0,T]}\kappa_2]\psi \ \text{ in } B_R,\quad \psi=0 \ \text{ on } \partial B_R$$

respectively. It is well known that

$$\lim_{R\to+\infty}\lambda_1(d,\min_{[0,T]}\kappa_1,R,T)=0,\quad \lim_{R\to0^+}\lambda_1(d,\min_{[0,T]}\kappa_1,R,T)=+\infty \tag{3.10}$$

and

$$\lim_{R\to+\infty}\lambda_1(d,\max_{[0,T]}\kappa_2,R,T)=0,\quad \lim_{R\to0^+}\lambda_1(d,\max_{[0,T]}\kappa_2,R,T)=+\infty. \tag{3.11}$$

Clearly (3.6) follows from (3.9)-(3.11). □

In view of Lemma 3.2, for any fixed $d>0$ and $\alpha\in C^{\nu_0/2,\nu_0}(\mathbb{R}\times[0,\infty))$ satisfying (1.2), there is a unique $R^* := R^*(d,\alpha,T)$ such that

$$\lambda_1(d,\alpha,R^*,T)=1 \tag{3.12}$$

and

$$1>\lambda_1(d,\alpha,R,T)\ \text{ for } R>R^*,\quad 1<\lambda_1(d,\alpha,R,T)\ \text{ for } R<R^*. \tag{3.13}$$

Let (u,h) be the unique solution of (1.1), and $h_\infty=\lim_{t\to\infty}h(t)$. The spreading-vanishing dichotomy is a consequence of the following two lemmas.

Lemma 3.3. *If $h_\infty < +\infty$, then $h_\infty \le R^*$ and*

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0.$$

Proof. The proof of this lemma follows that of Lemma 2.2 in [9].

We first show that $h_\infty < \infty$ implies $h_\infty \le R^*$. Otherwise $h_\infty \in (R^*, \infty)$ and there is $\tilde T > 0$ such that $h(t) > h_\infty - \varepsilon > R^*$ for all $t \ge \tilde T$ and some small $\varepsilon > 0$. Thus,

$$\lambda_1(d,\alpha,h(t),T) < 1 \ \text{ for all } t \ge \tilde T.$$

Consider the problem

$$\begin{cases} w_t - d\Delta w = w(\alpha(t,r) - \beta(t,r)w), & t \ge \tilde T, \ \ r \in [0, h_\infty - \varepsilon], \\ w_r(t,0) = 0, \ \ w(t, h_\infty - \varepsilon) = 0, & t \ge \tilde T, \\ w(\tilde T, r) = u(\tilde T, r), & r \in [0, h_\infty - \varepsilon]. \end{cases} \tag{3.14}$$

It is well known that this logistic problem admits a unique positive solution $\underline{w} = \underline{w}_\varepsilon(t,r)$. Moreover, the fact $\lambda_1(d,\alpha,h_\infty-\varepsilon,T) < 1$ implies that the trivial steady state 0 of (3.14) is linearly unstable. Therefore, it follows from Theorem 28.1 in [20] that

$$\underline{w}(t+nT, r) \to V_{h_\infty-\varepsilon}(t,r) \ \text{ in } C^{1,2}([0,T]\times[0,h_\infty-\varepsilon]) \text{ as } n\to\infty \tag{3.15}$$

where $V_{h_\infty-\varepsilon}(t,r)$ is the unique positive T-periodic solution of the problem

$$\begin{cases} V_t - d\Delta V = V\Big[\alpha(t,r) - \beta(t,r)V\Big], & (t,r) \in [0,T]\times[0,h_\infty-\varepsilon), \\ V_r(t,0) = V(t,h_\infty-\varepsilon) = 0, & t \in [0,T], \\ V(0,r) = V(T,r), & r \in [0, h_\infty-\varepsilon]. \end{cases} \tag{3.16}$$

Using the comparison principle for parabolic equations, we obtain

$$u(t,r) \ge \underline{w}(t,r) \quad \text{for } t > \tilde T,\ r \in [0, h_\infty - \varepsilon]. \tag{3.17}$$

This implies that

$$\underline{\lim}_{n\to+\infty} u(t+nT, r) \ge V_{h_\infty-\epsilon}(t,r) \ \text{ for } (t,r) \in [0,T]\times[0,h_\infty-\varepsilon]. \tag{3.18}$$

On the other hand, consider the problem

$$\begin{cases} w_t - d\Delta w = w(\alpha(t,r) - \beta(t,r)w), & t \ge \tilde T, \ \ r \in [0, h_\infty], \\ w_r(t,0) = 0, \ \ w(t, h_\infty) = 0, & t \ge \tilde T, \\ w(\tilde T, r) = \tilde u(\tilde T, r), & r \in [0, h_\infty] \end{cases} \tag{3.19}$$

where

$$\tilde u(\tilde T, r) = \begin{cases} u(\tilde T, r) & \text{for } r \in [0, h(\tilde T)], \\ 0 & \text{for } r \in (h(\tilde T), h_\infty]. \end{cases}$$

Similarly, (3.19) admits a unique positive solution $\overline{w}(t,r)$ with

$$\overline{w}(t+nT, r) \to V_{h_\infty}(t,r) \ \text{ in } C^{1,2}([0,T]\times[0,h_\infty]) \text{ as } n\to\infty \tag{3.20}$$

where V_{h_∞} is the unique positive T-periodic solution of the problem

$$\begin{cases} V_t - d\Delta V = V\Big[\alpha(t,r) - \beta(t,r)V\Big], & (t,r) \in [0,T]\times[0,h_\infty], \\ V_r(t,0) = V(t,h_\infty) = 0, & t \in [0,T], \\ V(0,r) = V(T,r), & r \in [0, h_\infty]. \end{cases} \tag{3.21}$$

The comparison principle implies that

$$u(t,r) \le \overline{w}(t,r) \ \text{ for } t > \tilde T,\ r \in [0, h(t)] \tag{3.22}$$

and hence

$$\overline{\lim}_{n\to+\infty} u(t+nT,r) \le V_{h_\infty}(t,r) \text{ for } (t,r)\in[0,T]\times[0,h_\infty]. \tag{3.23}$$

For any $0<\varepsilon_1<\varepsilon_2$, we easily see from the comparison principle that

$$V_{h_\infty-\varepsilon_1} \ge V_{h_\infty-\varepsilon_2} \quad \text{for } (t,r)\in[0,T]\times[0,h_\infty-\varepsilon_2].$$

Then it follows that

$$V_{h_\infty-\varepsilon} \to V_{h_\infty} \text{ in } [0,T]\times[0,h_\infty) \text{ as } \epsilon\to 0^+$$

since V_{h_∞} is the unique positive solution of (3.21). Thus, (3.18), (3.23) and the arbitrariness of ε imply

$$\lim_{n\to\infty} u(t+nT,r) = V_{h_\infty}(t,r) \text{ for } (t,r)\in[0,T]\times[0,h_\infty),$$

or equivalently,

$$\lim_{t\to\infty}[u(t,r)-V_{h_\infty}(t,r)] = 0 \text{ for } r\in[0,h_\infty). \tag{3.24}$$

As in the proof of Lemma 2.2 in [9], we may straighten the free boundary and use parabolic regularity for the new problem to obtain

$$\|\tilde{u}(t,\cdot)-\tilde{V}_{h_\infty}\|_{C^2([0,h_0])} \to 0 \text{ as } t\to\infty,$$

where $\tilde{u}$ denotes the transformed u, and $\tilde{V}_{h_\infty}$ denotes the transformed V_{h_∞} under the transformation which changes $[0,h(t)]$ into $[0,h_0]$, as indicated in [9]. Changing back to u and V_{h_∞} we obtain

$$\|u(t,\cdot)-V_{h_\infty}\|_{C^2([0,h(t)])} \to 0 \text{ as } t\to\infty.$$

It follows that

$$\lim_{t\to\infty}[h'(t)+\mu(V_{h_\infty})_r(t,h_\infty)] = -\lim_{t\to\infty}\mu[u_r(t,h(t))-(V_{h_\infty})_r(t,h(t))] = 0.$$

Hence $h'(t)\ge\delta>0$ for all large t and some fixed $\delta>0$. But this implies $h_\infty=\infty$, a contradiction. Thus we must have $h_\infty\le R^*$.

We are now ready to show that $\|u(t,\cdot)\|_{C([0,h(t)])}\to 0$ as $t\to\infty$. Let $\overline{u}(t,r)$ denote the unique positive solution of the problem

$$\begin{cases} \overline{u}_t - d\Delta\overline{u} = \overline{u}[\alpha(t,r)-\beta(t,r)\overline{u}], & t>0,\ 0<r<h_\infty,\\ \overline{u}_r(t,0)=0, \quad \overline{u}(t,h_\infty)=0, & t>0,\\ \overline{u}(0,r)=\tilde{u}_0(r), & 0\le r\le h_\infty \end{cases} \tag{3.25}$$

where

$$\tilde{u}_0(r) = \begin{cases} u_0(r), & 0\le r\le h_0,\\ 0, & r\ge h_0. \end{cases}$$

The comparison principle implies that

$$0\le u(t,r)\le\overline{u}(t,r) \quad \text{for } t>0 \text{ and } r\in[0,h(t)].$$

Since $h_\infty\le R^*$, we see that $1\le\lambda_1(d,\alpha,h_\infty,T)$, and it follows from Theorem 28.1 of [20] that $\overline{u}(t,r)\to 0$ uniformly for $r\in[0,h_\infty]$ as $t\to+\infty$. Hence,

$$\lim_{t\to+\infty}\|u(t,\cdot)\|_{C([0,h(t)])} = 0.$$

□

Lemma 3.4. *If $h_\infty = +\infty$, then*

$$\lim_{n\to+\infty} u(t+nT, r) = \hat{U}(t,r) \ \ \textit{locally uniformly for } (t,r) \in [0,T]\times[0,+\infty) \tag{3.26}$$

where $\hat{U}(t,|x|)$ is the unique positive T-periodic (radial) solution of the equation

$$\begin{cases} U_t - d\Delta U = U[\alpha(t,|x|) - \beta(t,|x|)U], & (t,x)\in[0,T]\times\mathbb{R}^N, \\ U(0,x) = U(T,x), & x\in\mathbb{R}^N. \end{cases} \tag{3.27}$$

Proof. Existence and uniqueness of a positive solution $\hat{U}$ of (3.27) follows from Theorem 1.3 of [28] (by choosing both γ and τ there to be 0). It must be radially symmetric since (3.27) is invariant under rotations with respect to the spatial variables around the origin of $\mathbb{R}^N$. (Under the extra condition $\inf U > 0$, the above conclusions also follow from Proposition 1.7 of [3].)

To show (3.26), we use a squeezing argument similar in spirit to [14]. We first consider the T-periodic Dirichlet problem

$$\begin{cases} v_t - d\Delta v = v[\alpha(t,|x|) - \beta(t,|x|)v], & (t,x)\in[0,T]\times B_R, \\ v(t,x) = 0, & (t,x)\in[0,T]\times\partial B_R, \\ v(0,x) = v(T,x), & x\in B_R \end{cases} \tag{3.28}$$

and the T-periodic boundary blow-up problem

$$\begin{cases} w_t - d\Delta w = w[\alpha(t,|x|) - \beta(t,|x|)w], & (t,x)\in[0,T]\times B_R, \\ w(t,x) = \infty, & (t,x)\in[0,T]\times\partial B_R, \\ w(0,x) = w(T,x), & x\in B_R. \end{cases} \tag{3.29}$$

When R is large, it is known from [28] that these problems admit unique T-periodic positive solutions $v_R(t,x)$ and $w_R(t,x)$ respectively. Moreover, $v_R(t,\cdot)$ and $w_R(t,\cdot)$ are radially symmetric for fixed $t\in[0,T]$. Furthermore, the proof of Theorem 1.3 in [28] implies that

$$v_R \nearrow \hat{U} \ \text{ locally uniformly for } (t,r)\in[0,T]\times[0,+\infty) \text{ as } R\to+\infty,$$

$$w_R \searrow \hat{U} \ \text{ locally uniformly for } (t,r)\in[0,T]\times[0,+\infty) \text{ as } R\to+\infty.$$

By Lemma 3.2, we can choose an increasing sequence of positive numbers R_m with $R_m\to+\infty$ as $m\to\infty$ such that $\lambda_1(d,\alpha,R_m,T) < 1$ for all $m\geq 1$. Since both v_{R_m} and w_{R_m} converge to $\hat{U}$ locally uniformly in $[0,T]\times\mathbb{R}^N$, we can find $\hat{T}_m > 0$ such that $h(t)\geq R_m$ for $t\geq\hat{T}_m$. Arguing as in the proof of Lemma 3.3, we see that the problem

$$\begin{cases} w_t - d\Delta w = w[\alpha(t,r) - \beta(t,r)w], & t\geq\hat{T}_m, \ \ r\in[0,R_m], \\ w_r(t,0) = 0, \ \ w(t,R_m) = 0, & t\geq\hat{T}_m, \\ w(\hat{T}_m, r) = u(\hat{T}_m, r), & r\in[0,R_m] \end{cases} \tag{3.30}$$

admits a unique positive solution $w_m(t,r)$, which satisfies that

$$w_m(t+nT, r) \to v_{R_m}(t,r) \tag{3.31}$$

uniformly for $(t,r)\in[0,T]\times[0,R_m]$ as $n\to+\infty$.

By the comparison principle, we have

$$w_m(t,r) \leq u(t,r) \ \text{ for } t\geq\hat{T}_m \text{ and } r\in[0,R_m].$$

Therefore,

$$\underline{\lim}_{n\to+\infty} u(t+nT,r) \geq v_{R_m}(t,r) \ \text{ uniformly for } (t,r)\in[0,T]\times[0,R_m].$$

Sending $m\to\infty$, we obtain

$$\underline{\lim}_{n\to+\infty} u(t+nT,r) \geq \hat{U}(t,r) \ \text{ locally uniformly for } (t,r)\in[0,T]\times[0,\infty). \tag{3.32}$$

Analogously, by arguments similar to those in the proof of Theorem 1.3 of [28], we see that

$$\overline{\lim}_{n\to+\infty} u(t+nT,r) \le w_{R_m}(t,r) \text{ uniformly for } (t,r)\in[0,T]\times[0,R_m],$$

which implies (by sending $m\to\infty$)

$$\overline{\lim}_{n\to+\infty} u(t+nT,r) \le \hat{U}(t,r) \text{ locally uniformly for } (t,r)\in[0,T]\times[0,\infty). \tag{3.33}$$

Clearly (3.26) is a consequence of (3.32) and (3.33). □

Combining Lemmas 3.3 and 3.4, we immediately obtain the following spreading-vanishing dichotomy:

Theorem 3.5. *Let $(u(t,r),h(t))$ be the solution of the free boundary problem* (3.2). *Then the following alternative holds:*

Either

(i) Spreading: $h_\infty=+\infty$ *and*

$$\lim_{t\to\infty}[u(t,r)-\hat{U}(t,r)]=0 \text{ locally uniformly for } r\in[0,\infty)$$

or

(ii) Vanishing: $h_\infty\le R^*$ *and* $\lim_{t\to+\infty}\ ||u(t,\cdot)||_{C([0,h(t)])}=0.$

We now determine when each of the two alternatives occurs. We divide the discussion into two cases: (a) $h_0\ge R^*$, (b) $h_0<R^*$.

For case (a), due to $h'(t)>0$ for $t>0$, we must have $h_\infty>R^*$. Then from Lemma 3.3 we immediately obtain the following result.

Theorem 3.6. *If $h_0\ge R^*$, then $h_\infty=+\infty$. Thus spreading always occurs in this case.*

In order to study case (b), and also for later applications, we need a comparison principle which can be used to estimate both $u(t,r)$ and the free boundary $r=h(t)$. For future applications, we also include a more general class of problems by replacing the special nonlinear term of (1.1) by the function $g(t,r,u)$ in (3.2).

Lemma 3.7. *Suppose that $\mathcal{T}\in(0,\infty)$, $\overline{k}\in C^1([0,\mathcal{T}])$, $\overline{v}\in C^{1,2}(D^*_{\mathcal{T}})$ with $D^*_{\mathcal{T}}=\{(t,r)\in\mathbb{R}^2 : 0\le t\le\mathcal{T}, 0\le r\le\overline{k}(t)\}$, and*

$$\begin{cases} \overline{v}_t-d\Delta\overline{v}\ge g(t,r,\overline{v}), & t>0,\ 0<r<\overline{k}(t),\\ \overline{v}_r(t,0)\le 0,\ \ \overline{v}(t,\overline{k}(t))=0, & t>0,\\ \overline{k}'(t)\ge -\mu\overline{v}_r(t,\overline{k}(t)), & t>0. \end{cases}$$

If $k_0\le\overline{k}(0)$ and $v_0(r)\le\overline{v}(0,r)$ for $r\in[0,k_0]$, then the solution (v,k) of the free boundary problem

$$\begin{cases} v_t-d\Delta v=g(t,r,v), & t>0,\ 0<r<k(t),\\ v_r(t,0)=0,\ \ v(t,k(t))=0, & t>0,\\ k'(t)=-\mu v_r(t,k(t)), & t>0,\\ v(0,r)=v_0(r), & 0\le r\le k_0 \end{cases} \tag{3.34}$$

satisfies

$$k(t)\le\overline{k}(t)\ \ \forall t\in[0,\mathcal{T}],\quad v(t,r)\le\overline{v}(t,r) \text{ for } t\in[0,\mathcal{T}] \text{ and } r\in[0,k(t)].$$

Proof. The proof of this lemma is similar to that of Lemma 3.5 in [12] and Lemma 2.6 in [9]. So we omit the details. □

Remark 3.8. *The pair* $(\overline{v}, \overline{k})$ *in Lemma 3.7 is usually called an upper solution of the free boundary problem* (3.34)*. We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 3.7 for lower solutions.*

Now we consider case (b), where $h_0 < R^*$. We first examine the case that μ is large, then we look at the case that $\mu > 0$ is small, and finally we use Lemma 3.7 and Remark 3.8 to show the existence of a critical μ^* so that spreading occurs if $\mu > \mu^*$ and vanishing happens if $\mu \in (0, \mu^*]$.

Firstly, using Lemma 3.7 and exactly the same arguments as those in the proof of Lemma 2.8 of [9], we have

Lemma 3.9. *Suppose* $h_0 < R^*$*. Then there exists* $\mu^0 > 0$ *depending on* u_0 *such that spreading occurs if* $\mu \geq \mu^0$*.*

On the other hand, we also have the following assertion.

Lemma 3.10. *Suppose* $h_0 < R^*$*. Then there exists* $\mu_0 > 0$ *depending on* u_0 *such that vanishing happens if* $\mu \leq \mu_0$*.*

Proof. We are going to construct a suitable upper solution to (1.1) and then apply Lemma 3.7.

For $t > 0$ and $r \in [0, \sigma(t)]$, where

$$\sigma(t) = h_0 \tau(t), \quad \tau(t) = (1 + \delta - \frac{\delta}{2} e^{-\gamma t}),$$

we define

$$w(t, r) = M e^{-\gamma t} V\Big(\int_0^t \tau^{-2}(s) ds,\ \frac{h_0}{\sigma(t)} r \Big),$$

where M, δ, γ are positive constants to be chosen later and $V(t, |x|)$ is the first eigenfunction of the problem

$$\begin{cases} V_t - d\Delta V = \lambda_1(d, \alpha, h_0, T)\alpha(t, |x|)V, & (t, x) \in [0, T] \times B_{h_0}, \\ V = 0, & (t, x) \in [0, T] \times \partial B_{h_0}, \\ V(t, x) \text{ is } T\text{-periodic in } t, \end{cases}$$

with $V > 0$ and $\|V\|_{L^\infty((0,T)\times B_{h_0})} = 1$. Since $h_0 < R^*$, we have

$$\lambda_1(d, \alpha, h_0, T) > 1.$$

Moreover, by the moving-plane argument in [8] we have

$$V_r(t, r) < 0 \ \text{ for } 0 < r \leq h_0 \text{ and } t \in [0, T).$$

(Since $V(\cdot, r)$ is a periodic function, $w(t, r)$ is defined for all $t > 0$.)

In the following calculations, we will use the notations $\xi = \int_0^t \tau^{-2}(s)ds$, $\eta = r\tau^{-1}(t)$ and $V = V(\xi,\eta)$. Thus $w(t,r) = Me^{-\gamma t}V(\xi,\eta)$ and

$$\begin{aligned}
&w_t - d\Delta w - w[\alpha(t,|x|) - \beta(t,|x|)w] \\
&= Me^{-\gamma t}[-\gamma V + V_\xi \tau^{-2}(t) - rV_\eta \tau^{-2}(t)\tau'(t) - d\tau^{-2}(t)V_{\eta\eta} - \frac{d(N-1)}{r\tau(t)}V_\eta \\
&\qquad -V(\alpha(t,r) - \beta(t,r)Me^{-\gamma t}V)] \\
&= Me^{-\gamma t}[-\gamma V - rV_\eta \tau^{-2}(t)\tau'(t) + \tau^{-2}(t)\lambda_1(d,\alpha,h_0,T)\alpha(\xi,\eta)V \\
&\qquad -V(\alpha(t,r) - \beta(t,r)Me^{-\gamma t}V)] \\
&\geq Me^{-\gamma t}V[-\gamma + \tau^{-2}(t)\lambda_1(d,\alpha,h_0,T)\alpha(\xi,\eta) - \alpha(t,r)] \\
&\geq Me^{-\gamma t}V[-\gamma + \frac{\lambda_1(d,\alpha,h_0,T)}{(1+\delta)^2}\alpha(\xi,\eta) - \alpha(t,r)] \\
&= Me^{-\gamma t}V\Big[-\gamma + \Big(\frac{\lambda_1(d,\alpha,h_0,T)}{(1+\delta)^2}\frac{\alpha(\xi,\eta)}{\alpha(t,r)} - 1\Big)\alpha(t,r)\Big].
\end{aligned}$$

Clearly

$$1 + \frac{\delta}{2} \leq \tau(t) \leq 1+\delta, \quad h_0(1+\frac{\delta}{2}) \leq \sigma(t) \leq h_0(1+\delta).$$

Therefore,

$$\left(1+\frac{\delta}{2}\right)^{-2} t \geq \xi \geq (1+\delta)^{-2}t, \quad \left(1+\frac{\delta}{2}\right)^{-1} r \geq \eta \geq (1+\delta)^{-1}r.$$

Hence, due to $1 < \lambda_1(d,\alpha,h_0,T)$, we can choose $\delta > 0$ sufficiently small such that

$$\varrho := \min_{t>0, r\in[0,\sigma(t)]} \frac{\lambda_1(d,\alpha,h_0,T)}{(1+\delta)^2}\frac{\alpha(\xi,\eta)}{\alpha(t,r)} - 1 > 0. \tag{3.35}$$

Setting $\gamma = \varrho\kappa_1$, we deduce

$$w_t - d\Delta w - w[\alpha(t,r) - \beta(t,r)w] \geq 0 \text{ for } t>0,\ r\in[0,\sigma(t)].$$

We now choose $M > 0$ sufficiently large such that

$$u_0(r) \leq MV\Big(0, \frac{r}{(1+\delta/2)}\Big) = w(0,r) \text{ for } r\in[0,h_0].$$

Then

$$\sigma'(t) = \frac{1}{2}h_0\gamma\delta e^{-\gamma t},$$

$$-\mu w_r(t,\sigma(t)) = \mu Me^{-\gamma t}\Big|V_\eta\Big(\int_0^t \tau^{-2}(s)ds,\ h_0\Big)\Big|\tau^{-1}(t) \leq \frac{C_0\mu M}{1+\frac{\delta}{2}}e^{-\gamma t},$$

where

$$C_0 = \max_{t\in[0,\infty)}\Big|V_\eta\Big(\int_0^t \tau^{-2}(s)ds,\ h_0\Big)\Big|.$$

Thus, if we choose

$$\mu_0 = \frac{\delta(1+\delta/2)\gamma h_0}{2MC_0},$$

then

$$\sigma'(t) \geq -\mu w_r(t,\sigma(t)) \text{ for } 0<\mu\leq\mu_0$$

and thus $(w(t,r),\sigma(t))$ satisfies

$$\begin{cases} w_t - d\Delta w \geq w[\alpha(t,r)-\beta(t,r)w], & t>0,\ 0<r<\sigma(t), \\ w(t,\sigma(t)=0, \quad \sigma'(t) \geq -\mu w_r(t,\sigma(t)), & t>0, \\ w_r(t,0)=0, & t>0, \\ \sigma(0)=(1+\frac{\delta}{2})h_0 > h_0. \end{cases}$$

Applying Lemma 3.7, we obtain that $h(t) \leq \sigma(t)$ and $u(t,r) \leq w(t,r)$ for $0 \leq r \leq h(t)$ and $t>0$. These imply that $h_\infty \leq \lim_{t\to\infty} \sigma(t) = h_0(1+\delta) < \infty$. □

Based on Lemmas 3.9 and 3.10, we can apply the same arguments as those in the proof of Theorem 2.10 of [9] to obtain a threshold value $\mu^* > 0$ of μ such that the alternatives in the spreading-vanishing dichotomy are determined by μ^* for the case $h_0 < R^*$, as stated in the following theorem.

Theorem 3.11. *If $h_0 < R^*$, then there is $\mu^* > 0$ depending on u_0 such that vanishing occurs if $0 < \mu \leq \mu^*$, and spreading happens if $\mu > \mu^*$.*

4. Spreading speed

In this section we study the spreading speed of the expanding front $r = h(t)$ when spreading occurs. By (1.2), we have that

$$\alpha^\infty(t) := \overline{\lim}_{r\to+\infty}\alpha(t,r) \leq \kappa_2, \quad \alpha_\infty(t) := \underline{\lim}_{r\to+\infty}\alpha(t,r) \geq \kappa_1,$$

$$\beta^\infty(t) := \overline{\lim}_{r\to+\infty}\beta(t,r) \leq \kappa_2, \quad \beta_\infty(t) := \underline{\lim}_{r\to+\infty}\beta(t,r) \geq \kappa_1,$$

and $\alpha^\infty(t), \alpha_\infty(t), \beta^\infty(t), \beta_\infty(t)$ are T-periodic functions. We assume that these functions are Hölder continuous.

We will need some simple variants of Lemma 3.7 and Remark 3.8, whose proofs are similar to the original ones and therefore omitted.

Lemma 4.1. *Let $d_1(s), d_2(s)$ and $l(s)$ be Hölder continuous functions for $s \geq 0$, all positive except possibly $d_2(s)$. Let $g(t,r,u)$ be a function satisfying (3.1). Suppose that $\mathcal{T} \in (0,\infty)$, $\overline{h} \in C^1([0,\mathcal{T}])$, $\overline{u} \in C^{1,2}(D^*_{\mathcal{T}})$ with $D^*_{\mathcal{T}} = \{(t,r) \in \mathbb{R}^2 : 0 \leq t \leq \mathcal{T}, 0 \leq r \leq \overline{h}(t)\}$, and*

$$\begin{cases} \overline{u}_t - d_1(r)\overline{u}_{rr} - d_2(r)\overline{u}_r \geq g(t,r,\overline{u}), & t \in (0,\mathcal{T}],\ 0<r<\overline{h}(t), \\ \overline{u}(t,\overline{h}(t))=0, \quad \overline{h}'(t) \geq -\mu\overline{u}_r(t,\overline{h}(t)), & t \in (0,\mathcal{T}], \\ \overline{u}(t,0) \geq l(t), & t \in (0,\mathcal{T}]. \end{cases}$$

If $h \in C^1([0,\mathcal{T}])$ and $u \in C^{1,2}(D_{\mathcal{T}})$ with $D_{\mathcal{T}} = \{(t,r) \in \mathbb{R}^2 :\ 0 \leq t \leq \mathcal{T},\ 0 \leq r \leq h(t)\}$ satisfy

$$0 < h(0) \leq \overline{h}(0),\ 0 < u(0,r) \leq \overline{u}(0,r) \ \text{ for } 0 \leq r \leq h(0),$$

and

$$\begin{cases} u_t - d_1(r)u_{rr} - d_2(r)u_r = g(t,r,u), & t \in (0,\mathcal{T}],\ 0<r<h(t), \\ u(t,h(t))=0, \ h'(t) = -\mu u_r(t,h(t)), & t \in (0,\mathcal{T}], \\ u(t,0)=l(t), & t \in (0,\mathcal{T}], \end{cases} \tag{4.1}$$

then

$$h(t) \leq \overline{h}(t) \ \text{ for } t \in (0,\mathcal{T}], \quad u(t,r) \leq \overline{u}(t,r) \ \text{ for } (t,r) \in (0,\mathcal{T}] \times (0,h(t)).$$

Similar to Remark 3.8, we have the following analogue of Lemma 4.1:

Lemma 4.2. *Let $d_1(s), d_2(s)$, $l(s)$, and $g(t,r,u)$ be as in Lemma 4.1. Suppose that $\mathcal{T} \in (0,\infty)$, $\underline{h} \in C^1([0,\mathcal{T}])$, $\underline{u} \in C^{1,2}(D^\dagger_{\mathcal{T}})$ with $D^\dagger_{\mathcal{T}} = \{(t,r) \in \mathbb{R}^2 : 0 \le t \le \mathcal{T}, 0 \le r \le \underline{h}(t)\}$, and*

$$\begin{cases} \underline{u}_t - d_1(r)\underline{u}_{rr} - d_2(r)\underline{u}_r \le g(t,r,\underline{u}), & t \in (0,\mathcal{T}],\ 0 < r < \underline{h}(t), \\ \underline{u}(t,\underline{h}(t)) = 0, \quad \underline{h}'(t) \le -\mu \underline{u}_r(t,\underline{h}(t)), & t \in (0,\mathcal{T}], \\ \underline{u}(t,0) \le l(t), & t \in (0,\mathcal{T}]. \end{cases}$$

If $h \in C^1([0,\mathcal{T}])$, $u \in C^{1,2}(D^\dagger_{\mathcal{T}})$ satisfy (4.1) and

$$h(0) \ge \underline{h}(0), \quad u(0,r) \ge \underline{u}(0,r) \ge 0 \text{ for } 0 \le r \le \underline{h}(0),$$

then

$$h(t) \ge \underline{h}(t) \text{ in } [0,\mathcal{T}],\ u(r,t) \ge \underline{u}(r,t) \text{ for } t \in [0,\mathcal{T}] \text{ and } r \in (0,\underline{h}(t)).$$

We also need the following result.

Lemma 4.3. *Suppose that $d > 0$, $c(s)$ and $l(s)$ are Hölder continuous functions for $s \ge 0$ with $l(s)$ positive, and $a(t), b(t)$ are Hölder continuous positive T-periodic functions. Let $v \in C^{1,2}(D)$ $(D = \{(t,r) : 0 \le r \le \sigma(t), t \ge 0\})$ be a solution of*

$$\begin{cases} v_t - dv_{rr} + c(r)v_r = v(a(t) - b(t)v), & t > 0, 0 < r < \sigma(t), \\ v(t,0) = l(t), \quad v(t,\sigma(t)) = 0, & t > 0, \\ v(0,r) = v_0(r) \ge 0, & 0 < r < \sigma(0). \end{cases} \tag{4.2}$$

Suppose that

$$\lim_{r\to\infty} c(r) = 0, \quad \lim_{t\to\infty} \sigma(t) = \infty$$

and

$$\lim_{n\to\infty} l(t+nT) = \ell(t) \ge V(t) \text{ uniformly for } t \in [0,T],$$

where $\ell(t)$ is a T-periodic function and $V(t)$ is the unique positive solution of (2.11). Then

$$\underline{\lim}_{n\to\infty} v(t+nT,r) \ge V(t) \text{ locally uniformly for } (t,r) \in [0,T] \times [0,\infty).$$

Proof. By the maximum principle, $v(t,r) > 0$ for $t > 0$ and $0 \le r < \sigma(t)$. For any given $R > 0$ and small $\epsilon > 0$, we can find $T_R > 0$ such that $\sigma(t) > R$ and $l(t) \ge \ell(t) - \epsilon > 0$ for all $t \ge T_R$. We now consider the auxiliary problem

$$\begin{cases} w_t - dw_{rr} + c(r)w_r = w(a(t) - b(t)w), & t > T_R, 0 < r < R, \\ w(t,0) = \ell(t) - \epsilon, \quad w(t,R) = 0, & t > T_R, \\ w(T_R,r) = v(T_R,r), & 0 < r < R. \end{cases} \tag{4.3}$$

It is well known that the logistic equation (4.3) admits a unique positive solution $w(t,r)$. By the comparison principle we see

$$w(t,r) \le v(t,r) \text{ for } t > T_R \text{ and } 0 \le r \le R.$$

Clearly $w \equiv 0$ is a lower solution to (4.3), and for any large positive constant M, $w \equiv M$ is an upper solution to (4.3). Thus the unique solution of problem (4.3) with initial function $v(T_R,r)$ replaced by 0, which we denote by $w_*(t,r)$, is increasing in t, and the solution $w^*(t,r)$ of the same problem with initial function $v(T_R,r)$ replaced by M is decreasing in t. Moreover,

$$\lim_{n\to\infty} w_*(t+nT,r) = \underline{w}(t,r) \ \ \forall (t,r) \in [0,T] \times [0,R], \tag{4.4}$$

$$\lim_{n\to\infty} w^*(t+nT,r) = \overline{w}(t,r) \ \ \forall (t,r) \in [0,T] \times [0,R], \tag{4.5}$$

with $\underline{w} \le \overline{w}$, and both $\underline{w}$ and $\overline{w}$ are positive solutions of the problem

$$(4.6)\qquad \begin{cases} w_t - dw_{rr} + c(r)w_r = w(a(t) - b(t)w), & (t,r) \in [0,T] \times [0,R], \\ w(t,0) = \ell(t) - \epsilon, \;\; w(t,R) = 0, & t \in [0,T], \\ w(0,r) = w(T,r), & r \in [0,R]. \end{cases}$$

Since the nonlinear term in (4.6) is concave, it is well known (see [20]) that $\underline{w} \equiv \overline{w} \equiv w_R^\epsilon$, the unique positive solution of (4.6). Hence

$$\lim_{n\to\infty} w_*(t+nT,r) = \lim_{n\to\infty} w^*(t+nT,r) = w_R^\epsilon(t,r).$$

By the comparison principle, the solution of (4.3) satisfies

$$w_*(t,r) \le w(t,r) \le w^*(t,r) \;\text{ for } t \ge T_R \text{ and } 0 \le r \le R.$$

This implies that

$$\lim_{n\to\infty} w(t+nT,r) = w_R^\epsilon(t,r) \text{ uniformly in } [0,T]\times[0,R].$$

Thus,

$$\underline{\lim}_{n\to\infty} v(t+nT,r) \ge w_R^\epsilon(t,r) \quad \text{uniformly for } (t,r) \in [0,T]\times[0,R].$$

Letting $\epsilon \to 0$ we deduce

$$\underline{\lim}_{n\to\infty} v(t+nT,r) \ge w_R^0(t,r) \quad \text{for } (t,r) \in [0,T]\times[0,R],$$

where $w_R^0(t,r)$ is the unique positive solution of (4.6) with $\epsilon = 0$. (The uniqueness of positive solutions to (4.6) implies the continuous dependence of w_R^ϵ on ϵ.)

A simple upper and lower solution argument shows that $w_R^0(t,r)$ is increasing in R, and it has a constant upper bound independent of R. Using this fact and a standard regularity consideration, we find that as R increases to infinity, $w_R^0(t,r)$ increases to the minimal positive solution $W(t,r)$ of

$$(4.7)\qquad \begin{cases} W_t - dW_{rr} + c(r)W_r = W(a(t) - b(t)W), & (t,r) \in [0,T] \times (0,\infty), \\ W(t,0) = \ell(t), & t \in [0,T], \\ W(0,r) = W(T,r), & r \in [0,\infty). \end{cases}$$

Thus,

$$(4.8)\qquad \underline{\lim}_{n\to\infty} v(t+nT,r) \ge W(t,r) \;\text{ locally uniformly for } (t,r) \in [0,T]\times[0,\infty).$$

We show next that

$$W(t,r) \ge V(t) \;\text{ for } (t,r) \in [0,T]\times[0,\infty).$$

Let R_n be a positive sequence increasing to ∞ as $n \to \infty$, and then define $c_n(r) = c(R_n + r)$. Since $c(r) \to 0$ as $r \to \infty$, clearly $c_n(r) \to 0$ locally uniformly in $\mathbb{R}$. It follows that the first eigenvalue $\lambda_1^n(R)$ of

$$-du_{rr} + c_n(r)u_r = \lambda u \text{ in } [-R,R], \; u(-R) = u(R) = 0$$

converges to $d\pi^2/(4R^2)$ as $n \to \infty$. Fix $R > 0$ large enough such that $d\pi^2/(4R^2) < \underline{a} = \min_{t\in\mathbb{R}} a(t)$, then the logistic problem

$$-du_{rr} + c_n(r)u_r = \underline{a}u - \|b\|_\infty u^2 \text{ in } (-R,R), \; u(-R) = u(R) = 0$$

has a unique positive solution u_n for all large n, and a simple regularity argument shows that $u_n \to u_0$ as $n \to \infty$ uniformly in $[-R,R]$, where u_0 is the unique positive solution of

$$-du_{rr} = \underline{a}u - \|b\|_\infty u^2 \text{ in } (-R,R), \; u(-R) = u(R) = 0.$$

We now define $W_n(t,r) = W(t, R_n + r)$ for $r \in [-R_n, R_n]$. Clearly W_n a positive solution of the problem

$$\left\{ \begin{array}{ll} w_t - dw_{rr} + c_n(r)w_r = w(a(t) - b(t)w), & (t,r) \in [0,T] \times (-R_n, R_n), \\ w(t, -R_n) = \ell(t), \;\; w(t, R_n) = W(t, 2R_n), & t \in [0,T], \\ w(0,r) = w(T,r), & r \in [-R_n, R_n]. \end{array} \right. \tag{4.9}$$

On the other hand, u_n extended to 0 outside $[-R, R]$ is a lower solution of (4.9), and any large positive constant is an upper solution of (4.9). Moreover, as before, since the nonlinear term in (4.9) is concave, it has a unique positive solution. Thus $W_n(t,r) \geq u_n(r)$ in $[-R, R]$.

Applying the L^p estimate to (4.9) we find that $\{W_n\}$ is bounded in $W_p^{1,2}([0,T] \times [-M, M])$ for any $p > 1$ and any $M > 0$. Hence by passing to a subsequence we may assume that $W_n \to W^*$ in $C^{1,2}_{\text{loc}}([0,T] \times \mathbb{R})$, and W^* is a weak solution of

$$\left\{ \begin{array}{ll} w_t - dw_{rr} = w(a(t) - b(t)w), & (t,r) \in [0,T] \times \mathbb{R}, \\ w(0,r) = w(T,r), & r \in \mathbb{R}. \end{array} \right. \tag{4.10}$$

Since $W_n \geq u_n$ we deduce by letting $n \to \infty$ that $W^* \geq u_0$ in $[0,T] \times [-R, R]$. Thus W^* must be a positive solution of (4.10). However, by [28], $w \equiv V(t)$ is the unique positive solution of (4.10). Thus we must have $W^* \equiv V(t)$. It follows that $W_n(t,r) \to V(t)$ locally uniformly in $(t,r) \in [0,T] \times \mathbb{R}$. In particular, $W(t, R_n) = W_n(t, 0) \to V(t)$ uniformly in $t \in [0,T]$. This implies that $W(t,r) \to V(t)$ uniformly in $t \in [0,T]$ as $r \to \infty$.

For any $\sigma \in (0,1)$, we can find $R_\sigma > 0$ large such that $W(t,r) > \sigma V(t)$ for $r \geq R_\sigma$. Fix an arbitrary $R > R_\sigma$ and consider the problem

$$\left\{ \begin{array}{ll} w_t - dw_{rr} + c(r)w_r = w(a(t) - b(t)w), & (t,r) \in [0,T] \times (0, R), \\ w(t, 0) = \ell(t), \; w(t, R) = W(t, R) & t \in [0,T], \\ w(0,r) = w(T,r), & r \in [0, R]. \end{array} \right. \tag{4.11}$$

Clearly W is the unique positive solution of (4.11). On the other hand, it is easily seen that $\sigma V(t)$ is a lower solution of (4.11), and any large constant is an upper solution of (4.11). It follows that $W(t,r) \geq \sigma V(t)$ in $[0,T] \times [0,R]$. Since $R \geq R_\sigma$ is arbitrary, this implies that $W(t,r) \geq \sigma V(t)$ in $[0,T] \times [0,\infty)$. Letting $\sigma \to 1$, we deduce $W(t,r) \geq V(t)$ in $[0,T] \times [0,\infty)$.

We may now use (4.8) to obtain

$$\underline{\lim}_{n\to\infty} v(t + nT, r) \geq V(t) \;\text{ locally uniformly for } (t,r) \in [0,T] \times [0,\infty).$$

This completes the proof. □

We are now ready to prove the first main result of this section.

Theorem 4.4. *Suppose that* (u, h) *is the unique solution of* (1.1) *and* $h_\infty = +\infty$; *then*

$$\overline{\lim}_{t\to\infty} \frac{h(t)}{t} \leq \frac{1}{T} \int_0^T k_0(\mu, \alpha^\infty, \beta_\infty)(t)dt, \tag{4.12}$$

$$\underline{\lim}_{t\to\infty} \frac{h(t)}{t} \geq \frac{1}{T} \int_0^T k_0(\mu, \alpha_\infty, \beta^\infty)(t)dt, \tag{4.13}$$

where $k_0(\mu, \cdot, \cdot)$ *is given in Theorems 2.4 and 2.5.*

Proof. We divide the proof into three steps.

Step 1. The unique positive (radial) solution $\hat{U}$ of (3.27) satisfies

$$\overline{\lim}_{r\to\infty} \hat{U}(t,r) \leq \overline{V}(t), \quad \underline{\lim}_{r\to\infty} \hat{U}(t,r) \geq \underline{V}(t) \;\text{ for } t \in [0,T], \tag{4.14}$$

where $\overline{V}(t)$ and $\underline{V}(t)$ are, respectively, the unique positive T-periodic solutions of

$$\frac{d\overline{V}}{dt} = \overline{V}[\alpha^\infty(t) - \beta_\infty(t)\overline{V}] \text{ in } [0,T], \quad \overline{V}(0) = \overline{V}(T) \tag{4.15}$$

and

$$\frac{d\underline{V}}{dt} = \underline{V}[\alpha_\infty(t) - \beta^\infty(t)\underline{V}] \text{ in } [0,T], \quad \underline{V}(0) = \underline{V}(T). \tag{4.16}$$

For any small $\varepsilon > 0$, there is $R_* := R(\varepsilon) > 1$ such that for $r \ge R_*$,

$$\alpha(t,r) \le \alpha_\varepsilon^\infty(t) := \alpha^\infty(t) + \varepsilon, \quad \alpha(t,r) \ge \alpha_\infty^\varepsilon(t) := \alpha_\infty(t) - \varepsilon,$$

$$\beta(t,r) \le \beta_\varepsilon^\infty(t) := \beta^\infty(t) + \varepsilon, \quad \beta(t,r) \ge \beta_\infty^\varepsilon(t) := \beta_\infty(t) - \varepsilon.$$

For $R > R_*$, we consider the problem

$$\begin{cases} z_t - d\Big[z_{rr} + \frac{N-1}{r} z_r\Big] = z[\alpha_\varepsilon^\infty(t) - \beta_\infty^\varepsilon(t) z], & (t,r) \in (0,T) \times (R_*, R), \\ z(t,R_*) = v_R(t,R_*), \ \ z(t,R) = 0, & t \in [0,T], \\ z(0,r) = z(T,r), & r \in [R_*, R], \end{cases} \tag{4.17}$$

where v_R is the unique solution of (3.28). Clearly v_R is a lower solution of (4.17) and any large constant M is an upper solution of (4.17). Hence it has a positive solution. Since the nonlinear term is concave, the positive solution is unique, which we denote by z_R^ε. Therefore,

$$v_R(t,r) \le z_R^\varepsilon(t,r) \le M \ \ \forall (t,r) \in [0,T] \times [R_*, R].$$

Much as before (see also the proof of Theorem 1.3 of [28]),

$$z_R^\varepsilon \nearrow \hat{Z}^\varepsilon \text{ locally uniformly in } [0,T] \times [R_*, \infty) \text{ as } R \to \infty,$$

where $\hat{Z}^\varepsilon$ satisfies

$$\begin{cases} z_t - d\Big[z_{rr} + \frac{N-1}{r} z_r\Big] = z[\alpha_\varepsilon^\infty(t) - \beta_\infty^\varepsilon(t) z], & (t,r) \in (0,T) \times (R_*, \infty), \\ z(t,R_*) = \hat{U}(t,R_*), & t \in [0,T], \\ z(0,r) = z(T,r), & r \in [R_*, \infty). \end{cases}$$

Recalling that $v_R \to \hat{U}$ as $R \to \infty$, we deduce

$$\hat{U}(t,r) \le \hat{Z}^\varepsilon(t,r) \ \ \forall (t,r) \in [0,T] \times [R_*, \infty). \tag{4.18}$$

Making use of Lemma 3.2 in [28] we easily deduce that

$$\lim_{r\to\infty} \hat{Z}^\varepsilon(t,r) = \overline{V}_\varepsilon(t) \ \text{ uniformly for } t \in [0,T],$$

where $\overline{V}_\varepsilon$ is the unique positive T-periodic solution of the problem

$$\frac{dV}{dt} = V[\alpha_\varepsilon^\infty(t) - \beta_\infty^\varepsilon(t) V] \text{ in } [0,T], \quad V(0) = V(T). \tag{4.19}$$

Thus,

$$\overline{\lim}_{r\to\infty} \hat{U}(t,r) \le \overline{V}_\varepsilon(t) \ \text{ for } t \in [0,T]. \tag{4.20}$$

Since $\overline{V}_\varepsilon$ varies continuously in ε, letting $\varepsilon \to 0$ we obtain

$$\overline{\lim}_{r\to\infty} \hat{U}(t,r) \le \overline{V}(t) \ \text{ for } t \in [0,T].$$

This proves the first inequality in (4.14). The second inequality in (4.14) can be obtained similarly by considering the problem

$$\begin{cases} \underline{z}_t - d\Big[\underline{z}_{rr} + \frac{N-1}{r}\underline{z}_r\Big] = \underline{z}[\alpha^{\varepsilon}_{\infty}(t) - \beta^{\infty}_{\varepsilon}(t)\underline{z}], & (t,r) \in (0,T)\times(R_*,R), \\ \underline{z}(t,R_*) = v_R(t,R_*), \ \ \underline{z}(t,R) = 0, & t \in [0,T], \\ \underline{z}(0,r) = \underline{z}(T,r), & r \in [R_*,R], \end{cases} \tag{4.21}$$

from which we obtain

$$\underline{\lim}_{r\to\infty}\hat{U}(t,r) \geq \underline{V}_{\varepsilon}(t) \ \text{ for } t \in [0,T], \tag{4.22}$$

where $\underline{V}_{\varepsilon}$ is the unique positive solution of (4.19) but with $(\alpha^{\infty}_{\varepsilon}, \beta^{\varepsilon}_{\infty})$ replaced by $(\alpha^{\varepsilon}_{\infty}, \beta^{\infty}_{\varepsilon})$. Letting $\varepsilon \to 0$ one gets

$$\underline{\lim}_{r\to\infty}\hat{U}(t,r) \geq \underline{V}(t) \ \text{ for } t \in [0,T].$$

Step 2. We prove (4.12).

By (4.14) there exists $R^\star := R^\star(\varepsilon) > R_* > 1$ such that

$$\underline{V}_{\frac{\varepsilon}{2}}(t) \leq \hat{U}(t,r) \leq \overline{V}_{\frac{\varepsilon}{2}}(t) \ \ \forall (t,r) \in [0,T]\times[R^\star,\infty).$$

Since $h_\infty = +\infty$ and $\lim_{n\to\infty} u(t+nT,r) = \hat{U}(t,r)$, there exists a positive integer $N = N(R^\star)$ such that with $\mathcal{T} := NT$,

$$h(\mathcal{T}) > 3R^\star \ \text{ and } \ u(t+\mathcal{T}, 2R^\star) < \overline{V}_{\varepsilon}(t) \quad \text{for all } t \geq 0.$$

Setting

$$\tilde{u}(t,r) = u(t+\mathcal{T}, r+2R^\star) \ \text{ and } \tilde{h}(t) = h(t+\mathcal{T}) - 2R^\star$$

and denoting

$$\tilde{\Delta}u = u_{rr} + \frac{N-1}{r+2R^\star}u_r,$$

we obtain

$$\begin{cases} \tilde{u}_t - d\tilde{\Delta}\tilde{u} = \tilde{u}\Big[\alpha(t,r+2R^\star) - \beta(t,r+2R^\star)\tilde{u}\Big], & t>0, \ 0<r<\tilde{h}(t), \\ \tilde{u}(t,0) = u(t+\mathcal{T},2R^\star), \quad \tilde{u}(t,\tilde{h}(t)) = 0, & t>0, \\ \tilde{h}'(t) = -\mu\tilde{u}_r(t,\tilde{h}(t)), & t>0, \\ \tilde{u}(0,r) = u(\mathcal{T}, r+2R^\star), & 0<r<\tilde{h}(0). \end{cases} \tag{4.23}$$

By our choice of $R^\star$, for $r \geq 0$,

$$\alpha(t,r+2R^\star) \leq \alpha^{\infty}_{\varepsilon}(t), \quad \beta(t,r+2R^\star) \geq \beta^{\varepsilon}_{\infty}(t).$$

Let $u^*(t)$ be the unique solution of the problem

$$\frac{\mathrm{d}u^*}{\mathrm{d}t} = u^*(\alpha^{\infty}_{\varepsilon}(t) - \beta^{\varepsilon}_{\infty}(t)u^*) \quad \text{ for } \ t>0; \ u^*(0) = \max\{\overline{V}_{\varepsilon}, ||\tilde{u}(0,\cdot)||_\infty\}. \tag{4.24}$$

Then

$$u^*(t) \geq \overline{V}_{\varepsilon}(t) \ \text{ for all } t \geq 0 \text{ and } \lim_{n\to\infty} u^*(t+nT) = \overline{V}_{\varepsilon}(t).$$

Now we have

$$u^*(0) \geq \tilde{u}(0,r), \ \ \tilde{u}(t,0) \leq \overline{V}_{\varepsilon}(t) \leq u^*(t), \ \tilde{u}(t,\tilde{h}(t)) = 0 \leq u^*(t),$$

and

$$u^*_t - d\tilde{\Delta}u^* = u^*(\alpha^{\infty}_{\varepsilon}(t) - \beta^{\varepsilon}_{\infty}(t)u^*) \geq u^*[\alpha(t,r+2R^\star) - \beta(t,r+2R^\star)u^*].$$

Hence we can apply the comparison principle to deduce

$$\tilde{u}(t,r) \leq u^*(t) \ \text{ for } 0<r<\tilde{h}(t), \ t>0. \tag{4.25}$$

As a consequence, there exists $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_\varepsilon = \tilde{N}T > 0$ (with an integer $\tilde{N}$) such that

$$\tilde{u}(t,r) \leq \overline{V}_\varepsilon(t)(1-\varepsilon)^{-1}, \quad \forall t \geq \tilde{\mathcal{T}},\ r \in [0,\tilde{h}(t)].$$

Let $U_\varepsilon = U_{\alpha_\varepsilon^\infty,\beta_\infty^\varepsilon,k^\varepsilon}$ denote the unique positive solution of (2.1) with $a(t) = \alpha_\varepsilon^\infty(t)$, $b(t) = \beta_\infty^\varepsilon(t)$ and $k(t) = k^\varepsilon(t) := k_0(\mu,\alpha_\varepsilon^\infty,\beta_\infty^\varepsilon)(t)$. Since

$$U_\varepsilon(t,r) \to \overline{V}_\varepsilon(t) \text{ in } [0,T] \text{ as } r \to +\infty,$$

there exists $R_0^\star := R_0^\star(\varepsilon) > 2R^\star$ such that

$$U_\varepsilon(t,r) > \overline{V}_\varepsilon(t)(1-\varepsilon) \text{ for } (t,r) \in [0,T] \times [R_0^\star,\infty).$$

We now define

$$\xi(t) = (1-\varepsilon)^{-2}\int_0^t k^\varepsilon(s)ds + R_0^\star + \tilde{h}(\tilde{\mathcal{T}}) \quad \text{for } t \geq 0,$$
$$w(t,r) = (1-\varepsilon)^{-2}U_\varepsilon(t,\xi(t)-r) \text{ for } t \geq 0,\ 0 \leq r \leq \xi(t).$$

Then

$$\xi'(t) = (1-\varepsilon)^{-2}k^\varepsilon(t),$$

$$-\mu w_r(t,\xi(t)) = \mu(1-\varepsilon)^{-2}(U_\varepsilon)_r(t,0) = (1-\varepsilon)^{-2}k^\varepsilon(t),$$

and so we have

$$\xi'(t) = -\mu w_r(t,\xi(t)).$$

Clearly

$$w(t,\xi(t)) = 0,\ \xi(0) = R_0^\star + \tilde{h}(\tilde{\mathcal{T}}) > \tilde{h}(\tilde{\mathcal{T}}).$$

Moreover, for $0 \leq r \leq \tilde{h}(\tilde{\mathcal{T}})$,

$$w(0,r) = (1-\varepsilon)^{-2}U_\varepsilon(0,\xi(0)-r) \geq (1-\varepsilon)^{-2}U_\varepsilon(0,R_0^\star) \geq \overline{V}_\varepsilon(0)(1-\varepsilon)^{-1} \geq \tilde{u}(\tilde{\mathcal{T}},r)$$

and $w(0,r) > 0$ for $\tilde{h}(\tilde{\mathcal{T}}) < r < \xi(0)$. It is also easily seen that for $t > 0$,

$$w(t,0) = (1-\varepsilon)^{-2}U_\varepsilon(t,\xi(t)) \geq (1-\varepsilon)^{-2}U_\varepsilon(t,R_0^\star) \geq \overline{V}_\varepsilon(t)(1-\varepsilon)^{-1} \geq \tilde{u}(t+\tilde{\mathcal{T}},0).$$

Direct calculations show that, for $t > 0$ and $0 < r < \xi(t)$, with $\rho = \xi(t) - r$,

$$\begin{aligned}
w_t - d\tilde{\Delta}w &= (1-\varepsilon)^{-2}\left[(U_\varepsilon)_t + (U_\varepsilon)_\rho\xi'(t) - d(U_\varepsilon)_{\rho\rho} + \frac{d(N-1)}{r+2R^\star}(U_\varepsilon)_\rho\right] \\
&= (1-\varepsilon)^{-2}\left[(U_\varepsilon)_t + (1-\varepsilon)^{-2}k^\varepsilon(t)(U_\varepsilon)_\rho - d(U_\varepsilon)_{\rho\rho} + \frac{d(N-1)}{r+2R^\star}(U_\varepsilon)_\rho\right] \\
&\geq (1-\varepsilon)^{-2}((U_\varepsilon)_t + k^\varepsilon(t)(U_\varepsilon)_\rho - d(U_\varepsilon)_{\rho\rho}) \quad (\text{due to } (U_\varepsilon)_\rho \geq 0) \\
&= (1-\varepsilon)^{-2}U_\varepsilon(\alpha_\varepsilon^\infty(t) - \beta_\infty^\varepsilon(t)U_\varepsilon) \\
&= w[\alpha_\varepsilon^\infty(t) - (1-\varepsilon)^2\beta_\infty^\varepsilon(t)w] \\
&\geq w[\alpha_\varepsilon^\infty(t) - \beta_\infty^\varepsilon(t)w].
\end{aligned}$$

Hence we can use Lemma 4.1 to conclude that

$$\tilde{u}(t+\tilde{\mathcal{T}},r) \leq w(t,r), \quad \tilde{h}(t+\tilde{\mathcal{T}}) \leq \xi(t) \text{ for } t \geq 0,\ 0 \leq r \leq \tilde{h}(t+\tilde{\mathcal{T}}).$$

It follows that

$$\begin{aligned}\overline{\lim}_{t\to+\infty}\frac{h(t)}{t} &= \overline{\lim}_{t\to+\infty}\frac{\tilde{h}(t-\mathcal{T})+2R^\star}{t}\\ &\le \lim_{t\to+\infty}\frac{\xi(t-(\mathcal{T}+\tilde{\mathcal{T}}))+2R^\star}{t}\\ &= \lim_{t\to+\infty}\frac{(1-\varepsilon)^{-2}\int_0^t k^\varepsilon(s)ds+R_0^\star+\tilde{h}(\tilde{\mathcal{T}})+2R^\star}{t}\\ &= (1-\varepsilon)^{-2}\frac{1}{T}\int_0^T k^\varepsilon(t)dt.\end{aligned}$$

Since $\varepsilon>0$ can be arbitrarily small, and $k^\varepsilon(t)\to k_0(\mu,\alpha^\infty,\beta_\infty)(t)$ as $\varepsilon\to 0$, we deduce

$$\overline{\lim}_{t\to+\infty}\frac{h(t)}{t}\le\frac{1}{T}\int_0^T k_0(\mu,\alpha^\infty,\beta_\infty)(t)dt.$$

Step 3. We show

$$\underline{\lim}_{t\to\infty}\frac{h(t)}{t}\ge\frac{1}{T}\int_0^T k_0(\mu,\alpha_\infty,\beta^\infty)(t)dt$$

by constructing a suitable lower solution.

To this end, we denote

$$k_\varepsilon(t)=k_0(\mu,\alpha_\infty^\varepsilon,\beta_\varepsilon^\infty)(t)\ \text{ and }\ Z_\varepsilon(t,r)=U_{\alpha_\infty^\varepsilon,\beta_\varepsilon^\infty,k_\varepsilon}(t,r).$$

We consider the auxiliary problem

$$\text{(4.26)}\qquad \begin{cases} v_t-d\tilde{\Delta}v=v(\alpha_\infty^\varepsilon(t)-\beta_\varepsilon^\infty(t)v), & t>0,\ 0<r<\tilde{h}(t),\\ v(t,0)=\tilde{u}(t,0),\quad v(t,\tilde{h}(t))=0, & t>0,\\ v(0,r)=\tilde{u}(0,r), & r\in[0,\tilde{h}(0)],\end{cases}$$

where $\tilde{u}$ and $\tilde{h}$ are defined as before. Since

$$\lim_{n\to\infty}\tilde{u}(t+nT,0)\to\hat{U}(t,2R^\star)>\underline{V}_{\varepsilon/2}\quad\forall t\in[0,T]$$

we can apply Lemma 4.3 to (4.26) to conclude that

$$\text{(4.27)}\qquad \underline{\lim}_{n\to\infty}v(t+nT,r)\ge\underline{V}_\varepsilon(t)\ \text{ locally uniformly for } (t,r)\in[0,T]\times[0,\infty).$$

Since

$$\alpha(t,r+2R^\star)\ge\alpha_\infty^\varepsilon(t),\quad \beta(t,r+2R^\star)\le\beta_\varepsilon^\infty(t)\quad\forall t\in[0,T],$$

from the comparison principle we deduce

$$\tilde{u}(t,r)\ge v(t,r)\ \text{ for } t>0,\ r\in[0,\tilde{h}(t)],$$

and hence, in view of (4.27), we have

$$\text{(4.28)}\qquad \underline{\lim}_{n\to\infty}\tilde{u}(t+nT,r)\ge\underline{V}_\varepsilon(t)\ \text{ locally uniformly for } (t,r)\in[0,T]\times[0,\infty).$$

Define

$$\eta(t)=(1-\varepsilon)^2\int_0^t k_\varepsilon(s)ds+\tilde{h}(0)\quad\text{for } t\ge 0,$$

and

$$w(t,r)=(1-\varepsilon)^2 Z_\varepsilon(t,\eta(t)-r)\quad\text{for }\ t\ge 0,\ \ 0\le r\le\eta(t).$$

Then

$$\eta'(t)=(1-\varepsilon)^2k_\varepsilon(t)\quad\forall t\in[0,T],$$
$$-\mu w_r(t,\eta(t))=\mu(1-\varepsilon)^2(Z_\varepsilon)_r(t,0)=(1-\varepsilon)^2k_\varepsilon(t)\quad\forall t\in[0,T],$$

and so we have

$$\eta'(t) = -\mu w_r(t, \eta(t)).$$

Clearly, $w(t, \eta(t)) = 0$. Since

$$(Z_\varepsilon)_r(t, r) \geq 0 \text{ for } (t, r) \in [0, T] \times [0, \infty)$$

and

$$\lim_{r \to +\infty} Z_\varepsilon(t, r) = \underline{V}_\varepsilon(t) \quad \forall t \in [0, T],$$

we must have

$$Z_\varepsilon(t, r) \leq \underline{V}_\varepsilon(t) \ \ \forall (t, r) \in [0, T] \times (0, \infty).$$

Therefore, due to (4.28) we can find some $\hat{\mathcal{T}} = \hat{\mathcal{T}}(\varepsilon) = \hat{\mathcal{N}}T > 0$ (with an integer $\hat{\mathcal{N}}$) such that

$$\tilde{u}(t + \hat{\mathcal{T}}, 0) \geq w(t, 0) \text{ for } t \geq 0 \tag{4.29}$$

and

$$\tilde{u}(\hat{\mathcal{T}}, r) \geq w(0, r) \quad \text{for } r \in [0, \eta(0)]. \tag{4.30}$$

Direct calculations yield (with the notation $\theta = \eta(t) - r$)

$$\begin{aligned} w_t - d\tilde{\Delta} w &= (1-\varepsilon)^2 \Big[(Z_\varepsilon)_t + (Z_\varepsilon)_\theta \eta'(t) - d(Z_\varepsilon)_{\theta\theta} + \frac{d(N-1)}{r + 2R^\star} (Z_\varepsilon)_\theta \Big] \\ &= (1-\varepsilon)^2 \Big[\Big((1-\varepsilon)^2 k_\varepsilon(t) + \frac{d(N-1)}{r + 2R^\star} \Big) (Z_\varepsilon)_\theta + (Z_\varepsilon)_t - d(Z_\varepsilon)_{\theta\theta} \Big] \\ &\leq (1-\varepsilon)^2 \Big[k_\varepsilon(t)(Z_\varepsilon)_\theta + (Z_\varepsilon)_t - d(Z_\varepsilon)_{\theta\theta} \Big] \quad \text{(since } (Z_\varepsilon)_\theta \geq 0) \\ &\leq w[\alpha_\infty^\varepsilon(t) - \beta_\varepsilon^\infty(t) w] \quad \text{for } t \geq 0,\ 0 \leq r \leq \eta(t), \end{aligned}$$

where we have used the fact that for large $R^\star$,

$$(1-\varepsilon)^2 k_\varepsilon(t) + \frac{d(N-1)}{r + 2R^\star} \leq k_\varepsilon(t).$$

Hence, we can use Lemma 4.2 to conclude that

$$\tilde{u}(t + \hat{\mathcal{T}}, r) \geq w(t, r), \ \ \tilde{h}(t + \hat{\mathcal{T}}) \geq \eta(t) \quad \text{for } t \geq 0,\ 0 \leq r \leq \eta(t).$$

It follows that

$$\begin{aligned} \underline{\lim}_{t \to \infty} \frac{h(t)}{t} &= \underline{\lim}_{t \to \infty} \frac{\tilde{h}(t - \mathcal{T})}{t} \\ &\geq \lim_{t \to \infty} \frac{\eta(t - \mathcal{T} - \hat{\mathcal{T}})}{t} = (1-\varepsilon)^2 \frac{1}{T} \int_0^T k_\varepsilon(t) dt. \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, this implies

$$\underline{\lim}_{t \to \infty} \frac{h(t)}{t} \geq \frac{1}{T} \int_0^T k_0(\mu, \alpha_\infty, \beta^\infty)(t) dt.$$

The proof of the theorem is now complete. □

The result below follows trivially from Theorem 4.4.

Corollary 4.5. *Assume that* $h_\infty = +\infty$ *and*

$$\alpha(t, r) \to \alpha_*(t), \quad \beta(t, r) \to \beta_*(t) \ \ \textit{uniformly for } t \in [0, T] \textit{ as } r \to +\infty. \tag{4.31}$$

Then

$$\lim_{t \to \infty} \frac{h(t)}{t} = \frac{1}{T} \int_0^T k_0(\mu, \alpha_*, \beta_*)(t) dt.$$

Let $c_*(\mu)=\frac{1}{T}\int_0^T k_0(\mu,\alpha_*,\beta_*)(t)dt$. Our next result describes the large time behavior of the solution to (1.1) inside the ball $\{x:\ |x|<c_*(\mu)t\}$, which considerably improves the conclusion in (3.26).

Theorem 4.6. *Suppose that the assumptions of Corollary 4.5 hold, and $u(t,r)$, $\hat U(t,r)$ are as in (3.26). Then*

$$\lim_{t\to\infty}\max_{r\le[c_*(\mu)-\epsilon]t}|u(t,r)-\hat U(t,r)|=0 \tag{4.32}$$

for every small $\epsilon>0$.

Proof. Since (4.31) holds, we see that

$$\alpha^\infty(t)\equiv\alpha_\infty(t)\equiv\alpha_*(t),\quad \beta^\infty(t)\equiv\beta_\infty(t)\equiv\beta_*(t)\ \ \forall t\in[0,T],$$

and thus the proof of Theorem 4.4 implies that $\overline V(t)\equiv\underline V(t)\equiv V(t)$, where $V(t)$ is the unique positive solution of

$$\frac{dV}{dt}=V[\alpha_*(t)-\beta_*(t)]V,\quad V(0)=V(T).$$

Moreover, by Step 1 of the proof of Theorem 4.4, we have

$$\lim_{r\to\infty}\hat U(t,r)=V(t)\quad\text{uniformly for } t\in[0,T].$$

Hence for any given small $\epsilon>0$, there exists $R_\epsilon>0$ such that

$$\left|\hat U(t,r)-V(t)\right|<\epsilon\ \text{ for all } (t,r)\in\mathbb{R}\times[R_\epsilon,\infty). \tag{4.33}$$

We next make use of the estimates for $\tilde u(t,r)$ given in Step 2 of the proof of Theorem 4.4, and see that for any given small $\delta>0$, there exist positive numbers $\mathcal{T}^\delta$, R_1^δ and R_2^δ such that

$$u(t+\mathcal{T}^\delta,r+R_1^\delta)\le(1-\delta)^{-2}U^\delta(t,\xi(t)-r)\ \text{ for } t\ge0,\ 0\le r\le\xi(t), \tag{4.34}$$

where

$$\xi(t)=(1-\delta)^{-2}\int_0^t k^\delta(s)ds+R_2^\delta$$

and

$$U^\delta(t,r):=U_{\alpha_*+\delta,\beta_*-\delta,k^\delta}(t,r),\quad k^\delta(t):=k_0(\mu,\alpha_*+\delta,\beta_*-\delta)(t).$$

Similarly, by Step 3 of the proof of Theorem 4.4 there exist positive numbers $\tilde{\mathcal{T}}^\delta$, $\tilde R_1^\delta$, $\tilde R_2^\delta$ such that

$$u(t+\tilde{\mathcal{T}}^\delta,r+\tilde R_1^\delta)\ge(1-\delta)^2U_\delta(t,\eta(t)-r)\ \text{ for } t\ge0,\ 0\le r\le\eta(t), \tag{4.35}$$

where

$$\eta(t)=(1-\delta)^2\int_0^t k_\delta(s)ds+\tilde R_2^\delta$$

and

$$U_\delta(t,r):=U_{\alpha_*-\delta,\beta_*+\delta,k_\delta}(t,r),\quad k_\delta(t):=k_0(\mu,\alpha_*-\delta,\beta_*+\delta)(t).$$

Since

$$\lim_{\delta\to0}(1-\delta)^2k_\delta(t)=\lim_{\delta\to0}(1-\delta)^{-2}k^\delta(t)=k_0(\mu,\alpha_*,\beta_*)(t)\quad\text{uniformly for } t\in[0,T],$$

we can find $\delta_\epsilon\in(0,\epsilon)$ sufficiently small so that for all large t, say $t\ge T_\epsilon$,

$$|(1-\delta_\epsilon)^2\int_0^t k_{\delta_\epsilon}(s)ds-c_*(\mu)t|<\frac{\epsilon t}{2},\ \ |(1-\delta_\epsilon)^{-2}\int_0^t k^{\delta_\epsilon}(s)ds-c_*(\mu)t|<\frac{\epsilon t}{2}.$$

We now fix $\delta = \delta_\epsilon$ in U^δ, ξ, U_δ and η. Then clearly, for $t \geq T_\epsilon$,

$$\xi(t) - r \geq [c_*(\mu) - \epsilon]t - r + R_2^{\delta_\epsilon} + \frac{\epsilon}{2}t,$$

$$\eta(t) - r \geq [c_*(\mu) - \epsilon]t - r + \tilde{R}_2^{\delta_\epsilon} + \frac{\epsilon}{2}t.$$

By Proposition 2.1, we have

$$\lim_{r\to\infty} U^{\delta_\epsilon}(t,r) = V^{\delta_\epsilon}(t) \text{ uniformly for } t \in [0,T],$$

where $V^{\delta_\epsilon}(t)$ is the unique positive solution to

$$\frac{dV^{\delta_\epsilon}}{dt} = V^{\delta_\epsilon}[(\alpha_*(t) + \delta_\epsilon) - (\beta_*(t) - \delta_\epsilon)V^{\delta_\epsilon}], \quad V^{\delta_\epsilon}(0) = V^{\delta_\epsilon}(T);$$

and

$$\lim_{r\to\infty} U_{\delta_\epsilon}(t,r) = V_{\delta_\epsilon}(t) \text{ uniformly for } t \in [0,T],$$

where $V_{\delta_\epsilon}(t)$ is the unique positive solution of

$$\frac{dV_{\delta_\epsilon}}{dt} = V_{\delta_\epsilon}[(\alpha_*(t) - \delta_\epsilon) - (\beta_*(t) + \delta_\epsilon)V_{\delta_\epsilon}], \quad V_{\delta_\epsilon}(0) = V_{\delta_\epsilon}(T).$$

Thus, the monotonicity of $U^{\delta_\epsilon}(t,\cdot)$ implies that we can find $\hat{R}_1^\epsilon > 0$ such that for $r \geq \hat{R}_1^\epsilon$,

$$U^{\delta_\epsilon}(t,r) \leq V^{\delta_\epsilon}(t) \;\; \forall t \in \mathbb{R}$$

and

$$U_{\delta_\epsilon}(t,r) \geq V_{\delta_\epsilon}(t) - \epsilon \;\; \forall t \in \mathbb{R}.$$

It follows that, if

$$0 \leq r \leq [c_*(\mu) - \epsilon]t \quad \text{and} \quad t \geq \max\left\{\frac{2}{\epsilon}\hat{R}_1^\epsilon, T_\epsilon\right\},$$

then

$$u(t + \mathcal{T}^{\delta_\epsilon}, r + R_1^{\delta_\epsilon}) \leq (1-\delta_\epsilon)^{-2}U^{\delta_\epsilon}(t, \xi(t) - r) \leq (1-\epsilon)^{-2}V^{\delta_\epsilon}(t),$$

and

$$u(t + \tilde{\mathcal{T}}^{\delta_\epsilon}, r + \tilde{R}_1^{\delta_\epsilon}) \geq (1-\delta_\epsilon)^2 U_{\delta_\epsilon}(t, \eta(t) - r) \geq (1-\epsilon)^2[V_{\delta_\epsilon}(t) - \epsilon].$$

Without loss of generality, we may assume that $\mathcal{T}^{\delta_\epsilon}$ and $\tilde{\mathcal{T}}^{\delta_\epsilon}$ are both integer multiples of T. Thus, from the above inequalities we obtain

$$(1-\epsilon)^2[V_{\delta_\epsilon}(t) - \epsilon] \leq u(t,r) \leq (1-\epsilon)^{-2}V^{\delta_\epsilon}(t) \tag{4.36}$$

if

$$t \geq \frac{2}{\epsilon}\hat{R}_1^\epsilon + \max\{T_\epsilon, \mathcal{T}^{\delta_\epsilon}, \tilde{\mathcal{T}}^{\delta_\epsilon}\}$$

and

$$0 \leq r - R_1^{\delta_\epsilon} \leq [c_*(\mu) - \epsilon]t, \quad 0 \leq r - \tilde{R}_1^{\delta_\epsilon} \leq [c_*(\mu) - \epsilon]t.$$

We now take

$$\mathcal{T}_1^\epsilon := \frac{1}{\epsilon}\max\{2\hat{R}_1^\epsilon, R_1^{\delta_\epsilon}, \tilde{R}_1^{\delta_\epsilon}\} + \max\{T_\epsilon, \mathcal{T}^{\delta_\epsilon}, \tilde{\mathcal{T}}^{\delta_\epsilon}\}, \quad \tilde{R}_\epsilon := \max\{R_\epsilon, R_1^{\delta_\epsilon}, \tilde{R}_1^{\delta_\epsilon}\},$$

and find that (4.36) holds if

$$t \geq \mathcal{T}_1^\epsilon \text{ and } \tilde{R}_\epsilon \leq r \leq [c_*(\mu) - 2\epsilon]t.$$

In view of (4.33), this implies that, for such t and r,

$$|u(t,r) - \hat{U}(t,r)| \leq I(\epsilon),$$

where

$$I(\epsilon) = \epsilon + \max_{t\in[0,T]} \left\{ \left|(1-\epsilon)^2(V_{\delta_\epsilon}(t)-\epsilon) - V(t) - \epsilon\right|, \left|(1-\epsilon)^{-2}V^{\delta_\epsilon}(t) - V(t) + \epsilon\right| \right\}.$$

By (3.26),

$$\lim_{n\to\infty} u(t+nT, r) = \hat{U}(t,r) \text{ uniformly for } (t,r)\in[0,T]\times[0,\tilde{R}_\epsilon].$$

Hence we can find $\mathcal{T}_2^\epsilon > \mathcal{T}_1^\epsilon$ such that

$$|u(t,r) - \hat{U}(t,r)| \le I(\epsilon) \text{ for } t \ge \mathcal{T}_2^\epsilon \text{ and } 0 \le r \le \tilde{R}_\epsilon.$$

So finally we find that for all $t \ge \mathcal{T}_2^\epsilon$ and $0 \le r \le [c_*(\mu) - 2\epsilon]t$,

$$|u(t,r) - \hat{U}(t,r)| \le I(\epsilon).$$

Since $I(\epsilon) \to 0$ as $\epsilon \to 0$, this implies that (4.32) holds. The proof is now complete. □

We now use Theorem 4.6 to study the dynamical behavior of the solution of (1.1) when the parameter μ is large. Let (u^μ, h^μ) be the unique solution of (1.1). Suppose that the assumptions of Corollary 4.5 hold. Then for all large μ, spreading occurs and

$$\lim_{t\to\infty} \frac{h^\mu(t)}{t} = \overline{k_0(\mu)}.$$

By Theorem 2.8, $\overline{k_0(\mu)}$ increases to $2\sqrt{\overline{a}d}$ as μ increases to ∞. We have the following theorem for the behavior of u^μ.

Theorem 4.7. *Suppose that the assumptions of Corollary 4.5 hold. Then for any given small $\epsilon > 0$, there exists a large $\mu_\epsilon > 0$ such that*

$$\lim_{t\to\infty} \max_{r\le[2\sqrt{\overline{a}d}-\epsilon]t} |u^\mu(t,r) - \hat{U}(t,r)| = 0 \quad \textit{for all } \mu \ge \mu_\epsilon. \tag{4.37}$$

Proof. For any given small $\epsilon > 0$, since $\lim_{\mu\to\infty} \overline{k_0(\mu)} = 2\sqrt{\overline{a}d}$, we can find $\mu_\epsilon \ge \mu_0$ such that

$$\overline{k_0(\mu)} > 2\sqrt{\overline{a}d} - \epsilon/2 \;\; \forall \mu \ge \mu_\epsilon. \tag{4.38}$$

Moreover, by Theorem 4.6 for any $\mu \ge \mu_\epsilon$,

$$\lim_{t\to\infty} \max_{r\le[\overline{k_0(\mu)}-\epsilon/2]t} |u^\mu(t,r) - \hat{U}(t,r)| = 0. \tag{4.39}$$

On the other hand, for any $\mu \ge \mu_\epsilon$, we see from (4.38) that if $r \le [2\sqrt{\overline{a}d} - \epsilon]t$, then

$$r \le [\overline{k_0(\mu)} - \epsilon/2]t.$$

Therefore, it follows from (4.39) that for any $\mu \ge \mu_\epsilon$,

$$\lim_{t\to\infty} \max_{r\le[2\sqrt{\overline{a}d}-\epsilon]t} |u^\mu(t,r) - \hat{U}(t,r)| = 0.$$

□

References

[1] D.G. Aronson and H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in *Partial Differential Equations and Related Topics*, Lecture Notes in Math., Vol. 446, Springer, Berlin, 1975, pp. 5–49.

[2] D.G. Aronson and H.F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, Adv. Math. 30 (1978), 33–76.

[3] H. Berestycki, F. Hamel and G. Nadin, Asymptotic spreading in heterogeneous diffusive excitable media, J. Funct. Anal., 255 (2008), no. 9, 2146-2189.

[4] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems. I. Periodic framework, J. Eur. Math. Soc. 7 (2005), 173-213.

[5] H. Berestycki, F. Hamel and L. Roques, Analysis of the periodically fragmented environment model. I. Species persistence, J. Math. Biol. **51** (2005), 75-113.

[6] H. Berestycki, F. Hamel and L. Roques, Analysis of the periodically fragmented environment model. II. Biological invasions and pulsating travelling fronts, J. Math. Pures Appl. 84 (2005), 1101-1146.

[7] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: analysis of a free boundary model, Netw. Heterog. Media 7(2012), 583-603.

[8] E.N. Dancer and P. Hess, The symmetry of positive solutions of periodic-parabolic problems, J. Comput. Appl. Math. 52 (1994), no. 1-3, 81–89.

[9] Y. Du and Z.M. Guo, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary II, J. Diff. Eqns. 250(2011), 4336-4366.

[10] Y. Du and Z.M. Guo, The Stefan problem for the Fisher-KPP equation, J. Diff. Eqns. 253(2012), 996-1035.

[11] Y. Du and X. Liang, Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model, preprint.

[12] Y. Du and Z.G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal. 42(2010), 377-405.

[13] Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, preprint.

[14] Y. Du and L. Ma, Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions, *J. London Math. Soc.* 64 (2001), 107-124.

[15] Y. Du and H. Matano, Convergence and sharp thresholds for propagation in nonlinear diffusion problems, J. Eur. Math. Soc. 12 (2010), 279–312.

[16] Y. Du, H. Matano and K. Wang, Regularity and asymptotic behavior of nonlinear Stefan problems, preprint.

[17] Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, preprint.

[18] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* **7** (1937), 335–369.

[19] J.-S. Guo and C.-H. Wu, On a free boundary problem for a two-species competition system, J. Dyn. Diff. Eqns. 24(2012), 873-895.

[20] P. Hess, *Periodic-parabolic Boundary Value Problems and Positivity*, Pitman Res., Notes in Mathematics **247**, Longman Sci. Tech., Harlow, 1991.

[21] Y. Kaneko and Y. Yamada, A free boundary problem for a reaction-diffusion equation appearing in ecology. Adv. Math. Sci. Appl. 21(2011), no. 2, 467-492.

[22] P. Koch Medina and G. Schätti, Long-time behaviour for reaction-diffusion equations on $\mathbb{R}^N$, Nonlinear Anal. 25 (1995), 831-870.

[23] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov, Ètude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bull. Univ. Moscou Sér. Internat. **A1** (1937), 1–26; English transl. in: Dynamics of Curved Fronts, P. Pelcé (ed.), Academic Press, 1988, 105–130.

[24] M.A. Lewis and P. Kareiva, Allee dynamics and the spreading of invasive organisms, Theor. Population Bio. 43 (1993), 141-158.

[25] X. Liang and X-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.* **60** (2007), no. 1, 1–40.

[26] J. L. Lockwood, M.F. Hoopes and M.P. Marchetti, *Invasion Ecology*, Blackwell Publishing, 2007.

[27] G. Nadin, The principal eigenvalue of a space-time periodic parabolic operator, Ann. Mat. Pura Appl. 188 (2009), 269-295.

[28] R. Peng and Dong Wei, The periodic-parabolic logistic equation on $\mathbb{R}^N$, Discrete Contin. Dyn. Syst. 32(2012), no. 2, 619-641.

[29] R. Peng and X.-Q. Zhao, The diffusive logistic model with a free boundary and seasonal succession, Discrete Contin. Dyn. Syst. 33 (2013), 2007-2031.
[30] N. Shigesada and K. Kawasaki, *Biological Invasions: Theory and Practice*, Oxford Series in Ecology and Evolution, Oxford Univ. Press., Oxford, 1997.
[31] J.G. Skellam, Random dispersal in theoretical populations, Biometrika 38 (1951), 196-218.
[32] H.F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, J. Math. Biol. 45 (2002), no. 6, 511–548.
[33] J.X. Xin, Front propagation in heterogeneous media, SIAM Rev. 42 (2000), no. 2, 161–230.