REGULARITY AND ASYMPTOTIC BEHAVIOR OF NONLINEAR STEFAN PROBLEMS*

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ABSTRACT. We study the following nonlinear Stefan problem

$$\begin{cases} u_t - d\Delta u = g(u) & \text{for } x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu|\nabla_x u|^2 & \text{for } x \in \Gamma(t), t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^n$ $(n \geq 2)$ is bounded by the free boundary $\Gamma(t)$, with $\Omega(0) = \Omega_0$, μ and d are given positive constants. The initial function u_0 is positive in Ω_0 and vanishes on $\partial\Omega_0$. The class of nonlinear functions $g(u)$ includes the standard monostable, bistable and combustion type nonlinearities. We show that the free boundary $\Gamma(t)$ is smooth outside the closed convex hull of Ω_0, and as $t \to \infty$, either $\Omega(t)$ expands to the entire $\mathbb{R}^n$, or it stays bounded. Moreover, in the former case, $\Gamma(t)$ converges to the unit sphere when normalized, and in the latter case, $u \to 0$ uniformly. When $g(u) = au - bu^2$, we further prove that in the case $\Omega(t)$ expands to $\mathbb{R}^n$, $u \to a/b$ as $t \to \infty$, and the spreading speed of the free boundary converges to a positive constant; moreover, there exists $\mu^* \geq 0$ such that $\Omega(t)$ expands to $\mathbb{R}^n$ exactly when $\mu > \mu^*$.

1. Introduction

In this paper, we study the following nonlinear Stefan problem

$$(1.1) \qquad \begin{cases} u_t - d\Delta u = g(u) & \text{for } x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu|\nabla_x u|^2 & \text{for } x \in \Gamma(t), t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^n$ $(n \geq 2)$ is bounded by the free boundary $\Gamma(t)$, with $\Omega(0) = \Omega_0$, μ and d are given positive constants. We assume that Ω_0 is a bounded domain that agrees with the interior of its closure $\overline{\Omega}_0$, $\partial\Omega_0$ satisfies the interior ball condition, and $u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0)$ is positive in Ω_0 and vanishes on $\partial\Omega_0$. For the nonlinear function g, we make the following assumptions:

$$(1.2) \qquad \begin{cases} \text{(i) } g(0) = 0 \text{ and } g \in C^{1,\alpha}([0,\delta_0]) \text{ for some } \delta_0 > 0 \text{ and } \alpha \in (0,1), \\ \text{(ii) } g(u) \text{ is locally Lipschitz in } [0,\infty),\ g(u) \leq 0 \text{ in } [M,\infty) \text{ for some } M > 0. \end{cases}$$

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We note that these conditions are satisfied by standard monostable, bistable and combustion type nonlinearities. Less restrictions on g will be assumed in the main body of the paper when it is possible to do so.

By [8], (1.1) has a unique weak solution $u(t,x)$ defined for all $t>0$; the free boundary is understood as $\Gamma(t)=\partial\Omega(t)$, $\Omega(t)=\{x: u(t,x)>0\}$. The following theorems are the main results of this paper.

Theorem 1.1. *For any fixed $t>0$, $\tilde\Gamma(t):=\Gamma(t)\setminus\overline{\mathrm{co}}(\Omega_0)$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^n$, and $\tilde\Gamma:=\{(t,x): x\in\tilde\Gamma(t),\ t>0\}$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^{n+1}$. In particular, the free boundary is always $C^{2,\alpha}$ smooth if Ω_0 is convex.*

Here $\overline{\mathrm{co}}(\Omega_0)$ stands for the closed convex hull of Ω_0.

Theorem 1.2. *$\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0\subset\Omega(t)\subset\Omega(s)$ if $0<t<s$. Moreover, $\Omega_\infty:=\cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^n$, or it is a bounded set. Furthermore, when $\Omega_\infty=\mathbb{R}^n$, for all large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^n$, and there exists a continuous function $M(t)$ such that*

$$\Gamma(t)\subset\{x: M(t)-\frac{d_0}{2}\pi\le|x|\le M(t)\}; \tag{1.3}$$

and when Ω_∞ is bounded, $\lim_{t\to\infty}\|u(t,\cdot)\|_{L^\infty(\Omega(t))}=0$.

Here d_0 is the diameter of Ω_0.

Theorem 1.3. *If $g(u)=au-bu^2$ with a,b positive constants, then there exists $\mu^*\ge 0$ such that $\Omega_\infty=\mathbb{R}^n$ if $\mu>\mu^*$, and Ω_∞ is bounded if $\mu\in(0,\mu^*]$. Moreover, when $\Omega_\infty=\mathbb{R}^n$, the following holds:*

$$\lim_{t\to\infty}\frac{M(t)}{t}=k_0(\mu),\ \lim_{t\to\infty}\max_{|x|\le ct}\left|u(t,x)-\frac{a}{b}\right|=0\ \forall c\in(0,k_0(\mu)),$$

where $k_0(\mu)$ is a positive increasing function of μ satisfying $\lim_{\mu\to\infty}k_0(\mu)=2\sqrt{ad}$.

There exists $R^*>0$ such that $\mu^*>0$ if $\overline{\Omega}_0$ is contained in a ball with radius R^*, and $\mu^*=0$ if Ω_0 contains a ball of radius R^* (see Theorem 5.11). The asymptotic spreading speed $k_0(\mu)$ is determined by a class of traveling wave solutions, called semi-wave solutions in [7] and [3]; detailed analysis of the function $k_0(\mu)$ and the associated semi-wave solutions can be found in [3].

Problem (1.1) reduces to the classical one phase Stefan problem when $g(u)\equiv 0$, which describes the melting of ice in contact with water, with $u(x,t)$ representing the temperature of the water. In the setting of (1.1), the water region $\Omega(t)$ is surrounded by ice, and the free boundary $\Gamma(t)=\partial\Omega(t)$ represents the interphase between water and ice. A nonlinear Stefan problem of the form (1.1) may arise if water is replaced by a chemically reactive and heat diffusive liquid surrounded by ice, with $g(u)$ representing the reaction. As explained below, in this work, u may also be viewed as the population density of an invasive species.

In the classical Stefan problem, it is often assumed that the water region $\Omega(t)$ is bounded by two surfaces: a fixed surface Γ_0, where a Dirichlet boundary condition is prescribed ($u=\phi(t,x)$ for $x\in\Gamma_0$ and $t>0$), and a moving surface $\Gamma_1(t)$ representing the water ice interphase. But we will only consider the situation described by (1.1).

The classical one phase Stefan problem has been extensively investigated in the past 50 years (see, for example, [4, 11, 12, 13, 14, 16, 20] and the references therein). In contrast, the nonlinear Stefan problem is much less studied.

Problem (1.1) is also closely related to the following Cauchy problem:

$$\begin{cases} U_t - d\Delta U = g(U) & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ U(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $u_0(x)$ is given in (1.1) but extended to $\mathbb{R}^n$ with value 0 outside Ω_0. It was shown in [8] (Theorem 5.4) that if u_μ denotes the unique weak solution of (1.1), with $\Omega_\mu(t) = \{x : u_\mu(t,x) > 0\}$, then as $\mu \to \infty$, $\Omega_\mu(t) \to \mathbb{R}^n$ $(\forall t > 0)$ and

$$u_\mu \to U \text{ in } C^{(1+\theta)/2,1+\theta}_{loc}((0,\infty) \times \mathbb{R}^n)\ (\forall \theta \in (0,1)),$$

where U is the unique solution of (1.4).

The Cauchy problem (1.4) arises in a variety of applied problems and has been extensively studied. For example, in the classical work [1], for monostable, bistable or combustion type nonlinearities, it was shown that if $\liminf_{t\to\infty} U(t,x) > 0$, then there exists $c^* > 0$ such that, for any small $\epsilon > 0$,

$$\lim_{t\to\infty} \max_{|x|\geq(c^*+\epsilon)t} U(t,x) = 0$$

and

$$\lim_{t\to\infty} \min_{|x|\leq(c^*-\epsilon)t} U(t,x) > 0.$$

The number c^* is usually called the spreading speed, and is determined by certain traveling wave solutions associated to (1.4). In particular,

$$c^* = 2\sqrt{ad} = \lim_{\mu\to\infty} k_0(\mu)$$

if $g(u) = au - bu^2$.

Our work here was motivated by recent research on the following special case of (1.1),

$$\begin{cases} u_t - d\Delta u = au - bu^2 & \text{for } x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu|\nabla_x u|^2 & \text{for } x \in \Gamma(t), t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0. \end{cases} \tag{1.5}$$

Problem (1.5) was introduced in [9, 7, 8] to better understand the spreading of invasive species, where u represents the population density of the species, and the free boundary stands for the spreading front (see [3] for a deduction of the free boundary condition based on ecological assumptions).

In space dimension 1, and in several space dimensions with radial symmetry, it was proved in [9] and [7] that problem (1.5) exhibits a spreading-vanishing dichotomy: as $t \to \infty$, either $\Omega(t)$ expands to the entire $\mathbb{R}^n$ and u converges to the positive steady-state a/b (spreading), or $\Omega(t)$ stays bounded and $u \to 0$ (vanishing). In these cases the free boundary and the solution are smooth due to the special geometry used, which greatly simplifies the analysis. It is natural to ask whether the spreading-vanishing phenomenon is retained in a general geometric setting. A positive answer to this question would suggest that the spreading-vanishing dichotomy is a rather robust phenomenon.

A first step in this direction was made in [8], where the existence and uniqueness of a weak solution for (1.1) with a general Ω_0 was established by adapting ideas from [12]. As mentioned above, it was also shown in [8] that as $\mu \to \infty$, the weak solution of (1.1) converges to the solution of the corresponding Cauchy problem (1.4). Moreover, for the special problem (1.5), it was shown in [8] that under suitable conditions on the initial values, as $t \to \infty$, $\Omega(t)$ expands to the entire space $\mathbb{R}^n$ and u converges to the positive equilibrium solution a/b, and under a set of different conditions $\Omega(t)$ remains bounded and u converges to 0. However, these two sets of conditions are not complementing to each other, and whether there is a sharp spreading-vanishing dichotomy as in the special cases studied in [9] and [7], was unclear. The regularity of the free boundary and the solution were not considered in [8]. These issues are now addressed here. In particular, our Theorem 1.3 gives a complete answer to the question on the spreading-vanishing dichotomy.

The formulation of weak solutions in [8] alone appears insufficient for the purpose of proving the regularity of the free boundary. In section 2, we give a new approach to the existence problem for (1.1), by using ideas of [14], where the classical one phase Stefan problem was formulated as a parabolic variational inequality suggested in [11]. However, unlike in the classical case, due to the reaction term $g(u)$ in our problem, a nonlocal term appears in the new weak formulation of our problem, which causes great difficulties. For example, comparison type of arguments are not directly applicable anymore, and hence a uniqueness result as in [14] is difficult to obtain. We show that any weak solution here corresponds to a weak solution in the sense of [8]. Thus it must be unique due to the result in [8], and the two formulations of weak solutions are equivalent.

The regularity of the free boundary of the weak solution is invesstigated in sections 3 and 4, where both weak formulations of (1.1) are employed. In section 3, we use the weak formulation of section 2 to show that if the free boundary is Lipschitz, then the techniques for proving C^1 and higher regularity of the free boundary developed by Caffarelli [4] and Kinderlehrer-Nirenberg [16] can be adapted to treat the case here. A crucial fact is that the nonlocal term in the equation is smooth enough near a free boundary point (see Lemmas 3.7 and 3.14).

The Lipschitz regularity of the free boundary outside the closed convex hull of Ω_0 is proved in section 4 by employing the weak formulation in [8]. This formulation allows us to apply a monotonicity method along the lines of [20], where the classical one phase Stefan problem was treated. Similar to [20], by a reflection and comparison argument we prove the monotonicity of the solution in certain spatial directions. The Lipschitz regularity of the free boundary is a consequence of this monotonicity property of the solution. Combined with the regularity results established in section 3, this proves Theorem 1.1.

The reflection and comparison argument also shows that for any point on $\Gamma(t)\setminus\overline{\text{co}}(\Omega_0)$, the inward normal line to $\Gamma(t)$ at that point intersects $\overline{\text{co}}(\Omega_0)$. It was demonstrated in [20] that such a normal line property implies some strong geometric constraints on the free boundary of the classical one phase Stefan problem. In section 5, we make use of this property and some novel techniques to prove Theorem 1.2. We first show by this normal line property that (1.3) holds for $\Gamma(t)\setminus\overline{\text{co}}(\Omega_0)$ if we assume that the free boundary is unbounded as $t\to\infty$. To show that (1.3) holds for $\Gamma(t)$, we need to understand the large-time behavior of $\Gamma(t)\cap\overline{\text{co}}(\Omega_0)$, where the regularity of the free boundary is unclear for non-convex Ω_0, and singularity may occur. We show that if $\Gamma(t)$ becomes unbounded as $t\to\infty$, then $\Gamma(t)\cap\overline{\text{co}}(\Omega_0)$ must be empty after a finite time (see Theorem 5.4). This relies on a new device based on the Harnack inequality. To prove that $u\to 0$ as $t\to\infty$ when $\Gamma(t)$ stays bounded, a situation where the regularity of the free boundary is again unclear unless Ω_0 is convex, we rely on an energy inequality (see Lemma 5.6). Theorem 1.3 is largely a consequence of Theorem 1.2 and results of [7] and [8].

2. Weak solutions

For the study of regularity of the weak solution of (1.1), the definition in [8] seems difficult to use directly. In this section, we give a different yet equivalent definition of weak solutions to (1.1), and then obtain some basic properties of the weak solutions. From now on in this paper, we will actually treat the following more general problem

$$\begin{cases} u_t - d\Delta u = g(x,u) & \text{for } x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu|\nabla_x u|^2 & \text{for } x \in \Gamma(t), t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases} \tag{2.1}$$

where g satisfies the following conditions:

$$\begin{cases} g \text{ is continuous for } (x,u) \in \mathbb{R}^n \times [0,+\infty), \\ g(x,0) \equiv 0 \text{ and } g(x,u) \text{ is locally Lipschitz in } u \text{ uniformly for } x \in \mathbb{R}^n, \\ \text{there exists } C > 0 \text{ such that } g(x,u) \le Cu \text{ for all } u \ge 0 \text{ and } x \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

Our assumptions on Ω_0 and u_0 are the same as in (1.1).

Following [14], for an arbitrarily given $\varepsilon > 0$, take a smooth function β_ε defined on $\mathbb{R}$, such that

$$\begin{cases} \beta_\varepsilon(t) = 0 \text{ for } t > \varepsilon, \\ \beta_\varepsilon(0) = -1, \\ \beta_\varepsilon^{'} > 0 \text{ and } \beta_\varepsilon^{''} \le 0 \text{ for } t < \varepsilon. \end{cases} \tag{2.3}$$

For given $T > 0$, take $R > 0$ large enough (in particular, $\Omega_0 \subset B_R(0)$). Define

$$f(x) = \begin{cases} u_0(x), & x \in \Omega_0, \\ -d/\mu, & x \in B_R(0) \setminus \Omega_0. \end{cases} \tag{2.4}$$

We denote by $f_\varepsilon(x)$ a family of functions smooth in $B_R(0)$, uniformly bounded, and decreasing to $f(x)$ as ε decreases to 0.

Now consider the following parabolic equation with a memory term

$$\begin{cases} (\partial_t - d\Delta)u_\varepsilon = g(x,u_\varepsilon) - d\mu^{-1}\beta_\varepsilon'\Big(\displaystyle\int_0^t u_\varepsilon(\tau,x)d\tau\Big)u_\varepsilon & \text{in } (0,T) \times B_R(0), \\ u_\varepsilon = 0 & \text{on } (0,T) \times \partial B_R(0), \\ u_\varepsilon = f_\varepsilon + d\mu^{-1} & \text{on } \{0\} \times B_R(0). \end{cases} \tag{2.5}$$

The existence and uniqueness of a global solution in $C^{1+\frac{\alpha}{2},2+\alpha}((0,T] \times \overline{B_R(0)})$ to (2.5) can be proved as usual; see for example [15].

Define $w_\varepsilon(t,x) = \int_0^t u_\varepsilon(\tau,x)d\tau$. Then noticing that

$$\int_0^t \partial_t u_\varepsilon(\tau,x)d\tau = u_\varepsilon(t,x) - u_\varepsilon(0,x) = \partial_t w_\varepsilon(t,x) - f_\varepsilon(x) - d\mu^{-1},$$

and

$$\int_0^t \beta_\varepsilon'(w_\varepsilon)u_\varepsilon d\tau = \int_0^{w_\varepsilon(t,x)} \beta_\varepsilon'(w)dw = \beta_\varepsilon(w_\varepsilon(t,x)) + 1,$$

we obtain, by integrating (2.5) over $(0,t)$, that

$$\begin{cases} (\partial_t - d\Delta)w_\varepsilon + \dfrac{d}{\mu}\beta_\varepsilon(w_\varepsilon) = \displaystyle\int_0^t g(x,\partial_t w_\varepsilon)d\tau + f_\varepsilon & \text{in } (0,T) \times B_R(0), \\ w_\varepsilon = 0 & \text{on } \partial_p((0,T) \times B_R(0)). \end{cases} \tag{2.6}$$

Here ∂_p denotes the parabolic boundary.

Proposition 2.1. *There exists $K(T) > 0$ such that $0 \leq \partial_t w_\varepsilon \leq K(T)$ in $(0, T) \times B_R(0)$.*

Proof. By (2.2) and the boundedness of u_ε, $\exists C_\varepsilon > 0$ such that

$$-C_\varepsilon u_\varepsilon \leq g(x, u_\varepsilon) \leq C u_\varepsilon.$$

In view of $\beta'_\varepsilon \geq 0$, we obtain from (2.5) that

$$-(C_\varepsilon + d\mu^{-1}\beta'_\varepsilon(w_\varepsilon))u_\varepsilon \leq (\partial_t - d\Delta)u_\varepsilon \leq C u_\varepsilon \text{ in } (0, T) \times B_R(0).$$

Thus we can apply the maximum principle to (2.5) to conclude that

$$0 \leq u_\varepsilon \leq K \text{ in } (0, T) \times B_R(0)$$

for some constant K depending on T but independent of ε (for all small $\varepsilon > 0$). □

A direct consequence is

Corollary 2.2. *There exists $K_1(T) > 0$ such that $0 \leq w_\varepsilon \leq K_1(T)$ in $(0, T) \times B_R(0)$.*

Denote

$$H(u)(t, x) := \int_0^t g(x, u(\tau, x))d\tau.$$

Because $\partial_t w_\varepsilon \geq 0$ and $|g(x, u)| \leq C_T\, u$ for $u \in [0, K(T)]$, a simple calculation shows that there exists another constant $C = C(T) > 0$ such that

$$|H(\partial_t w_\varepsilon)| \leq C w_\varepsilon. \tag{2.7}$$

In view of (2.7) and $-1 \leq \beta_\varepsilon(w_\varepsilon) \leq 0$, we may use (2.6) and Corollary 2.2 to obtain

$$|(\partial_t - d\Delta)w_\varepsilon| \leq K_2(T).$$

Then $\forall p > 1$, by the L^p estimate for parabolic equations, w_ε is uniformly bounded in $W_p^{1,2}((0, T) \times B_R(0))$. Thus we can find a subsequence of ε, say $\varepsilon_j \to 0$, such that w_{ε_j} converges to w weakly in $W_p^{1,2}((0, T) \times B_R(0))$, $\forall p > 1$. By the Sobolev embedding theorem, w_{ε_j} converges to w in $H_{1+\gamma}([0, T] \times \overline{B_R(0)})$, $\forall \gamma \in (0, 1)$. Here and in the rest of this paper, we use the notation

$$H_{k+\gamma}(\overline{\Omega}) = C^{\frac{k+\gamma}{2}, k+\gamma}(\overline{\Omega}) \quad \text{for } k = 0, 1, 2,\ \gamma \in (0, 1) \text{ and } \Omega \subset \mathbb{R}^{n+1}.$$

We will eventually show that $w = \lim_{\varepsilon \to 0} w_\varepsilon$ and it is uniquely determined, but for the time being, w just stands for the limit of w_ε along the sequence ε_j. By Proposition 2.1,

$$0 \leq w_t \leq K(T) \text{ in } (0, T) \times B_R(0). \tag{2.8}$$

Moreover w is continuous in $[0, T] \times \overline{B_R(0)}$, and is zero on the parabolic boundary of this set, so $\{w > 0\} := \{(t, x) \in (0, T) \times B_R(0) : w(t, x) > 0\}$ is an open set in $\mathbb{R}^{n+1}$. We denote

$$\widetilde{\Omega}(t) := \{w(t, \cdot) > 0\},$$

which is an open set in $\mathbb{R}^n$. From (2.8) we obtain

Proposition 2.3. *$\widetilde{\Omega}(t)$ is expanding as t increases, that is, for $0 < t_1 < t_2$, we have $\widetilde{\Omega}(t_1) \subset \widetilde{\Omega}(t_2)$.*

We also have

Proposition 2.4. *$\widetilde{\Omega}(t) \supset \Omega_0$ for $t > 0$.*

Proof. By (2.6), (2.7) and the definitions of f_ε and f, in $(0,T)\times\Omega_0$,

$$\partial_t w_\varepsilon - d\Delta w_\varepsilon \geq -Cw_\varepsilon + u_0.$$

Thus $w_\varepsilon \geq \underline{w}$, where $\underline{w}$ is the solution to the initial boundary value problem

$$\begin{cases} \underline{w}_t - d\Delta\underline{w} = -C\underline{w} + u_0 \text{ in } (0,+\infty)\times\Omega_0, \\ \underline{w} = 0 \text{ on } \partial_p((0,+\infty)\times\Omega_0). \end{cases}$$

Because $u_0 > 0$, we have $\underline{w} > 0$ in $(0,+\infty)\times\Omega_0$. By the comparison principle we have $w_\varepsilon \geq \underline{w}$ in $(0,T)\times\Omega_0$. It follows that $w \geq \underline{w} > 0$ in $(0,T)\times\Omega_0$. Hence $\widetilde{\Omega}(t) \supset \Omega_0$ for $t > 0$. □

In fact, by the interior ball condition on $\partial\Omega_0$, we have $\widetilde{\Omega}(t) \supset \overline{\Omega}_0$ for $t > 0$, which can be easily proved after the equivalence of the weak solution here and that in [8] is established; see Proposition 2.10.

In the following, we denote

$$u := w_t \text{ and } \{u > 0\} := \{(t,x) \in (0,T)\times B_R(0) : u(t,x) > 0\}.$$

Proposition 2.5. $\{u > 0\} = \{w > 0\}$, *and* $u \in H_{1+\gamma}(\{u > 0\})$ *for all* $\gamma \in (0,1)$.

Proof. Assume $(t_0,x_0) \in \{w > 0\}$ and so $2\delta := w(t_0,x_0) > 0$; then in some neighborhood $V = (t_0-\sigma, t_0+\sigma)\times B_\sigma(x_0)$ of (t_0,x_0) we have $w \geq \delta$, where the small positive constant σ depends only on δ due to $w \in H_{1+\gamma}((0,T]\times\overline{B_R(0)})$. By the uniform convergence of w_{ε_j}, for ε_j small,

$$w_{\varepsilon_j} \geq \frac{\delta}{2} \text{ in } V.$$

By the definition of β_ε, for all large j,

$$\beta_{\varepsilon_j}(w_{\varepsilon_j}) \equiv 0 \text{ in } V.$$

Thus in V, for all large j, w_{ε_j} satisfies the equation

$$(\partial_t - d\Delta)w_{\varepsilon_j} = \int_0^t g(x, \partial_t w_{\varepsilon_j}(\tau,x))d\tau + f_{\varepsilon_j}, \tag{2.9}$$

and $\partial_t w_{\varepsilon_j}$ satisfies

$$(\partial_t - d\Delta)\partial_t w_{\varepsilon_j} = g(x, \partial_t w_{\varepsilon_j}). \tag{2.10}$$

By the uniform bound of $\partial_t w_{\varepsilon_j}$, applying standard parabolic regularity theory, we can get a uniform bound for $\partial_t w_{\varepsilon_j}$ in $W_p^{1,2}(K)$ $(\forall p > 1)$ for any compact subset K of V. Because $\partial_t w_{\varepsilon_j}$ converges to $\partial_t w$ weakly in $L^2((0,T)\times B_R(0))$, we must have $\partial_t w_{\varepsilon_j}$ converges to $\partial_t w$ in $H_{1+\gamma,loc}(V)$ $(\forall\gamma \in (0,1))$. In particular, $u = w_t$ satisfies

$$(\partial_t - d\Delta)u = g(x,u) \text{ in } V.$$

Standard interior regularity shows that $u \in H_{1+\gamma,loc}(V)$ for any $\gamma \in (0,1)$. By Proposition 2.1, $u \geq 0$. Since $g(x,0) = 0$ and g is locally Lipschitz continuous in u, by the strong maximum principle, either $u(t_0,x_0) > 0$ or $u \equiv 0$ in $[t_0-\sigma, t_0]\times B_\sigma(x_0)$. If the latter happens, then $\forall t \in [t_0-\sigma, t_0]$, $w(t,x_0) \equiv w(t_0,x_0) = 2\delta$ (by integration in t). In particular $w(t_0-\sigma, x_0) = 2\delta > 0$. We may now repeat the above argument with (t_0,x_0) replaced by $(t_0-\sigma, x_0)$ to deduce that $w(t,x_0) \equiv 2\delta$ for $t \in [t_0-2\sigma, t_0]$. After finitely many steps we deduce $w(t,x_0) \equiv 2\delta$ in $(0,t_0]$. This contradicts the assumption that $w(0,x_0) = 0$. Therefore we must have $u(t_0,x_0) > 0$ and this proves $\{u > 0\} \supset \{w > 0\}$. Note also that the above argument implies that $u \in H_{1+\gamma}$ in $\{w > 0\}$. Since $w_t \geq 0$

and $w \geq 0$, we find that if $w(t_0, x_0) = 0$ with $t_0 > 0$, then $w(t, x_0) \equiv 0$ for $t \in [0, t_0]$ and therefore $u(t_0, x_0) = w_t(t_0, x_0) = 0$ whenever $w_t(t_0, x_0)$ exists. Thus we must have $\{u > 0\} = \{w > 0\}$ a.e., and $u \in H_{1+\gamma}(\{u > 0\})$. □

The following result implies that (2.7) holds for w, too.

Proposition 2.6. *$H(\partial_t w_{\varepsilon_j})$ converges to $H(w_t)$ uniformly in $(0, T) \times B_R(0)$.*

Proof. Assume the contrary; then by passing to a subsequence, we may assume that there exist $X_{\varepsilon_j} \in (0, T) \times B_R(0)$ and $\delta > 0$, such that

$$|H(\partial_t w_{\varepsilon_j})(X_{\varepsilon_j}) - H(w_t)(X_{\varepsilon_j})| \geq \delta, \ \forall j \geq 1.$$

Without loss of generality, we can assume X_{ε_j} converges to $X_0 \in [0, T] \times \overline{B_R(0)}$. We divide the problem into two cases.

Case 1. $w(X_0) \leq \frac{\delta}{6C}$, with C given in (2.7).

Because w_{ε_j} converges to w uniformly, for ε_j small enough, $w_{\varepsilon_j}(X_0) \leq \frac{\delta}{5C}$. Then by the uniform continuity of w and w_{ε_j}, for ε_j sufficiently small, $w_{\varepsilon_j}(X_{\varepsilon_j}) \leq \frac{\delta}{4C}$ and $w(X_{\varepsilon_j}) \leq \frac{\delta}{4C}$. By (2.7),

$$|H(\partial_t w_{\varepsilon_j})(X_{\varepsilon_j}) - H(w_t)(X_{\varepsilon_j})| \leq C[w_\varepsilon(X_\varepsilon) + w(X_\varepsilon] < \delta.$$

This is a contradiction.

Case 2. $w(X_0) > \frac{\delta}{6C}$.

Write $X_0 = (t_0, x_0)$ and $X_{\varepsilon_j} = (t_{\varepsilon_j}, x_{\varepsilon_j})$. Because w is nondecreasing in t and $w(0, x_0) = 0$, we can take a $t_1 \in (0, t_0)$ such that

$$\frac{\delta}{6C} < w(t_1, x_0) < \frac{\delta}{3C}.$$

By the uniform convergence of w_{ε_j}, for ε_j small we also have

$$\frac{\delta}{6C} < w_{\varepsilon_j}(t_1, x_{\varepsilon_j}) < \frac{\delta}{3C}.$$

In view of $\partial_t w \geq 0$, we have

$$w(t, x_0) > \frac{\delta}{6C} \text{ for } t \in [t_1, T].$$

Much as in the proof of Proposition 2.5, we can find a small neighborhood V of $[t_1, T] \times \{x_0\}$ in $(0, T] \times B_R(0)$ such that $\partial_t w_{\varepsilon_j} \to \partial_t w$ in $H_{1+\gamma}(\overline{V})$. It follows that, as $j \to \infty$,

$$\int_{t_1}^{t_{\varepsilon_j}} |g(x_{\varepsilon_j}, \partial_t w_{\varepsilon_j}(\tau, x_{\varepsilon_j})) - g(x_{\varepsilon_j}, \partial_t w(\tau, x_{\varepsilon_j}))| d\tau \to 0.$$

Hence

$$\begin{aligned} & |H(\partial_t w_{\varepsilon_j})(X_{\varepsilon_j}) - H(w_t)(X_{\varepsilon_j})| \\ \leq & \int_{t_1}^{t_{\varepsilon_j}} |g(x_{\varepsilon_j}, \partial_t w_{\varepsilon_j}(\tau, x_{\varepsilon_j})) - g(x_{\varepsilon_j}, \partial_t w(\tau, x_{\varepsilon_j}))| d\tau \\ & + \left|H(\partial_t w_{\varepsilon_j})(t_1, x_{\varepsilon_j})\right| + \left|H(w_t)(t_1, x_{\varepsilon_j})\right| \\ < & \ \delta, \end{aligned}$$

for ε_j small and we get a contradiction again. □

Finally we give the equation satisfied by w. For convenience of notation, we write $\Omega_{T,R} = (0, T) \times B_R(0)$.

Proposition 2.7. *The function w is a $W_p^{1,2}(\Omega_{T,R})$-solution of*

$$\begin{cases} w_t - d\Delta w = \int_0^t g(x, w_t(\tau,x))d\tau + d\mu^{-1}\chi_{\{w=0\}} + f & \text{in } \Omega_{T,R}, \\ w = 0 & \text{on } \partial_p(\Omega_{T,R}). \end{cases} \tag{2.11}$$

Proof. Take $\varphi \in C^\infty(\overline{\Omega}_{T,R})$ which vanishes near $[(0,T]\times\partial B_R(0)]\cup[\{T\}\times B_R(0)]$, multiply the equation of w_{ε_j} by φ and integrate by parts; it results

$$\iint_{\Omega_{T,R}} \big[w_{\varepsilon_j}(-\varphi_t - d\Delta\varphi) + d\mu^{-1}\beta_{\varepsilon_j}(w_{\varepsilon_j})\varphi - H(\partial_t w_{\varepsilon_j})\varphi - f_{\varepsilon_j}\varphi\big]dtdx = 0.$$

Without loss of generality, we may assume that as $\varepsilon_j \to 0$, $\beta_{\varepsilon_j}(w_{\varepsilon_j})$ converges to some β_∞ weakly in $L^2(\Omega_{T,R})$. Then in view of the definition of f_ε and the previous propositions, we obtain by letting $j \to \infty$,

$$\iint_{\Omega_{T,R}} \big[w(-\varphi_t - d\Delta\varphi) + d\mu^{-1}\beta_\infty\varphi - H(w_t)\varphi - f\varphi\big]dtdx = 0. \tag{2.12}$$

Since $w \in W_p^{1,2}(\Omega_{T,R})$, by standard parabolic regularity theory, (2.12) implies that w solves, in the $W_p^{1,2}$ sense,

$$\begin{cases} w_t - d\Delta w = H(w_t) - d\mu^{-1}\beta_\infty + f & \text{in } \Omega_{T,R}, \\ w = 0 & \text{on } \partial_p(\Omega_{T,R}). \end{cases} \tag{2.13}$$

To complete the proof, it remains to show $\beta_\infty = -\chi_{\{w=0\}}$ a.e. in $\Omega_{T,R}$. In fact, for any $(t_0, x_0) \in \{w > 0\}$, we have $\beta_{\varepsilon_j}(w_{\varepsilon_j}) = 0$ for all large j in a small neighborhood of (t_0, x_0), and so $\beta_\infty \equiv 0$ in this small neighborhood. It follows that $\beta_\infty \equiv 0$ in $\{w > 0\}$.

By (2.7), we have $H(w_t) = 0$ in $\{w = 0\}$. Moreover, $(\partial_t - d\Delta)w = 0$ a.e. in $\{w = 0\}$. So we obtain from (2.13) that $d\mu^{-1}\beta_\infty = f$ a.e in $\{w = 0\}$. By Proposition 2.4 we have $\Omega_0 \subset \widetilde{\Omega}(t)$ for $t > 0$. It follows that $\{w = 0\} \subset [0,T] \times (B_R(0) \setminus \Omega_0)$, and thus, by definition, $f = -d\mu^{-1}$ on $\{w = 0\}$. Therefore $\beta_\infty = -1$ a.e. in $\{w = 0\}$. It follows that $\beta_\infty = -\chi_{\{w=0\}}$. □

Proposition 2.8. *$w_t \in L^2(\Omega_{T,R}) \cap L^\infty(\Omega_{T,R})$ satisfies*

$$\iint_{\Omega_{T,R}} \big[w_t(-d\Delta\phi) - \alpha(w_t)\phi_t\big]dtdx - \int_{B_R(0)} \alpha(\tilde{u}_0)\phi(0,x)dx = \iint_{\Omega_{T,R}} g(x, w_t)\phi dtdx \tag{2.14}$$

for every function $\phi \in C^\infty(\overline{\Omega}_{T,R})$ that vanishes near $[(0,T] \times \partial B_R(0)] \cup [\{T\} \times B_R(0)]$, where

$$\alpha(\xi) = \xi - d\mu^{-1}\chi_{\{\xi\le 0\}},\ \tilde{u}_0 = u_0 \text{ in } \Omega_0,\ \tilde{u}_0 = 0 \text{ outside } \Omega_0. \tag{2.15}$$

Proof. From (2.11) we obtain

$$\iint_{\Omega_{T,R}} \big[w(-\varphi_t - d\Delta\varphi) - d\mu^{-1}\chi_{\{w=0\}}\varphi - H(w_t)\varphi - f\varphi\big]dtdx = 0$$

for every $\varphi \in C^\infty(\overline{\Omega}_{T,R})$ which vanishes near $[(0,T] \times \partial B_R(0)] \cup [\{T\} \times B_R(0)]$. Taking $\varphi = -\phi_t$, and using integration by parts in t, we deduce

$$\iint_{\Omega_{T,R}} \big[w_t(-\phi_t - d\Delta\phi) + d\mu^{-1}\chi_{\{w=0\}}\phi_t - g(x, w_t)\phi\big]dtdx = \int_{B_R(0)} f(x)\phi(0,x)dx, \tag{2.16}$$

where we have used $w = 0$ and $|H(w_t)| \le Cw = 0$ on $\{0\} \times B_R(0)$.

Clearly $f(x) = \alpha(\tilde{u}_0(x))$. Moreover, by Proposition 2.5, we have $\chi_{\{w=0\}} = \chi_{\{w_t=0\}}$ a.e. in $\Omega_{T,R}$. Therefore, due to $w_t \geq 0$, we have

$$w_t - d\mu^{-1}\chi_{\{w=0\}} = w_t - d\mu^{-1}\chi_{\{w_t=0\}} = \alpha(w_t).$$

Substituting these into (2.16) we obtain (2.14). □

If $w_t \in H^1(\Omega_{T,R}) \cap L^\infty(\Omega_{T,R})$, and if R is large enough so that $G = B_R(0)$ meets the requirement in [8], then the above proposition implies that w_t is a weak solution to (2.1) in the sense of [8] (see Definition 2.1 there). Therefore, under these assumptions, by the existence and uniqueness results in [8], w_t must coincide with the unique weak solution determined by Theorems 3.1 and 3.2 there. Since we have only proved $w_t \in L^2(\Omega_{T,R}) \cap L^\infty(\Omega_{T,R})$ here, we could not apply these results of [8] directly. However, the uniqueness proof of Theorem 3.2 in [8] does not use the fact that the weak solution there is in H^1. Checking this proof one finds that uniqueness also holds for solutions satisfying (2.14). Since the weak solution obtained in [8] also satisfies (2.14), we thus conclude that w_t coincides with the unique weak solution of [8][1]. This implies that w is the unique solution of (2.11), and $w_\varepsilon \to w$ weakly in $W^{1,2}_p(\Omega_{T,R})$ $(\forall p > 1)$ as $\varepsilon \to 0$.

Summarizing the above discussions, we have the following result.

Theorem 2.9. *For any given $T > 0$, suppose that $R > 0$ is chosen so large that $G = B_R(0)$ satisfies the requirements in Definition 2.1 of* [8]*, then w_ε obtained from* (2.6) *satisfies* $\lim_{\varepsilon\to 0} w_\varepsilon = w$ *weakly in* $W^{1,2}_p(\Omega_{T,R})$ $(\forall p > 1)$*, where w is the unique solution of* (2.11)*, and w_t is the unique weak solution of* (2.1) *as determined in* [8].

We are now in a position to improve the conclusion in Proposition 2.4.

Proposition 2.10. $\widetilde{\Omega}(t) \supset \overline{\Omega}_0$ *for* $t > 0$.

Proof. We have proved $\widetilde{\Omega}(t) \supset \Omega_0$ for $t > 0$. It remains to show that $w(t, x) > 0$ if $t > 0$ and $x \in \partial\Omega_0$. Otherwise, we can find $t_0 > 0$ and $x_0 \in \partial\Omega_0$ such that $w(t_0, x_0) = 0$. By the interior ball condition of $\partial\Omega_0$, we can find a ball $B = B_{R_0}(y_0) \subset \Omega_0$ that touches $\partial\Omega_0$ at x_0. Let v_0 be a C^2 radially symmetric function in B such that $0 < v_0 \lneqq u_0$ in B and $v_0 = 0$ on ∂B. Choose $C_0 > 0$ such that $g(w(t, x)) \geq -C_0 w(t, x)$ for $x \in \widetilde{\Omega}(t)$, $t \in [0, T]$ with $T > t_0$. We now consider the auxiliary radially symmetric problem

$$\begin{cases} v_t - d\Delta v = -C_0 v, & t > 0, \ \ 0 < r < h(t), \\ v_r(t, 0) = 0, \ \ v(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu v_r(t, h(t)), & t > 0, \\ h(0) = R_0, \ \ v(0, r) = v_0(r), & 0 \leq r \leq R_0. \end{cases} \tag{2.17}$$

By Proposition 4.3 of [8], we know that (2.17) has a unique solution v defined for all $t > 0$, and the Hopf boundary lemma guarantees that $h'(t) > 0$ for all $t > 0$. The extended v (by 0) is a weak solution of (2.1) with $g(x, u)$ replaced by $-C_0 u$. Hence we can apply Theorem 4.2 of [8] to obtain $0 < v(t, |x - y_0|) \leq w_t(t, x)$ in $\{(t, x) : |x - y_0| < h(t), 0 \leq t \leq T\}$. Since $x_0 \in \partial B_{R_0}(y_0)$ and $h'(t) > 0$, we find that $|x_0 - y_0| = R_0 < h(t_0)$ and hence $w_t(t, x_0) \geq v(t, R_0) > 0$ for all t close to t_0. This implies that $w(t_0, x_0) > 0$, a contradiction. □

[1]Alternatively, as in section 3 of [14], it is possible to prove separately that $w_t \in H^1(\Omega_{T,R})$ by estimating the H^1 norm of w_ε. Moreover, we can get an energy inequality similar to Lemma 3.4 therein for $\partial_t w_\varepsilon$, which shows that for any t, $w_t(t, \cdot) \in H^1(B_R(0))$.

3. Regularity outside the convex hull of Ω_0

In this section we discuss the regularity of the free boundary $\partial\{w > 0\}$. We will show that it is smooth outside $\overline{\text{co}}(\Omega_0)$, the closed convex hull of Ω_0.

By Proposition 2.5, $\partial\{w > 0\} = \partial\{u > 0\}$. So the study of the regularity of the free boundary for (2.1) is equivalent to the study of that of (2.11). We will take advantage of the fact that the latter can be viewed as a perturbation of the one phase Stefan problem for which powerful techniques have already been developed. Let us recall that $\widetilde{\Omega}(t) = \Omega(t) = \{x \in B_R : u(t,x) > 0\}$ is an open set for each $t \in (0,T)$. Also, from

$$|H(w_t)(t,x)| \le Cw(t,x),$$

we easily see that

$$h(t,x) := H(w_t)(t,x) = \int_0^t g(x, w_t(\tau,x))d\tau \tag{3.1}$$

is continuous, and vanishes on $\{w = 0\}$.

Proposition 3.1. *Let $(t_0,x_0) \in \partial\{w > 0\}$ with $t_0 > 0$ and $h(t,x)$ be defined as above. Then there exists $r_0 > 0$ such that*

$$d\Delta w - w_t = (d\mu^{-1} - h)\chi_{\{w>0\}} \text{ in } P_{r_0}(t_0,x_0), \tag{3.2}$$

where

$$P_r(t,x) := (t - r^2, t + r^2) \times B_r(x).$$

Proof. By Proposition 2.10, we find that $x_0 \notin \overline{\Omega}_0$ and there exists $r_0 > 0$ small such that $B_{r_0}(x_0) \notin \Omega_0$. Thus for $x \in B_{r_0}(x_0)$ we have $f(x) = -d\mu^{-1}$ and

$$d\mu^{-1}\chi_{\{w=0\}} + f = d\mu^{-1}(\chi_{\{w=0\}} - 1) = -d\mu^{-1}\chi_{\{w>0\}} \text{ in } P_{r_0}(t_0,x_0).$$

Substituting this into (2.11) and recalling $h \equiv 0$ on $\{w = 0\}$, we immediately obtain (3.2). □

Using (3.2), as in [2] we may follow the arguments of [6], [5] or [4] to obtain the following results, where C denotes various constants which depend only on the space dimension n, the solution w and the nonlinear function g, but are independent of (t,x) in the given range.

Lemma 3.2. (Growth bound) $\exists C > 0$, *such that for any $(t_0,x_0) \in \{w = 0\}$ with $t_0 > 0$,*

$$\sup_{P_r(t_0,x_0)} w \le Cr^2 \text{ for all small } r > 0.$$

Proof. We may follow the first part of the proof of Lemma 4.2 in [6] and then argue as in the proof of Lemma 4.3 there. □

Lemma 3.3. (Nondegeneracy) $\exists C > 0$, *such that for any $(t_0,x_0) \in \partial\{w > 0\}$ with $t_0 > 0$,*

$$\sup_{x\in B_r(x_0)} w(t_0,x) \ge Cr^2 \text{ for all small } r > 0.$$

Proof. Since $w_t \ge 0$, $w(t,\cdot)$ satisfies

$$d\Delta w \ge (d\mu^{-1} - h)\chi_{\{w>0\}}.$$

In view of $h(t_0,x_0) = 0$ we find that $(d\mu^{-1} - h)\chi_{\{w>0\}} > (1/2)d\mu^{-1}$ in $P_r(t_0,x_0)$ for all small $r > 0$, say $r \in (0,r_0]$.

For $r \in (0, r_0/2]$, choose a sequence $(t_j, x_j) \in P_r(t_0, x_0) \cap \{w > 0\}$ such that $(t_j, x_j) \to (t_0, x_0)$ as $j \to \infty$. Then define

$$v_j(x) = w(t_j, x) - \frac{1}{4n\mu}|x - x_j|^2.$$

Clearly

$$\Delta v_j \geq 0 \text{ in } B_r(x_j) \cap \Omega(t_j),\ v_j(x_j) = w(t_j, x_j) > 0.$$

Therefore $\sup_{B_r(x_j)\cap\Omega(t_j)} v_j$ is positive and is achieved on the boundary of $B_r(x_j) \cap \Omega(t_j)$. On $B_r(x_j) \cap \partial\Omega(t_j)$, $w(t_j, x) = 0$ and so $v_j \leq 0$. Hence the positive supremum is achieved at some $y_j \in \partial B_r(x_j) \cap \Omega(t_j)$:

$$0 < \sup_{B_r(x_j)\cap\Omega(t_j)} v_j = v_j(y_j) = w(t_j, y_j) - \frac{1}{4n\mu}r^2.$$

It follows that

$$\sup_{B_r(x_j)} w(t_j, x) \geq \sup_{B_r(x_j)\cap\Omega(t_j)} w(t_j, x) \geq w(t_j, y_j) \geq \frac{1}{4n\mu}r^2.$$

Since w is continuous, letting $j \to \infty$ we obtain $\sup\limits_{B_r(x_0)} w(t_0, x) \geq \frac{1}{4n\mu}r^2$. □

A simple consequence of Lemma 3.3 is the following result, which indicates that $\Omega(t)$ expands continuously as t increases.

Proposition 3.4. *Let $t_0 \in [0, T)$. For $\epsilon > 0$ small, $\Omega(t_0 + \epsilon)$ is contained in a small neighborhood of $\Omega(t_0)$.*

Proof. Otherwise $\exists x_i \in \partial\Omega(t_0+\epsilon_i)$ with $\epsilon_i > 0$ and $\epsilon_i \to 0$ such that $\mathrm{dist}(x_i, \Omega(t_0)) \geq \delta > 0$. Since x_i is a bounded sequence, by passing to a subsequence we may assume that $x_i \to x_0$ as $i \to \infty$. Thus $\mathrm{dist}(x_0, \Omega(t_0)) \geq \delta$ and $w(t_0, x) \equiv 0$ in $B_{\delta/2}(x_0)$.

On the other hand, by Lemma 3.3, there is a constant $C > 0$, such that

$$\sup_{B_{\delta/4}(x_i)} w(t_0 + \epsilon_i, x) \geq C\delta^2 > 0 \text{ for all } i \geq 1.$$

Letting $i \to \infty$ and using the continuity of w we obtain

$$\sup_{B_{\delta/4}(x_0)} w(t_0, x) \geq C\delta^2 > 0.$$

This contradiction completes the proof. □

A direct corollary is

Corollary 3.5.

$$\partial\Omega(t) = \Big\{x : (t, x) \in \partial\{w > 0\}\Big\},\ \forall t > 0.$$

3.1. **Lipschitz-Hölder regularity.** From now on, we assume that

$$g(x, u) \equiv g(u) \text{ is independent of } x. \tag{3.3}$$

We have the following result.

Theorem 3.6. *Let* (3.3) *hold, and $t_0 > 0$, $x_0 \in \Gamma(t_0) \setminus \overline{\mathrm{co}}(\Omega_0)$. Then there exists a fixed open cone $K_0 \subset \mathbb{R}^n$ (depending on x_0) with vertex at the origin, and a small $r_0 > 0$, such that the following three conclusions hold:*

(i) Monotonicity:
For any $\tilde{x}$ *and* $x \in B_{r_0}(x_0)$, $\tilde{x} - x \in K_0$ *implies* $w_t(t, \tilde{x}) \leq w_t(t, x)$ $(\forall t > 0)$ *and hence* $w(t, \tilde{x}) \leq w(s, x)$ $(\forall t > 0)$.

(ii) Cone property:
For any $(t, x) \in \partial\{w > 0\} \cap P_{r_0}(t_0, x_0)$,

$$\begin{cases} (x + K_0) \cap B_{r_0}(x_0) \subset \{z : w(t, z) = 0\}, \\ (x - K_0) \cap B_{r_0}(x_0) \subset \{z : w(t, z) > 0\}. \end{cases}$$

(iii) Lipschitz-Hölder representation of the free boundary:
There exists a coordinate system $(s, y) \in \mathbb{R} \times \mathbb{R}^n$, *with* (t_0, x_0) *as its origin,* $s = t - t_0$, *and the* y_1 *direction parallel to the axis of* K_0, *such that* $\partial\{w > 0\} \cap P_{r_0}(t_0, x_0)$ *can be expressed as*

$$y_1 < f(s, y'),\ (s, y) \in N_0,$$

with f *Lipschitz continuous in* y', $\frac{1}{2}$*-Hölder continuous in* s, *and* $f(0, 0) = 0$, *where* $y' = (y_2, ..., y_n)$, *and* N_0 *is a small neighborhood of* $(0, 0) \in \mathbb{R}^1 \times \mathbb{R}^{n-1}$.

Therefore we may write

$$K_0 = \{y : y_1 > \delta_0 |y'|\},\ \delta_0 > 0. \tag{3.4}$$

The proof of Theorem 3.6 uses the monotonicity method and is given in section 4 below.

3.2. C^1 **regularity in space variables.** In this subsection, we assume that (3.3) holds and make use of Theorem 3.6 to show that the free boundary is C^1 in space (for fixed time t) and the solution w is C^2 in the space variables in $\{w > 0\}$ up to the boundary near a free boundary point (t_0, x_0). This is achieved by showing that Caffarelli's result in [4] can be applied to the setting here.

We now fix such a point (t_0, x_0), and consider the free boundary in $P_{r_0}(t_0, x_0)$. It is convenient to use the new coordinate system (s, y) given in conclusion (iii) of Theorem 3.6, and so (t_0, x_0) is replaced by (0,0), and the conclusions (i) and (ii) in Theorem 3.6 become

$$w_s(s, \tilde{y}) \leq w_s(s, y), w(s, \tilde{y}) \leq w(s, y) \text{ if } s > 0, \tilde{y} - y \in K_0 \text{ and } y, \tilde{y} \in B_{r_0}(0), \tag{3.5}$$

and

$$\begin{cases} (y + K_0) \cap B_{r_0}(0) \subset \{z : w(s, z) = 0\}, \\ (y - K_0) \cap B_{r_0}(0) \subset \{z : w(s, z) > 0\}, \end{cases} \forall (s, y) \in \partial\{w > 0\} \cap P_{r_0}(0, 0). \tag{3.6}$$

To further simplify the notations, we normalize the parameters in (3.2). Through a simple scaling change of w, t and h ($t \to dt$, $w \to \mu w$, $h \to d^{-1}\mu h$), the constants d and $d\mu^{-1}$ in (3.2) can both be reduced to 1. Therefore, without loss of generality, in the rest of this section we assume that w satisfies

$$\Delta w - w_s = (1 - h)\chi_{\{w>0\}} \text{ in } P_{r_0}(0, 0). \tag{3.7}$$

Recall that in the new coordinate system $(0, 0) \in \partial\{w > 0\}$.

Lemma 3.7. *The functions* $h(s, y)$ *and* $w_s(s, y)$ *in* (3.7) *are both Hölder continuous in* $P_{r_0}(0, 0)$ *provided that* $r_0 > 0$ *is small enough.*

Proof. Since h and w_t are identically 0 outside $\overline{\{w>0\}}$, it suffices to show that they are Hölder continuous over $\overline{\{w>0\}} \cap P_{r_0}(0,0)$. In this region, $u(s,y) = w_s(s,y)$ satisfies

$$u_s - \Delta u = g(u) \text{ in } \{w>0\} \cap P_{r_0}(0,0), \; u = 0 \text{ on } \partial\{w>0\} \cap P_{r_0}(0,0).$$

Since $g(0) = 0$, and $g(u)$ is locally Lipschitz continuous and u is bounded in the L^∞ norm, we may write $g(u) = c(s,y)u$ with $c \in L^\infty$. The Lipschitz-Hölder smoothness of $\partial\{w>0\} \cap P_{r_0}(0,0)$ in property (iii) of Theorem 3.6 allows us to use standard interior and boundary parabolic regularity (see Theorem 6.33 in [19]) to conclude that u is Hölder continuous over $\overline{\{w>0\}} \cap P_{r_0/2}(0,0)$. Thus u (extended by 0 outside $\{w>0\}$) is Hölder continuous in $P_{r_0}(0,0)$.

Recall that in the original (t,x) coordinates

$$h(t,x) = \int_0^t g(u(\tau,x))d\tau.$$

To deduce the Hölder continuity of $h(t,x)$ near (t_0,x_0), we need to consider the smoothness of $u(t,x)$ for all $t \in (0,t_0]$. Our above discussion shows that the extended u is Hölder continuous in $P_{r_0/2}(t_0,x_0)$. We show next that the extended u is Hölder continuous in $[0, t_0+r] \times B_r(x_0)$ for some $r>0$.

Since $x_0 \notin \overline{\text{co}}(\Omega_0)$, we can find $r>0$ small such that $\overline{B_r(x_0)} \cap \overline{\Omega}_0 = \emptyset$. By Proposition 3.4, there exists $t_1 \in (0,t_0)$ such that $u(t,x) = 0$ for all $t \in [0,t_1]$ and $x \in B_r(x_0)$. Thus u is in particular Hölder continuous over $[0,t_1] \times B_r(x_0)$.

For each $(t,x) \in [t_1,t_0] \times \overline{B_r(x_0)}$, if $w(t,x)>0$, then we can apply the interior regularity to the above equation for u to see that u is Hölder continuous in a small neighborhood of (t,x). If $(t,x) \in \partial\{w>0\}$, then we can apply Theorem 3.6 with (t_0,x_0) replaced by (t,x) and repeat the above argument to conclude that u is Hölder continuous near (t,x). If $(t,x) \notin \overline{\{w>0\}}$, then u is identically 0 in a neighborhood of (t,x). Thus we can use a finite covering argument to conclude that u is Hölder continuous in a small neighborhood of $[t_1,t_0] \times \overline{B_r(x_0)}$.

Hence u is Hölder continuous in $[0,t_0+r] \times B_r(x_0)$ for some small $r>0$. The Hölder continuity of $h(s,y)$ near $(0,0)$ is now obvious. □

Lemma 3.8. *There exists $C>0$ such that, for $j,k \in \{1,...,n\}$,*

$$|w_{y_j y_k}(s,y)| \le C \;\forall (s,y) \in \{w>0\} \cap P_{r_0}(0,0).$$

Proof. Since h is Hölder continuous in $P_{r_0}(0,0)$, away from the free boundary in $\{w>0\} \cap P_{r_0}(0,0)$, we can apply classical Schauder estimates to see that $w \in H_{2+\sigma}$. Therefore it suffices to show that for any sequence $(s_i,y_i) \in \{w>0\} \cap P_{r_0}(0,0)$, $(s_i,y_i) \to (s_0,y_0) \in \partial\{w>0\} \cap P_{r_0}(0,0)$, $|w_{y_j y_k}(s_i,y_i)|$ has a bound that does not depend on the choice of the sequence.

Denote $d_i = d_p((s_i,y_i), \partial\{w>0\})$, where d_p denotes the parabolic distance. Then define

$$w_i(s,y) = d_i^{-2} w(s_i + d_i^2 s, y_i + d_i y).$$

Clearly

$$\Delta w_i - \partial_s w_i = 1 - h_i \text{ in } P_1(0,0),$$

where $h_i(s,y) = h(s_i + d_i^2 s, y_i + d_i y)$, and thus h_i is uniformly Hölder continuous in $P_1(0,0)$. Moreover, by Lemma 3.2,

$$w_i(s,y) \le C(1+|s|^2+|y|) \text{ in } P_1(0,0)$$

for all i. Therefore we can apply classical interior Schauder estimates to the equation of w_i to conclude that

$$|(w_i)_{y_j y_k}| \le C \text{ in } P_{1/2}(0,0) \text{ for all } i \ge 1,$$

where C only depends on $\|w\|_\infty$. In particular,

$$|w_{y_j y_k}(s_i, y_i)| = |(w_i)_{y_j y_k}(0,0)| \le C$$

for all i. □

From (3.6) we find that $(0,0)$ is a density point on the free boundary. With the help of Lemmas 3.7 and 3.8, we can apply Caffarelli's result [4] as in Lemma 9.11 on page 236 of [13] to obtain the following result.

Theorem 3.9. *The function $y_1 = f(s, y_2, ..., y_n)$ in Theorem 3.6 is a C^1 function in $(y_2, ..., y_n)$, uniformly with respect to s. Moreover, $w_{y_i y_j}$ $(i, j \in \{1, ..., n\})$ are all continuous in y, uniformly with respect to s, for $(s, y) \in \overline{\{w > 0\}} \cap P_{r_0}(0,0)$.*

3.3. **Higher regularity.** In this subsection, we will apply the partial hodograph-Legendre transformation introduced by Kinderlehrer and Nirenberg [16] to obtain higher regularity for the free boundary and the solution w. In order to do this, we first need to obtain L^∞ bound for $|w_{sy_i}|$ $(i = 1, ..., n)$ and $|w_{ss}|$ in $\{w > 0\} \cap P_{r_0}(0,0)$. Recall that in the new (s, y) coordinate system, $(0,0) \in \partial\{w > 0\}$ and $0 \notin \overline{\text{co}}(\Omega_0)$.

Let $(s, y) \in \{w > 0\} \cap P_{r_0}(0,0)$. Since $0 \notin \overline{\text{co}}(\Omega_0)$ and $|y| < r_0$, by shrinking r_0 we may assume that $y \notin \overline{\text{co}}(\Omega_0)$. Thus there is a first time moment $\tau(y) \in (-t_0, s)$ such that (τ, y) enters $\{w > 0\}$ as τ increases across $\tau(y)$, namely $w(\tau, y) = 0$ for $\tau \le \tau(y)$, and $w(\tau, y) > 0$ for $\tau > \tau(y)$.

Since $\Omega(s) = \{y : (s, y) \in \{w > 0\}\}$ is expanding continuously as s increases (Propositions 2.3 and 3.4), there exists $\delta > 0$ small such that $\Omega(s) \cap B_{r_0}(0) = \emptyset$ for $s \le -t_0 + \delta$ provided that r_0 is small enough so that $B_{2r_0}(0) \cap \overline{\text{co}}(\Omega_0) = \emptyset$. The choice of δ implies that $\tau(y) > -t_0 + \delta$ whenever $(s, y) \in \{w > 0\} \cap P_{r_0}(0,0)$. Moreover, for each such (s, y), $d_y(\tau) := \text{dist}(y, \partial\Omega(\tau))$ is a nondecreasing function of τ for $\tau > \tau(y)$ (due to the fact that $\Omega(\tau)$ is expanding). It follows that, for $(s, y) \in \{w > 0\} \cap P_{r_0}(0,0)$,

$$\text{dist}\big((\tau, y), \partial\{w > 0\}\big) \le d_y(\tau) \le d_y(0) < r_0 \ \forall \tau \in (\tau(y), -r_0^2].$$

For $\tau \in (-r_0^2, s]$, we have $(\tau, y) \in P_{r_0}(0,0)$ and hence

$$\text{dist}((\tau, y), \partial\{w > 0\}) \le \text{dist}((\tau, y), (0,0)) \le r_0.$$

Thus by Lemma 3.7 (applied to all points (τ, y) in $\{w > 0\}$ near the free boundary with $\tau \in [-t_0 + \delta, s]$), $|w_s(\tau, y)| \le C r_0^\sigma$ for some $\sigma \in (0,1)$ and $C > 0$ independent of (s, y). Therefore we have the following result.

Lemma 3.10. *There exist $\sigma \in (0,1)$, $\delta > 0$ and $C > 0$ such that for all small $r_0 > 0$ and all $(s, y) \in \{w > 0\} \cap P_{r_0}(0,0)$,*

$$\tau(y) \ge -t_0 + \delta, \ u(\tau, y) = w_s(\tau, y) \le C r_0^\sigma \ \ \forall \tau \in (-t_0, s].$$

Due to (3.5), we can find $k_1 > 0$ large enough such that for any fixed s, $w(s, y)$ and $w_s(s, y)$ are nonincreasing in the direction $y - z_0$ for $y \in B_{r_0}(0)$, where $z_0 = (-k_1, 0, ..., 0) \in \mathbb{R}^n$. We now establish a polar coordinate system $(\rho, \theta) = (\rho, \theta_1, ..., \theta_{n-1})$ with origin at z_0, and write

$$\Delta w = w_{\rho\rho} + \frac{n-1}{\rho} w_\rho + \frac{1}{\rho^2} \Delta_{S^{n-1}} w,$$

where $\Delta_{S^{n-1}}$ denotes the Laplace-Beltrami operator on the unit sphere $\{\rho = 1\}$. The choice of z_0 ensures that $\partial_\rho w \leq 0$ for $y \in B_{r_0}(0)$. We define, as in [17],

$$v_0 = -\rho \partial_\rho w.$$

Clearly $v_0 \geq 0$ in $\{w > 0\} \cap P_{r_0}(0,0)$. Since all the partial derivatives of w vanish on $\partial\{w > 0\}$, we have $v_0 = 0$ on $\partial\{w > 0\} \cap P_{r_0}(0,0)$.

Lemma 3.11. *There exists $M_0 > 0$ (depending on g) such that*

$$\partial_s v_0 - \Delta v_0 + M_0 v_0 \geq 1 \text{ in } \{w > 0\} \cap P_{r_0}(0,0).$$

Proof. Using the polar coordinates, one easily calculates $\Delta v_0 = -\rho^{-1}\partial_\rho(\rho^2 \Delta w)$. It follows that

$$\begin{aligned}
&\partial_s v_0 - \Delta v_0 + M_0 v_0 \\
&= -\rho^{-1}\partial_\rho\big[\rho^2(w_s - \Delta w)\big] + 2w_s - M_0\rho\partial_\rho w \\
&= \rho^{-1}\partial_\rho\left[\rho^2 - \rho^2\int_{-t_0}^{s} g(w_s(\tau, y)d\tau\right] + 2w_s - M_0\rho\partial_\rho w \\
&\geq 2 - 2\int_{-t_0}^{s} g(w_s(\tau, y))d\tau - \rho\int_{-t_0}^{s}\partial_\rho\big[g(w_s(\tau, y)) + M_0 w_s(\tau, y)\big]d\tau.
\end{aligned}$$

We now choose $M_0 > 0$ such that $\tilde{g}(u) = g(u) + M_0 u$ is increasing in the interval $[0, \|w_s\|_\infty]$. It follows that

$$\partial_\rho \tilde{g}(w_s) = \tilde{g}'(w_s)\partial_\rho w_s \leq 0$$

in view of the monotonicity of w_s for $y \in B_{r_0}(0)$. Therefore

$$\begin{aligned}
\partial_s v_0 - \Delta v_0 + M_0 v_0 &\geq 2 - 2\int_{-t_0}^{s} g(w_s(\tau, y))d\tau \\
&\geq 2 - Cw(s, y) \geq 1
\end{aligned}$$

in $\{w > 0\} \cap P_{r_0}(0,0)$ provided that r_0 is small enough. □

Lemma 3.12. *There exist $c_1 > 0$ and $c_2 > 0$ such that for any $(s_0, y_0) \in \partial\{w > 0\} \cap P_{r_0/2}(0,0)$,*

$$0 \leq w_s(s, y) \leq c_1|y - y_0|^2 + c_2 v_0(s, y) \text{ in } \{w > 0\} \cap P_{r_0}(0,0).$$

Proof. Denote $\Omega_0 = \{w > 0\} \cap P_{r_0}(0,0)$ and denote by $\partial_p\Omega_0$ its parabolic boundary. On $\partial\{w > 0\} \cap \partial_p\Omega_0$, $w_s = 0$, and for $y \in \partial_p\Omega_0 \setminus \partial\{w > 0\}$, $|y - y_0| \geq c_0 > 0$. Therefore we can find $c_1 > 0$ such that

$$w_s(s, y) \leq c_1|y - y_0|^2 \ \forall (s, y) \in \partial_p\Omega_0.$$

We now choose $c_2 > 0$ such that

$$(\partial_s - \Delta + M_0)[c_1|y - y_0|^2 + c_2 v_0(s, y)] \geq -2nc_1 + c_2 \geq 1.$$

Next we compare w_s and $W := c_1|y - y_0|^2 + c_2 v_0(s, y)$ over Ω_0 by the maximum principle. Clearly $w_s \leq W$ on $\partial_p\Omega_0$. Since

$$(\partial_s - \Delta + M_0)w_s = g(w_s) + M_0 w_s \leq 1 \text{ in } \Omega_0$$

provided that r_0 is small enough, we conclude that $w_s \leq W$ in Ω_0. □

We are now ready to prove the L^∞ bound for the second order derivatives of w not covered by Lemma 3.8.

Lemma 3.13. *There exists $C > 0$ such that*

$$\Sigma_{i=1}^n |w_{sy_i}| \leq C \text{ in } \{w > 0\} \cap P_{r_0/4}(0,0).$$

If further $g \in C^{1,\alpha}([0, \delta_0])$, then we have

$$|w_{ss}| \leq C \text{ in } \{w > 0\} \cap P_{r_0/6}(0,0).$$

Proof. To simplify notations we will write P_{r_0} instead of $P_{r_0}(0,0)$, etc.

Step 1. Boundedness of $\Sigma_{i=1}^n |w_{sy_i}|$.

We follow the ideas of the proof of Theorem 6 in [4]. Choose a function $\varphi \in C_0^\infty(P_{r_0/3})$, $0 \leq \varphi \leq 1$, with $\varphi = 1$ in $P_{r_0/4}$. For $(s, y) \in \{w > 0\} \cap P_{r_0}$, we have

$$\begin{aligned}(\varphi u)_s - \Delta(\varphi u) &= \varphi g(u) + (\varphi_s + \Delta\varphi)u - \Sigma_{i=1}^n (2\varphi_{y_i} u)_{y_i} \\ &= a - \Sigma_{i=1}^n \partial_{y_i} b_i\end{aligned}$$

with

$$a = \varphi g(u) + (\varphi_s + \Delta\varphi)u, \quad b_i = 2\varphi_{y_i} u.$$

We note that a and b_i are well defined over P_{r_0}, and by Lemma 3.7, they are Hölder continuous, say $a, b_i \in C^\alpha(\overline{P_{r_0}})$. Therefore the problem

$$\begin{cases} v_s - \Delta v = a - \Sigma_{i=1}^n \partial_{y_i} b_i & \text{in } P_{r_0}, \\ v = 0 & \text{on } \partial_p P_{r_0} \end{cases}$$

has a unique solution $v \in H_{1+\alpha}(\overline{P_{r_0}})$ (see Theorem 6.45 in [19]).

We now consider the function $V = v - \varphi u$. Clearly

$$V_s - \Delta V = 0 \text{ in } \{w > 0\} \cap P_{r_0}.$$

For any unit vector $\xi \in \mathbb{R}^n$, consider the difference quotient

$$V_h(s, y) := \frac{1}{h}[V(s, y + h\xi) - V(s, y)].$$

Define

$$\Omega_0 = \{w > 0\} \cap P_{r_0/3}, \ \Omega_h = \{(s, y) \in \Omega_0 : \text{dist}(y, \partial\Omega(s)) > h\}.$$

Evidently

$$\partial_s V_h - \Delta V_h = 0 \text{ in } \Omega_h.$$

For any $(s, y) \in \Omega_0$ with $\text{dist}(y, \partial\Omega(s)) = h \in (0, \frac{r_0}{12})$, there exists $y_0 \in \partial\Omega(s)$ such that $|y - y_0| = h$. By Lemma 3.12,

$$0 < u(s, y) \leq c_1 |y - y_0|^2 + c_2 v_0(s, y) \leq c_3 h$$

since $v_0 = -\rho\partial_\rho w$ is a Lipschitz function in the space variables due to Lemma 3.8; similarly,

$$0 < u(s, y + h\xi) \leq C_1 |y + h\xi - y_0|^2 + c_2 v_0(s, y + h\xi) \leq c_4 h.$$

It follows that for $(s, y) \in \Omega_0$ with $\text{dist}(y, \partial\Omega(s)) = h \in (0, \frac{r_0}{12})$,

$$|u_h(s, y)| \leq \frac{1}{h}[u(s, y + h\xi) + u(s, y)] \leq c_3 + c_4.$$

Hence there exists $c_5 > 0$ such that

$$|V_h| \leq c_5 \text{ on } \partial\Omega_h \text{ for all small } h > 0.$$

Applying the maximum principle to V_h over Ω_h we deduce

$$|V_h| \leq c_5 \text{ in } \Omega_h.$$

Letting $h \to 0$ we obtain
$$|\partial_\xi V| \le c_5 \text{ in } \Omega_0,$$
which implies that
$$|\partial_\xi u| \le c_6 \text{ in } \{w > 0\} \cap P_{r_0/4},$$
and therefore
$$\Sigma_{i=1}^n |w_{sy_i}| \le nc_6 \text{ in } \{w > 0\} \cap P_{r_0/4}.$$

Step 2. Bound for $|w_{ss}|$.

We first observe from the estimate proved in Step 1 that
$$|\partial_s v_0| = |\rho \partial_{\rho s} w| \le c_7 \text{ in } \{w > 0\} \cap P_{r_0/4}.$$
It follows that for $(s-h, y) \in \partial\{w > 0\} \cap P_{r_0/4}$ and $h \in (0, \frac{r_0}{8})$,
$$0 < \sup_{|\tau - s| < h} u(\tau, y) \le \sup_{|\tau - s| < h} c_2 v_0(\tau, y) \le c_8 h.$$

Denote
$$\Omega^h := \{(s, y) \in \{w > 0\} \cap P_{r_0/5} : s - \tau(y) > 2h\},$$
and recall that $u(\tau(y), y) = 0$, $u(\tau, y) > 0$ for $\tau > \tau(y)$. For $(s, y) \in \Omega^h$, define
$$u^h(s, y) = \frac{1}{h} \int_s^{s+h} u(\tau, y) d\tau.$$
(It is crucial that we define u^h this way instead of using mollifiers as on page 266 of [17].) Clearly
$$\partial_s u^h = \frac{1}{h}[u(s+h, y) - u(s, y)],$$
and so for all small $h > 0$,
$$|\partial_s u^h| \le c_9 \text{ for } (s, y) \in \Omega^h \text{ with } s = \tau(y) + 2h,$$
and
$$|\nabla u^h| = \left| \frac{1}{h} \int_s^{s+h} \nabla u(\tau, y) d\tau \right| \le \sup_{\{w>0\} \cap P_{r_0/4}} |\nabla u| \le c_6 \ \ \forall (s, y) \in \Omega^h.$$

Choose a function $\zeta \in C_0^\infty(P_{r_0/5})$, $0 \le \zeta \le 1$, with $\zeta = 1$ in $P_{r_0/6}$, and define, with positive constants μ and σ to be specified,
$$W = \zeta^2 (\partial_s u^h)^2 + \mu |\nabla u^h|^2 + \sigma.$$
We are going to apply a Bernestern type argument to show that W has an upper bound in Ω^h that is independent of h.

Since we now assume that $g \in C^{1,\alpha}([0, \delta_0])$, by setting r_0 small enough, we may assume without loss of generality that $0 < u(s, y) < \delta_0$ in $\{w > 0\} \cap P_{r_0}$. Hence from the equation
$$u_s - \Delta u = g(u) \text{ in } \{w > 0\} \cap P_{r_0},$$
we see by the interior Schauder estimates that u_s and u_{y_i} $(i = 1, ..., n)$ belong to $H_{2+\alpha}(\{w > 0\} \cap P_{r_0})$. In particular, $W \in H_{2+\alpha}(\overline{\Omega^h})$.

Let us also observe that, for $(s, y) \in \Omega^h$ and all small $h > 0$,
$$\partial_s u^h - \Delta u^h = [g(u)]^h \ \text{ with } |[g(u)]^h| \le c_{10},$$
$$|\partial_s [g(u)]^h| = \frac{1}{h} |g(u(s+h, y)) - g(u(s, y))| \le c_{11} |\partial_s u^h|,$$
$$|\nabla [g(u)]^h| = |[g'(u) \nabla u]^h| \le c_{12}.$$

We compute, for $(s,y)\in\Omega^h$,

$$\begin{aligned}
&\Delta W+W-W_s\\
&=2\mu\Sigma_{i,j}(u^h)^2_{y_iy_j}+8\zeta\partial_s u^h\nabla\zeta\cdot\nabla(\partial_s u^h)+2\zeta^2|\nabla(\partial_s u^h)|^2\\
&\quad+2\Big(|\nabla\zeta|^2+\zeta\Delta\zeta-\zeta\zeta_s\Big)(\partial_s u^h)^2\\
&\quad\quad+2\mu\nabla u^h\cdot\nabla(\Delta u^h-\partial_s u^h)+2\zeta^2(\partial_s u^h)\partial_s\big(\Delta u^h-\partial_s u^h\big)\\
&\quad\quad\quad+\zeta^2(\partial_s u^h)^2+\mu|\nabla u^h|^2+\sigma\\
&\ge 2\mu\Sigma_{i,j}(u^h)^2_{y_iy_j}-8|\nabla\zeta|^2(\partial_s u^h)^2\\
&\quad+2\Big(|\nabla\zeta|^2+\zeta\Delta\zeta-\zeta\zeta_s\Big)(\partial_s u^h)^2\\
&\quad\quad-2\mu|\nabla u^h|\,|\nabla[g(u)]^h|-2\zeta^2|\partial_s u^h|\,|\partial_s[g(u)]^h|\\
&\quad\quad\quad+\zeta^2(\partial_s u^h)^2+\mu|\nabla u^h|^2+\sigma\\
&\ge 2\mu\Sigma_{i,j}(u^h)^2_{y_iy_j}-8|\nabla\zeta|^2(\partial_s u^h)^2\\
&\quad+2\Big(|\nabla\zeta|^2+\zeta\Delta\zeta-\zeta\zeta_s\Big)(\partial_s u^h)^2\\
&\quad\quad-2c_{12}\mu|\nabla u^h|-2\zeta^2c_{11}(\partial_s u^h)^2\\
&\quad\quad\quad+\zeta^2(\partial_s u^h)^2+\mu|\nabla u^h|^2+\sigma\\
&=2\mu\Sigma_{i,j}(u^h)^2_{y_iy_j}-(\partial_s u^h)^2\psi+\mu(|\nabla u^h|-c_{12})^2+\sigma-c_{12}^2\mu,
\end{aligned}$$

where ψ is a bounded function (independent of h). Since

$$(\partial_s u^h)^2=(\Delta u^h+[g(u)]^h)^2\le 2(\Delta u^h)^2+2|[g(u)]^h|^2\le 2n^2\Sigma_{i=1}^n(u^h)^2_{y_iy_i}+2c_{10}^2,$$

we easily see that if $\mu\ge n^2|\psi|$ and $\sigma\ge c_{12}^2\mu+2c_{10}^2|\psi|$, then

$$\Delta W+W-W_s\ge 0 \text{ in } \Omega^h \text{ for all small } h>0.$$

Applying the maximum principle to $e^{-s}W$, which satisfies $(e^{-s}W)_s-\Delta(e^{-s}W)\le 0$ in Ω^h, we obtain, for $(s,y)\in\Omega^h\cap P_{r_0/6}$,

$$\begin{aligned}
(\partial_s u^h)^2&\le\sup_{\Omega^h}W\le e^{r_0^2/36}\sup_{\Omega^h}(e^{-s}W)\le e^{r_0^2/36}\sup_{\partial\Omega^h}(e^{-s}W)\\
&\le e^{r_0^2}\sup_{\partial\Omega^h}W\le e^{r_0^2}(c_9^2+\mu c_6^2+\sigma).
\end{aligned}$$

Letting $h\to 0$ we obtain

$$|\partial_s u|^2\le e^{r_0^2}(c_9^2+\mu c_6^2+\sigma) \text{ for } (s,y)\in\{w>0\}\cap P_{r_0/6}.$$

The proof is complete. □

We next establish a key smoothness lemma for $h(s,y)$.

Lemma 3.14. *Suppose that $g\in C^{1,\alpha}([0,\delta_0])$. Then the function $h(s,y)$ is Lipschitz continuous in $\{w>0\}\cap P_{r_0}(0,0)$. Moreover, if $u(s,\cdot)=w_s(s,\cdot)\in C^{1,\alpha}(\overline{\Omega_{r_0}(s)})$ uniformly for $s\in[-t_0+\delta,r_0^2]$, where δ is given in Lemma 3.10 and*

$$\Omega_{r_0}(s)=\{y\in\Omega(s):\mathrm{dist}(y,\partial\Omega(s))<r_0\}\cap B_{r_0}(0),$$

then $h(s,\cdot)\in C^{1,\alpha}(\overline{\Omega_{r_0}(s)})$ uniformly for $s\in[-r_0^2,r_0^2]$.

Proof. **Step 1.** h is Lipschitz.

Since $\partial_s h(s,y) = g(u(s,y))$, it is clear that $\partial_s h$ is uniformly bounded in $P_{r_0}(0,0)$. (It is actually Lipschitz, recalling the conclusions in Lemma 3.13.)

Let $(s,y) \in \{w > 0\} \cap P_{r_0}(0,0)$, and $\nu \in \mathbb{R}^n$ be a unit vector. We now consider $\partial_\nu h(s,y)$. We first prove the following formula

$$\partial_\nu h(s,y) = \int_{\tau(y)}^{s} g'(u(\tau,y))\partial_\nu u(\tau,y)d\tau. \tag{3.8}$$

If $\tau(y)$ is a C^1 function, this formula would follow directly from differentiating the equation $h(s,y) = \int_{\tau(y)}^{s} g(u(\tau,y))d\tau$. Since it is unclear whether $\tau(y)$ is C^1, a proof is needed.

For small $\epsilon > 0, \sigma > 0$, we consider

$$\begin{aligned} I_\epsilon : &= \epsilon^{-1}\big[h(s,y+\epsilon\nu) - h(s,y)\big] \\ &= \epsilon^{-1}\int_{\tau(y+\epsilon\nu)}^{s} g(u(\tau,y+\epsilon\nu))d\tau - \epsilon^{-1}\int_{\tau(y)}^{s} g(u(\tau,y))d\tau \\ &= \int_{\tau(y)+\sigma}^{s} \epsilon^{-1}\big[g(u(\tau,y+\epsilon\nu)) - g(u(\tau,y)\big]d\tau \qquad [\,=: I_1\,] \\ &\quad + \int_{\tau(y)}^{\tau(y)+\sigma} \epsilon^{-1}\big[g(u(\tau,y+\epsilon\nu)) - g(u(\tau,y))\big]d\tau \qquad [\,=: I_2\,] \\ &\quad + \int_{\tau(y+\epsilon\nu)}^{\tau(y)} \epsilon^{-1} g(u(\tau,y+\epsilon\nu))d\tau. \qquad [\,=: I_3\,] \end{aligned}$$

We observe that $\limsup_{\epsilon\to 0}\tau(y+\epsilon\nu) = \tau^* \le \tau(y)$ for otherwise we would have $w(\tau^*,y) = 0$ with $\tau^* > \tau(y)$, contradicting the definition of $\tau(y)$. Therefore we may assume that $\tau(y+\epsilon\nu) < \tau(y)+\sigma$ for all small ϵ. By Lemmas 3.10 and 3.13, there exists $C > 0$ such that

$$|\partial_\nu u(\tau,y)| \le C \;\; \forall \tau \in [\tau(y), r_0^2],\; \forall y \in B_{r_0}(0). \tag{3.9}$$

It now follows easily that

$$\begin{aligned} I_1 &= \int_{\tau(y)+\sigma}^{s} g'(u(\tau,y))\partial_\nu u(\tau,y)d\tau + o_\epsilon(1) \\ &= \int_{\tau(y)}^{s} g'(u(\tau,y))\partial_\nu u(\tau,y)d\tau + o_\epsilon(1) + O(\sigma), \end{aligned}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ uniformly in σ, and $|O(\sigma)| \le C\sigma$ for some $C > 0$ independent of ϵ.

To estimate I_2, we note that for $\tau \in [\tau(y), \tau(y)+\sigma]$,

$$|g(u(\tau,y+\epsilon\nu)) - g(u(\tau,y))| \le C_1|u(\tau,y+\epsilon\nu) - u(\tau,y)| \le C_2\epsilon$$

due to the fact that $u = w_s$ is Lipschitz in view of Lemma 3.13. It follows that

$$I_2 = O(\sigma).$$

Since $(\tau(y),y)\in\{w=0\}$, we have, noting $0\le g(u(\tau,y+\epsilon\nu))\le Cu(\tau,y+\epsilon\nu)$,

$$\begin{aligned}|I_3| &\le \epsilon^{-1}\Big|\int_{\tau(y+\epsilon\nu)}^{\tau(y)} Cw_s(\tau,y+\epsilon\nu)d\tau\Big|\\ &= \epsilon^{-1}Cw(\tau(y),y+\epsilon\nu)\\ &\le \epsilon^{-1}C_3\Big(\mathrm{dist}\big[(\tau(y),y+\epsilon\nu),(\tau(y),y)\big]\Big)^2 \quad [\text{ by Lemma 3.2 }]\\ &\le C_3\epsilon.\end{aligned}$$

Thus $I_3=o_\epsilon(1)$ and

$$\big|I_\epsilon-\int_{\tau(y)}^{s} g'(u(\tau,y))\partial_\nu u(\tau,y)d\tau\big| = o_\epsilon(1)+O(\sigma).$$

Letting $\epsilon\to 0$ followed by letting $\sigma\to 0$, we obtain (3.8).

Using (3.9) and (3.8) we obtain

$$|\partial_\nu h(s,y)|\le C\ \forall (s,y)\in\{w>0\}\cap P_{r_0}(0,0).$$

This proves the first part of the lemma.

Step 2. $h(s,\cdot)$ is $C^{1,\alpha}$.

Suppose now $u(s,\cdot)\in C^{1,\alpha}(\overline{\Omega_{r_0}(s)})$ uniformly for $s\in[-t_0+\delta,r_0^2]$. Fix $s\in[-r_0^2,r_0^2]$ and $y\in\Omega_{r_0}(s)$. For small $\epsilon>0$ and fixed unit vector $\eta\in\mathbb{R}^n$, we may assume that $y+\epsilon\eta\in\Omega_{r_0}(s)$. For definiteness, we assume that $\tau(y)<\tau(y+\epsilon\eta)$. (The other case is handled similarly.) Then

$$\begin{aligned}J_\epsilon :&= \epsilon^{-\alpha}\big[\partial_\nu h(s,y+\epsilon\eta)-\partial_\nu h(s,y)\big]\\ &= \int_{\tau(y+\epsilon\eta)}^{s}\epsilon^{-\alpha}g'(u(\tau,y+\epsilon\eta))\partial_\nu u(\tau,y+\epsilon\eta)d\tau-\int_{\tau(y)}^{s}\epsilon^{-\alpha}g'(u(\tau,y))\partial_\nu u(\tau,y)d\tau\\ &= \int_{\tau(y+\epsilon\eta)}^{s}\epsilon^{-\alpha}\Big[g'(u(\tau,y+\epsilon\eta))\partial_\nu u(\tau,y+\epsilon\eta)-g'(u(\tau,y))\partial_\nu u(\tau,y)\Big]d\tau \qquad [=: J_1\ ]\\ &\quad -\int_{\tau(y)}^{\tau(y+\epsilon\eta)}\epsilon^{-\alpha}g'(u(\tau,y))\partial_\nu u(\tau,y)d\tau. \qquad [=: -J_2\ ]\end{aligned}$$

To simplify notations, for fixed $\tau\in[\tau(y),s]$ we write

$$G(z)=g'(u(\tau,z)),\ U(z)=\partial_\nu u(\tau,z).$$

Then

$$\begin{aligned}&\epsilon^{-\alpha}\big[G(y+\epsilon\eta)-G(y)\big]\\ &=\frac{g'(u(\tau,y+\epsilon\eta))-g'(u(\tau,y))}{|u(\tau,y+\epsilon\eta)-u(\tau,y)|^\alpha}\left(\frac{|u(\tau,y+\epsilon\eta)-u(\tau,y)|}{\epsilon}\right)^\alpha\\ &\le C_1\end{aligned}$$

for some $C_1>0$ independent of ϵ and τ, since g' is C^α and $u(\tau,\cdot)$ is C^1.

It follows that

$$\begin{aligned}|J_1| &\le (t_0+r_0^2)\epsilon^{-\alpha}|G(y+\epsilon\eta)U(y+\epsilon\eta)-G(y)U(y)|\\ &\le (t_0+r_0^2)\big(\epsilon^{-\alpha}|G(y+\epsilon\eta)-G(y)|\cdot|U(y+\epsilon\eta)|\\ &\qquad\qquad +\epsilon^{-\alpha}|U(y+\epsilon\eta)-U(y)|\cdot|G(y)|\big)\\ &\le C_2.\end{aligned}$$

To estimate J_2, we observe that $\tau \in (\tau(y), \tau(y+\epsilon\eta))$ implies $y \in \Omega(\tau)$ and $y+\epsilon\eta \notin \Omega(\tau)$. Therefore the line segment in $\mathbb{R}^n$ joining y and $y+\epsilon\eta$ intersects $\partial\Omega(\tau)$ at some point $z(\tau) \in \partial\Omega(\tau)$. By the known smoothness of the free boundary, $z(\tau)$ is a continuous function of $\tau \in [\tau(y), \tau(y+\epsilon\eta)]$, with $z(\tau(y)) = y$ and $z(\tau(y+\epsilon\eta) = y+\epsilon\eta$. More importantly we have $|z(\tau) - y| \le \epsilon$. We thus similarly have, for fixed $\tau \in [\tau(y), \tau(y+\epsilon\eta)]$,

$$\epsilon^{-\alpha}|G(y)U(y) - G(z(\tau))U(z(\tau))| \le C_3$$

for some $C_3 > 0$ independent of ϵ and τ. This implies that if the integrand function $G(y)U(y)$ in J_2 is replaced by $G(z(\tau))U(z(\tau))$, the change in J_2 is bounded by a constant, namely

$$\Big|J_2 - \int_{\tau(y)}^{\tau(y+\epsilon\eta)} \epsilon^{-\alpha} g'(u(\tau, z(\tau))\partial_\nu u(\tau, z(\tau))d\tau\Big| \le \tilde{C}_3.$$

Moreover, $(\tau, z(\tau)) \in \partial\{w > 0\}$ implies $u(\tau, z(\tau)) = 0$ and $g'(u(\tau, z(\tau)) = g'(0)$. So

$$\int_{\tau(y)}^{\tau(y+\epsilon\eta)} \epsilon^{-\alpha} g'(u(\tau, z(\tau))\partial_\nu u(\tau, z(\tau))d\tau = \epsilon^{-\alpha} g'(0) \int_{\tau(y)}^{\tau(y+\epsilon\eta)} \partial_\nu u(\tau, z(\tau))d\tau.$$

We further have

$$\epsilon^{-\alpha}\big|\partial_\nu u(\tau, z(\tau)) - \partial_\nu u(\tau, y)\big| \le \epsilon^{-\alpha} C_4|z(\tau) - y|^\alpha \le C_4,$$

and due to $\partial_\nu u(\tau, y) = \partial_\tau\big[w_\nu(\tau, y)\big]$, and the fact that $w_\nu = 0$ on the free boundary, we have

$$\begin{aligned}
&\epsilon^{-\alpha}\Big|\int_{\tau(y)}^{\tau(y+\epsilon\eta)} \partial_\nu u(\tau, y)d\tau\Big| \\
&= \epsilon^{-\alpha}\big|w_\nu(\tau(y+\epsilon\eta), y) - w_\nu(\tau(y), y)\big| \\
&= \epsilon^{-\alpha}\big|w_\nu(\tau(y+\epsilon\eta), y) - w_\nu(\tau(y+\epsilon\eta), y+\epsilon\eta)\big| \\
&\le C_5.
\end{aligned}$$

Thus we have

$$\epsilon^{-\alpha}\Big|\int_{\tau(y)}^{\tau(y+\epsilon\eta)} \partial_\nu u(\tau, z(\tau))d\tau\Big| \le C_6$$

and $|J_2| \le C_7$. Hence we have a constant C independent of s, ϵ and η such that $|J_\epsilon| \le C$.

For the remaining case $s \in [-r_0^2, r_0^2]$ and $y \in \partial\Omega(s)$, if $y+\epsilon\eta \in \partial\Omega(s)$, then $s = \tau(y) = \tau(y+\epsilon\nu)$ and from (3.8) we find

$$\partial_\nu h(s, y+\epsilon\eta) = \partial_\nu h(s, y) = 0;$$

if $y + \epsilon\,\eta \in \Omega_{r_0}(s)$, then using $s = \tau(y)$ and $\partial_\nu h(s, y) = 0$ we obtain

$$\begin{aligned}
&\epsilon^{-\alpha}\big[\partial_\nu h(s, y+\epsilon\eta) - \partial_\nu h(s, y)\big] \\
&= \int_{\tau(y+\epsilon\eta)}^{\tau(y)} \epsilon^{-\alpha} g'(u(\tau, y+\epsilon\eta))\partial_\nu u(\tau, y+\epsilon\eta)d\tau,
\end{aligned}$$

and our argument used for estimating J_2 above can be applied to obtain a bound for this quantity.

Thus for each $y \in \overline{\Omega_{r_0/2}(s)}$, we can find a small ball $B_\epsilon(y)$ (with ϵ depending on y) such that $\partial_\nu h(s, \cdot)$ is in $C^\alpha(\overline{B_\epsilon(y)} \cap \overline{\Omega_{r_0/2}(s)})$. The required conclusion now follows from a finite covering argument, recalling that the constants bounding the Hölder norms in each step of our arguments are independent of s. □

We are now ready to prove our higher regularity result.

Theorem 3.15. *Suppose that $g \in C^{1,\alpha}([0,\delta_0])$. Then $\partial\{w>0\}\cap P_{r_0}(0,0)$ is of class $C^{2,\alpha}$, and therefore the function $y_1 = f(s,y_2,...,y_n)$ in Theorem 3.9 can be chosen to be $C^{2,\alpha}$; moreover, $f_s > 0$.*

Proof. For clarity we divide the proof into several steps.

Step 1. The partial hodograph-Legendre transformation.

Through a suitable rotation of the y coordinate system around the origin, we may assume that the function f satisfies additionally $f_{y_i}(0,0,...,0) = 0$ for $i = 2,...,n$. It follows that $w_{y_iy_j}(0,0) = 0$ except for $w_{y_1y_1}(0,0) = 1$. We recall that w_{y_1} is Lipschitz continuous in $\overline{\{w>0\}}\cap P_{r_0}(0,0)$, and for fixed s, it is C^1 in y with modulus of continuity independent of s. As in [16], we extend w_{y_1} into a full neighborhood of $(0,0)$ keeping the above smoothness property, and consider the partial hodograph-Legendre transformation

$$\xi = (\xi_1,...,\xi_n) = (-w_{y_1}, y_2,...,y_n),\ v = \xi_1 y_1 + w = -y_1 w_{y_1} + w.$$

From [16] we know that for fixed s, $y \to \xi$ is a C^1 local diffeomorphism near 0, and the mapping $(s,y)\to(s,\xi)$ and its inverse are both Lipschitz continuous, and it changes the free boundary $y_1 = f(s,y_2,...,y_n)$ into part of the hyperplane $\{(s,\xi) : \xi_1 = 0\}$, with

$$\text{(3.10)}\qquad \begin{cases} v_s = w_s,\ v_{\xi_1} = y_1,\ v_{\xi_i} = w_{y_i}, \\ v_{s\xi_1} = -\frac{w_{sy_1}}{w_{y_1y_1}},\ v_{s\xi_i} = w_{sy_i}, \\ w_{y_1y_1} = -\frac{1}{v_{\xi_1\xi_1}},\ w_{y_1y_i} = \frac{v_{\xi_1\xi_i}}{v_{\xi_1\xi_1}},\ w_{y_jy_i} = v_{\xi_j\xi_i} - \frac{v_{\xi_1\xi_i}}{v_{\xi_1\xi_1}} v_{\xi_j\xi_1}, \end{cases} \qquad i,j\in\{2,...,n\}.$$

Hence (3.7) over $\{w>0\}\cap P_{r_0}(0,0)$ becomes

$$\Sigma_{i=2}^n v_{\xi_i\xi_i} - \frac{1}{v_{\xi_1\xi_1}} - \frac{1}{v_{\xi_1\xi_1}}\Sigma_{i=2}^n v_{\xi_1\xi_i}^2 - v_s = 1 - h(s, v_{\xi_1}, \xi_2,...,\xi_n)$$

in $N_0\cap\{(s,\xi) : \xi_1 < 0\}$, where N_0 is a small neighborhood of $(0,0)$ in $\mathbb{R}\times\mathbb{R}^n$. Furthermore, from the definition of v and (3.10), in view of Lemmas 3.8, 3.13 and Theorem 3.9, one finds that v, v_s and $v_{\xi_i}(i = 1,...,n)$ are Lipschitz in $N_0 \cap \{(s,\xi) : \xi_1 \le 0\}$, and for fixed s, $v(s,\xi)$ is C^2 in ξ with modulus of continuity independent of s. In particular, v belongs to $W^{1,2}_p(N_0\cap\{(s,\xi) : \xi_1 \le 0\})$ for $1 < p \le \infty$.

We now denote the above fully nonlinear equation as

$$\text{(3.11)}\qquad F(D^2 v) - v_s = 1 - h(s, v_{\xi_1}, \xi_2,...,\xi_n),$$

where D^2v is the Hessian of v in the space variables. To simplify notations, we write

$$\Omega_0 = \{w>0\}\cap P_{r_0}(0,0)\},\ \Gamma_0 = \partial\{w>0\}\cap P_{r_0}(0,0),$$

and use O and Σ to denote their images in the (s,ξ) space under the transformation $(s,y)\to(s,\xi)$. We note that Σ is contained in the hyperplane $\xi_1 = 0$, and Γ_0 can now be represented by

$$\text{(3.12)}\qquad y_1 = v_{\xi_1}(s, 0, y_2,...,y_n).$$

When r_0 in the above definitions is replaced by some $r_0' \in (0,r_0)$, we denote the corresponding sets by Ω_0', Γ_0', O' and Σ', respectively. We also write

$$\Omega_0(s) = \{y : (s,y)\in\Omega_0\},\ \Gamma_0(s) = \{y : (s,y)\in\Gamma_0\},\ \text{etc.}$$

Step 2. $v_{\xi_k}(s,\cdot)$ $(k = 1,...,n)$ belong to $C^{1,\gamma}(O(s)\cup\Sigma(s))$ for any $\gamma\in(0,1)$.

Since $h(s, y)$ is Lipschitz in $\Omega_0 \cup \Gamma_0$ and v_{ξ_1} is Lipschitz in $O \cup \Sigma$, we find that $h(s, v_{\xi_1}, \xi_2, ..., \xi_2)$ is Lipschitz in $O \cup \Sigma$. Thus the function

$$\tilde{h}(s, \xi) := 1 - h(s, v_{\xi_1}, \xi_2, ..., \xi_n)$$

is Lipschitz in $O \cup \Sigma$. Since v_s is Lipschitz in this set, $\hat{h} = v_s + \tilde{h}$ is also Lipschitz in $O \cup \Sigma$.

For fixed $s \in [-r_0^2, r_0^2]$, we may now rewrite (3.11) as

$$F(Dv^2) = \hat{h}(s, \cdot) \in C^{0,1} \text{ in } O(s),\ v = 0 \text{ on } \Sigma(s).$$

In the direction ξ_k, $k \neq 1$, the difference quotient of $v(s, \cdot)$,

$$\Delta_\epsilon^k v(s, \cdot) = \frac{v(s, \cdot + \epsilon e_k) - v(s, \cdot)}{\epsilon},$$

satisfies the equation

$$\Sigma a_{ij}^\epsilon (\Delta_\epsilon^k v)_{\xi_i \xi_j} = \Delta_\epsilon^k \hat{h} \text{ in } O'(s),\ \Delta_\epsilon^k v = 0 \text{ on } \Sigma'(s), \tag{3.13}$$

with

$$a_{ij}^\epsilon(s, y) = \int_0^1 \partial_{v_{\xi_i \xi_j}} F\big[(1-t) v_{\xi_i \xi_j}(s, y) + t v_{\xi_i \xi_j}(s, y + \epsilon e_k)\big] dt, \tag{3.14}$$

which is uniformly continuous in $O'(s) \cup \Sigma'(s)$, and the equation is uniformly elliptic in $O'(s)$ (see [16]). Therefore one can apply standard L^p theory to conclude that $\Delta_\epsilon^k v(s, \cdot)$ has a $W^{2,p}$ bound that is independent of ϵ, for any $p > 1$, since the right hand side of the differential equation is uniformly bounded in L^∞. It follows that $v_{\xi_k}(s, \cdot)$ belongs to $W^{2,p}(O(s))$ for any $p > 1$ and hence, by Sobolev embedding, $v_{\xi_k}(s, \cdot) \in C^{1,\gamma}(O(s) \cup \Sigma(s))$ for any $\gamma \in (0, 1)$. We further notice that the bounds for v_{ξ_k} in the norms of these spaces are independent of s. We finally obtain the same bound for v_{ξ_1} from the differential equation and the bound for v_{ξ_k}, $k = 2, ..., n$.

Step 3. $v \in C^2(O \cup \Sigma)$.

Using (3.12) we now see that $\Gamma_0(s) \in C^{1,\gamma}$ uniformly in $s \in [-r_0^2, r_0^2]$. Moreover the above smoothness conclusion on v implies that $w_{y_i y_j}(s, \cdot) \in C^\gamma(\Omega_0(s) \cup \Gamma_0(s))$ uniformly in $s \in [-r_0^2, r_0^2]$ (see page 351 of [16] for more details).

For fixed $s \in [-r_0^2, r_0^2]$, the function $u = w_s(s, \cdot)$ satisfies

$$\Delta u = w_{ss} - g(w_s) \in L^\infty \text{ in } \Omega_0(s),\ u = 0 \text{ on } \Gamma_0(s).$$

Since $\Gamma_0(s) \in C^{1,\gamma}$, by Lemma 3.1 of [17], $u(s, \cdot) \in C^{1,\gamma}(\Omega_0(s) \cup \Gamma_0(s))$, and its modulus of continuity is independent of s.

For later use, we note that since the above analysis can be applied near any point on $\partial\{w > 0\} \cap \big([-t_0 + \delta, r_0^2] \times B_{r_0}(0)\big)$, we find that $u(s, \cdot) \in C^{1,\gamma}(\overline{\Omega_{r_0'}(s)})$ uniformly for $s \in [-t_0 + \delta, r_0^2]$ (with r_0' sufficiently small).

We thus find that $w_{sy_i}(s, \cdot) \in C^\gamma(\Omega_0(s) \cup \Gamma_0(s))$ uniformly in s. Let us recall from Step 2 that the same conclusion holds for $w_{y_i y_j}$. On the other hand, since h and h_s are Hölder continuous, by standard interior parabolic estimate we know that $w \in C^2(\Omega_0)$. Therefore we would have $w \in C^2(\Omega_0 \cup \Gamma_0)$ if we can show that $w_{sy_i}, w_{y_i y_j}$ are all continuous along Γ_0. This can be done in the same way as on page 270 of [17]. We have thus proved that $w \in C^2(\Omega_0 \cup \Gamma_0)$. It follows that $v \in C^2(O \cup \Sigma)$.

Step 4. v_s and v_{ξ_k} $(k = 1, ..., n)$ belong to $H_{1+\gamma}(O \cup \Sigma)$ for any $\gamma \in (0, 1)$.

We now return to (3.11) and view it as a fully nonlinear parabolic equation of the form

$$F(D^2 v) - v_s = \tilde{h} \in C^{0,1} \text{ in } O,\ v = 0 \text{ on } \Sigma. \tag{3.15}$$

In the direction ξ_k, $k \neq 1$, the difference quotient $\Delta_\epsilon^k v$ satisfies

$$\Sigma a_{ij}^\epsilon (\Delta_\epsilon^k v)_{\xi_i \xi_j} - (\Delta_\epsilon^k v)_s = \Delta_\epsilon^k \tilde{h} \text{ in } O',\ \Delta_\epsilon^k v = 0 \text{ on } \Sigma', \tag{3.16}$$

with $a_{ij}^\epsilon(s,y)$ given by (3.14), which are uniformly continuous in $O' \cup \Sigma'$ (due to the continuity of $v_{\xi_i \xi_j}$ in $O \cup \Sigma$), and the equation is uniformly parabolic in $O' \cup \Sigma'$. Therefore one can apply standard L^p theory for linear parabolic equations to conclude that $\Delta_\epsilon^k v$ has a $W_p^{1,2}$ bound that is independent of small $\epsilon > 0$, for any $p > 1$, since $\Delta_\epsilon^k \tilde{h}$ is uniformly bounded in L^∞. It follows that v_{ξ_k} belongs to $W_p^{1,2}(O')$ for any $p > 1$, and hence, by Sobolev embedding, $v_{\xi_k} \in H_{1+\gamma}(O' \cup \Sigma')$ for any $\gamma \in (0,1)$. We can do the same in the direction of s to deduce that $v_s \in W_p^{1,2}(O')$, and the bound for v_{ξ_1} finally follows from the differential equation and the bound for v_s, v_{ξ_k}, $k = 2, ..., n$. Therefore $\partial_{\xi_i} v_{\xi_1}$ $(i = 1, ..., n)$ and $\partial_s v_{\xi_1}$ all belong to $C^\gamma(O' \cup \Sigma')$, for any $\gamma \in (0,1)$. In view of (3.12), we have proved that $\Gamma_0 \in C^{1,\gamma}$ for any $\gamma \in (0,1)$.

Step 5. Completion of the proof.

In Step 3 we have shown that $u(s,\cdot) \in C^{1,\gamma}(\overline{\Omega_{r_0'}(s)})$ uniformly for $s \in [-t_0 + \delta, r_0^2]$ (with r_0' sufficiently small). Therefore we may apply Lemma 3.14 to conclude that $h(s,\cdot) \in C^{1,\alpha}(\Omega_0'(s) \cup \Gamma_0'(s))$ uniformly for $s \in [-(r_0')^2, (r_0')^2]$. From (3.8) it is clear that $\partial_\nu h(s,y)$ is Lipschitz continuous in s uniformly in y. From $\partial_s h(s,y) = g(w_s(s,y))$ and Lemma 3.13 we immediately see that $\partial_s h \in C^{0,1}(\Omega_0' \cup \Gamma_0')$. Therefore $h \in H_{1+\alpha}(\Omega_0' \cup \Gamma_0')$. It follows that $\tilde{h} \in H_{1+\alpha}(O' \cup \Sigma')$.

We now return to (3.15) and (3.16), and notice that due to Step 4 and the above discussion on $\tilde{h}$, the terms a_{ij}^ϵ and $\Delta_\epsilon^k \tilde{h}$ are uniformly bounded in $H_\alpha(O' \cup \Sigma')$. Therefore we may apply standard Hölder estimates to conclude that $v_{\xi_k} \in H_{2+\alpha}(O' \cup \Sigma')$. The estimates for v_s and v_{ξ_1} are obtained in a similar fashion as before. Therefore $\partial_{\xi_i} v_{\xi_1}, \partial_s v_{\xi_1} \in C^{1,\alpha}(O' \cup \Sigma')$, which implies that $\Gamma_0 \in C^{2,\alpha}$.

From the equation

$$u_s - \Delta u = g(u) \text{ in } \Omega_0,\ u = 0 \text{ on } \Gamma_0$$

we deduce, by the strong maximum principle, $w_{y_1 s} = \partial_{y_1} u < 0$ on Γ_0.

Rewriting (3.12) as $y_1 = f(s, y_2, ..., y_n)$ and recalling that w_{y_1} vanishes on the free boundary, we have

$$w_{y_1}(s, f(s,y'), y') \equiv 0.$$

Differentiating this identity with respect to s we obtain

$$\partial_s f = -\frac{w_{y_1 s}}{w_{y_1 y_1}} > 0$$

since $w_{y_1 y_1} > 0$ on Γ_0 due to $w_{y_1 y_1}(0,0) = 1$ and r_0 is small. □

Corollary 3.16. *Suppose* (2.2) *and* (3.3) *hold, and* $g \in C^{1,\alpha}([0,\delta_0])$ *for some small* $\delta_0 > 0$. *Then for any* $t > 0$, $\tilde{\Gamma}(t) := \Gamma(t) \setminus \overline{\text{co}}(\Omega_0)$ *is a* $C^{2,\alpha}$ *hypersurface in* $\mathbb{R}^n$, *and* $\tilde{\Gamma} := \{(t,x) : x \in \tilde{\Gamma}(t),\ t > 0\}$ *is a* $C^{2,\alpha}$ *hypersurface in* $\mathbb{R}^{n+1}$.

4. The monotonicity method and Lipschitz smoothness

In this section, we prove Theorem 3.6 by the monotonicity method. This is where (3.3) is needed. More precisely the reflection argument to be used requires

$$g(x,u) \equiv g(u). \tag{4.1}$$

Let $(t_0, x_0) \in \partial\{w > 0\}$ with $t_0 > 0$ and $x_0 \notin \overline{\text{co}}(\Omega_0)$. We will first show that $\Gamma(t_0)$ is Lipschitz continuous near x_0. The Hölder continuity in t then follows from a blowing up argument.

The Lipschitz continuity of $\Gamma(t_0)$ near x_0 follows from a simple reflection and comparison argument as employed in [20]. This argument shows that $u = w_t$ is monotone in certain directions, which implies the Lipschitz continuity of $\Gamma(t_0)$ near x_0.

This monotonicity method was applied in [20] to the classical one phase Stefan problem corresponding to the weak formulation (2.11) in this paper. However, in our situation here, due to the nonlocal term $H(w_t)$ in (2.11), comparison arguments are difficult to apply directly. Instead, we will apply the comparison argument to $u = w_t$, which is the unique weak solution to (2.1) as defined in [8].

Fix (t_0, x_0) as above, then fix $T > t_0$. By Theorem 3.1 of [8], $u = w_t$ is the weak limit in $H^1((0,T)\times B_R)$ and strong limit in $L^2((0,T)\times B_R)$ of a sequence of approximate solutions u_m satisfying

$$\begin{cases} \partial_t[\alpha_m(u_m)] - d\Delta u_m = g(u_m) & \text{in } (0,T)\times B_R, \\ u_m = 0 & \text{on } (0,T)\times \partial B_R, \\ u_m(0,x) = \tilde{u}_0(x) & \text{in } B_R, \end{cases} \tag{4.2}$$

where B_R is a ball of radius R with center a fixed point in Ω_0, R is chosen large enough so that $\Omega(t) \subset B_R$ for $t \in (0,T]$ (see [8] for the choice of R), $\tilde{u}_0(x)$ is the zero extension of $u_0(x)$, and α_m is a sequence of smooth functions with the following properties:

$$\begin{cases} \alpha_m(\xi) \to \alpha(\xi) \text{ uniformly in any compact subset of } \mathbb{R}^1 \setminus \{0\}, \\ \alpha_m(0) \to -d\mu^{-1},\ \alpha_m'(\xi) \geq 1 \text{ for all } \xi \in \mathbb{R}^1, \\ \xi - d\mu^{-1} \leq \alpha_m(\xi) \leq \xi \text{ for all } \xi \in \mathbb{R}^1, \end{cases}$$

where $\alpha(\xi)$ is defined in (2.15).

For any given $z_0 \notin \overline{\text{co}}(\Omega_0)$, we can associate a uniquely determined open set of unit vectors S_{z_0} and an open cone C_{z_0} with vertex 0 in the following way:

$$S_{z_0} := \{\nu \in \mathbb{R}^N : \ |\nu| = 1\ ,\ \nu\cdot(x - z_0) < 0\ \forall x \in \overline{\text{co}}(\Omega_0)\},$$

$$C_{z_0} := \{\lambda\nu : \lambda \in (0,1),\ \nu \in S_{z_0}\}.$$

C_{z_0} has the following geometric characterization: For any $x \in z_0 + C_{z_0}$, let l_0 denote the straight line passing through z_0 and x; then the hyperplane passing through z_0 and normal to l_0 does not intersect $\overline{\text{co}}(\Omega_0)$.

Lemma 4.1. *For $s \in (0,T)$, $z \in B_R \setminus \overline{\text{co}}(\Omega_0)$ and $\nu \in S_z$, we have $\partial_\nu u_m(s,z) \leq 0$.*

Proof. Let $P = P_z$ be the hyperplane passing through z with normal vector ν. P divides B_R into two parts. Denote S^+ the part containing Ω_0 and S^- the other part. This is possible because by the definition of ν, $\overline{\text{co}}(\Omega_0) \subset \{x : \nu\cdot(x - z) < 0\}$.

For $x \in S^-$, let $\tilde{x}$ be the reflection point of x in P. We claim that for $(t,x) \in (0,T)\times S^-$,

$$u_m(t,x) \leq u_m(t,\tilde{x}).$$

In fact, this is true on the parabolic boundary $\partial_p((0,T)\times S^-)$, and both $u_m(t,x)$ and $v_m(t,x) := u_m(t,\tilde{x})$ satisfy the first equation in (4.2) over $(0,T)\times S^-$, so this claim follows from the comparison principle (see Lemma 3.2 in [8]). From this claim, we immediately obtain $\partial_\nu u_m(s,z) \leq 0$. □

Lemma 4.2. *For $(s,z) \in (0,T) \times [B_R \setminus \overline{\mathrm{co}}(\Omega_0)]$, and all $\nu \in S_z$, we have $\partial_\nu u(s,z) \le 0$. Moreover, for every $s_0 \in (0,T)$, $z_0 \in \Omega(s_0) \setminus \overline{\mathrm{co}}(\Omega_0)$ and $\nu \in S_{z_0}$, we have $\partial_\nu u(s_0,z_0) < 0$.*

Proof. Since $u_m \to u$ weakly in $H^1((0,T)\times B_R)$, the first part of the lemma follows directly from Lemma 4.1.

We now consider the second part. Recall that u is continuous in $\{w > 0\} = \{u > 0\}$. Therefore from $u(s_0, z_0) > 0$ we can find $r_0 > 0$ small such that for $(s,z) \in P_{r_0}(s_0,z_0)$, $u(s,z) \ge u(s_0,z_0)/2 > 0$ and $z \in \Omega(s) \setminus \overline{\mathrm{co}}(\Omega_0)$.

Fix $\nu \in S_{z_0}$. Since S_z varies continuously with z, we find that $\nu \in S_z$ for all z close to z_0. Thus by shrinking $r_0 > 0$ we may assume that $\nu \in S_z$ whenever $(s,z) \in P_{r_0}(s_0,z_0)$.

We may now apply Lemma 4.1 to conclude that $\partial_\nu u_m(s,z) \le 0$ for all $(s,z) \in P_{r_0}(s_0,z_0)$. By the definition of α and α_m, and by our choice of r_0, for all large m, $\alpha(u_m) = u_m$ in $P_{r_0}(s_0,z_0)$. This implies that $u_m \to u$ in $H_{2+\sigma,loc}(P_{r_0}(s_0,z_0))$ $(0<\sigma<1)$ by standard regularity theory for parabolic equations. It follows that $\partial_\nu u(s,z) \le 0$ in $P_{r_0}(s_0,z_0)$. Moreover, u satisfies

$$u_t - d\Delta u = g(u) \text{ in } P_{r_0}(s_0,z_0).$$

Denote $v = \partial_\nu u$ and we find that

$$v_t - d\Delta v = c(t,x)v \text{ and } v \le 0 \text{ in } P_{r_0}(s_0,z_0),$$

for some $c \in L^\infty(P_{r_0}(s_0,z_0))$. By the strong maximum principle we have either $v(s_0,z_0) < 0$ or $v(t,x) \equiv 0$ in $P^-_{r_0}(s_0,z_0) := \{(t,x) \in P_{r_0}(s_0,z_0) : t \le s_0\}$.

To complete the proof, it remains to show that the second alternative cannot happen. Suppose by way of contradiction that $v \equiv 0$ in $P^-_{r_0}(s_0,z_0)$. Then $u(s_0,z_0+r\nu) \equiv u(s_0,z_0)$ for $r \in [0,r_0]$. Since $\Omega(s_0) \subset B_R$, we can find a maximal $r^* > 0$ such that $u(s_0,z_0+r\nu) \equiv u(s_0,z_0) > 0$ for $r \in [0,r^*]$. Set $z^* = z_0 + r^*\nu$. Since $\nu \in S_{z_0}$, the hyperplane in $\mathbb{R}^n$ that passes through z^* and is perpendicular to ν does not intersect $\overline{\mathrm{co}}(\Omega_0)$, which indicates that $\nu \in S_{z^*}$. Hence we can repeat the argument used above but with z_0 replaced by z^* to conclude that for some $r_1 > 0$ small, either $\partial_\nu u(s_0,z^*) < 0$ or $\partial_\nu u(s_0,x) \equiv 0$ in $P^-_{r_1}(s_0,z^*)$. However, from the definition of z^* we see that $\partial_\nu u(s_0,z^*) = 0$ since $u(s_0,\cdot)$ takes the constant value $u(s_0,z_0)$ on the line segment connecting z_0 and z^*. Thus the second alternative must happen, which implies $u(s_0,z_0+r\nu) \equiv u(s_0,z_0)$ for $r \in [0,r^*+r_1]$, a contradiction to the maximality of r^*. This completes the proof. □

For any small $\delta > 0$, let

$$W_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \overline{\mathrm{co}}(\Omega_0)) \le \delta\}.$$

We now associate to each $z_0 \notin \overline{\mathrm{co}}(\Omega_0)$ the unique open set $S^\delta_{z_0}$ and open cone $C^\delta_{z_0}$ which are obtained by replacing $\overline{\mathrm{co}}(\Omega_0)$ with W_δ in the definitions of S_{z_0} and C_{z_0}, respectively. It is easily seen that for each $\delta > 0$ and $z_0 \notin \overline{\mathrm{co}}(\Omega_0)$, there exists $\epsilon > 0$ small (depending on $\operatorname{dist}(z_0, \overline{\mathrm{co}}(\Omega_0))$ and δ) such that

$$S^\delta_{\tilde z} \subset S^{\delta/2}_z \subset S_{z_0} \text{ and } C^\delta_{\tilde z} \subset C^{\delta/2}_z \subset C_{z_0} \text{ if } z,\, \tilde z \in B_\epsilon(z_0).$$

This property will be used in the proof of the next result.

Lemma 4.3. *Suppose that $t_0 \in (0,T)$, $x_0 \in \Gamma(t_0) \setminus \overline{\mathrm{co}}(\Omega_0)$ and $\delta > 0$ is small. Then there exists $\epsilon > 0$ small such that $u(t_0,x) \equiv 0$ in $(x_0 + C^\delta_{x_0}) \cap B_\epsilon(x_0)$, and $u(t_0,x) > 0$ in $(x_0 - C^\delta_{x_0}) \cap B_\epsilon(x_0)$.*

Proof. We first choose $\epsilon > 0$ small so that $C_{x_0} \supset C^{\delta/2}_x \supset C^\delta_{\tilde x}$ for all $x, \tilde x \in B_{2\epsilon}(x_0)$, and $B_{2\epsilon}(x_0) \cap \overline{\mathrm{co}}(\Omega_0) = \emptyset$. We now show that $u(t_0,\cdot) \equiv 0$ in $(x_0 + C^\delta_{x_0}) \cap B_\epsilon(x_0)$. Otherwise there exists z_0 in this set such that $(t_0, z_0) \in \{u > 0\}$. We may now use Lemma 4.2 to deduce that $u(t_0, x) > u(t_0, z_0) > 0$ for $x \in (z_0 - C_{z_0}^{\delta/2}) \cap B_\epsilon(x_0)$. This implies that $(t_0, x_0) \in \{u > 0\}$, which clearly contradicts the assumption that $(t_0, x_0) \in \partial\{u > 0\}$.

We show next that $u(t_0, x) > 0$ in $(x_0 - C_{x_0}^\delta) \cap B_\epsilon(x_0)$. Since $\Gamma(t_0)$ is the boundary of the open set $\Omega(t_0)$, there exists $x_i \in \Omega(t_0)$ such that $x_i \to x_0$ as $i \to \infty$. By Lemma 4.2, we have $u(t_0, x) > 0$ in $(x_i - C_{x_0}^\delta) \cap B_\epsilon(x_0)$ for all large i. Letting $i \to \infty$ we find that $u(t_0, x) > 0$ in $(x_0 - C_{x_0}^\delta) \cap B_\epsilon(x_0)$. □

Because the cone C_z depends continuously on z, by Lemma 4.3 it is easily seen that for any $x_0 \in \Gamma(t_0) \setminus \overline{\text{co}}(\Omega_0)$ with $t_0 \in (0, T)$, in a neighborhood $B_r(x_0)$ of x_0, $\Gamma(t_0)$ can be represented by a Lipschitz graph, with the bound of its Lipschitz constant determined by the opening angle of $C_{x_0}^\delta$.

Fix such a pair (t_0, x_0) and fix $\delta > 0$ small. We now choose the coordinate system so that x_0 is the origin. Moreover, if $\nu_{x_0} \in S_{x_0}$ is the axis of C_{x_0}, we choose the x_n-axis to agree with ν_{x_0}. Let $r_0 > 0$ be a small number so that $C_x^\delta \subset C_{x_0}^{\delta/2}$ for all $x \in B_{r_0}(x_0)$. By the continuous dependence of $\Omega(t)$ on t (see Proposition 3.4), we can find $r_1 > 0$ small such that for $t \in (t_0 - r_1^2, t_0 + r_1^2)$, $A_t := \Gamma(t) \cap \{\lambda\nu_{x_0} : \lambda \in \mathbb{R}^1\} \subset B_{r_0/2}(x_0)$. We may now apply Lemma 4.3 to conclude that A_t consists of a single point, say $A_t = \{y^t\}$, and $\Gamma(t) \cap B_{r_0}(x_0)$ is a Lipschitz hypersurface of the form $x_n = f(t, x')$, with $y^t = f(t, 0)$, for x' varying in a small r-neighborhood U_r of $0 \in \mathbb{R}^{n-1}$, and $r > 0$ can be chosen to be independent of $t \in (t_0 - r_1^2, t + r_1^2)$.

Now we show that $f(t, x)$ is $\frac{1}{2}$–Hölder continuous with respect to t in $(t_0 - r_1^2, t_0 + r_1^2) \times U_{r/2}$.

Lemma 4.4. *$\exists C > 0$, such that*

$$|f(t, x') - f(s, x')| \le C|t - s|^{\frac{1}{2}} \textit{ for } t, s \in (t_0 - r_1^2, t_0 + r_1^2) \textit{ and } x' \in U_{r/2}.$$

Proof. Assume by way of contradiction that $\exists (t_j, x_j')$ and (s_j, x_j') such that t_j, $s_j \in (t_0 - r_1^2, t_0 + r_1^2)$, $x_j' \in U_{r/2}$ and

$$\frac{|f(t_j, x_j') - f(s_j, x_j')|}{|t_j - s_j|^{1/2}} \to +\infty \text{ as } j \to \infty.$$

Without loss of generality, assume $t_j - s_j = r_j^2 > 0$. Then by the monotonicity of $\Omega(t)$ we have $f(t_j, x_j') \ge f(s_j, x_j')$. Thus

$$\frac{f(t_j, x_j') - f(s_j, x_j')}{(t_j - s_j)^{1/2}} \to +\infty. \tag{4.3}$$

Denote $x_j = (x_j', f(t_j, x_j'))$ and $y_j = (x_j', f(s_j, x_j')$. Then define the rescaling

$$w_j(t, x) := \frac{1}{r_j^2} w(t_j + r_j^2 t, x_j + r_j x) \text{ for } (t, x) \in P_{r_j^{-1}}(0, 0),$$

where w is the solution of (2.11).

By Lemma 3.2, for all large j, w_j is uniformly bounded in any compact set K of $(-\infty, +\infty) \times \mathbb{R}^n$. By rescaling the equation of w, we see $(d\Delta - \partial_t)w_j$ is uniformly bounded in K. Thus for $\forall p > 1$, w_j is uniformly bounded in $W_p^{1,2}(K)$. After passing to a subsequence, we can assume w_j converges to w_∞ uniformly in any compact set of $(-\infty,+\infty)\times\mathbb{R}^n$. By Lemma 3.3, w_∞ is nontrivial. In particular,

$$\sup_{P_1(0,0)} w_\infty \geq C(n) > 0. \tag{4.4}$$

Because $(s_j,y_j)\in\partial\{w>0\}=\partial\{u>0\}$, we have, with $\tilde{x}_j := (0, \frac{f(s_j,x_j)-f(t_j,x_j)}{r_j})$,

$$(-1,\tilde{x}_j)\in\partial\{w_j>0\}=\partial\{u_j>0\},$$

where u_j denotes the corresponding rescaling of u. By the monotonicity of u_j, $\forall\lambda\geq 0$ such that $x\in\tilde{x}_j+\lambda S^\delta_{x_0}\subset B_{R/r_i}$,

$$(-1,x)\in\{u_j=0\}=\{w_j=0\},\ i.e.,\ w_j(-1,x)=0.$$

Passing to the limit and noticing our assumption (4.3), we see

$$w_\infty(-1,x)\equiv 0 \text{ for } x\in\mathbb{R}^n.$$

On the other hand, using (3.2), we obtain

$$(d\Delta-\partial_t)w_\infty = d\mu^{-1}\chi_{\{w_\infty>0\}}\geq 0 \text{ in } \mathbb{R}^1\times\mathbb{R}^n,$$

and by Lemma 3.2, $0\leq w_\infty(t,x)\leq C(n)r^2$ in $P_r^-(0,0)$ for all $r>0$. Combing these three facts we get $w_\infty\equiv 0$ in $(-1,\infty)\times\mathbb{R}^n$. This contradicts (4.4). □

Clearly Theorem 3.6 is a consequence of the above results.

As in [8], we know that when (2.2) holds, the weak solution u of (2.1) is defined for all $t>0$. Let us end this section by observing the following easy consequence of Corollary 3.16.

Theorem 4.5. *Apart from* (4.1) *if we assume further that* $g\in C^{1,\alpha}([0,\delta_0])$ *and* Ω_0 *is convex, then* $\Gamma(t)$ *is* $C^{2,\alpha}$ *for* $t>0$, *and the weak solution is classical.*

Proof. By Proposition 2.4, $\Gamma(t)\cap\overline{\Omega}_0=\emptyset$ for $t>0$. Hence we may apply Corollary 3.16 to conclude. □

5. The spreading-vanishing dichotomy

In this section, we study the asymptotic behavior of $\Gamma(t)$ and $u(t,x)$ as $t\to\infty$. We always assume that (4.1) holds. We will also need a further restriction on g:

$$g(x,u)=g(u)\leq 0 \text{ for all } u\geq M>0. \tag{5.1}$$

In Section 2, we have proven that $\Omega(t)$ is expanding in t; thus we can define the limit

$$\Omega_\infty=\bigcup_{t>0}\Omega(t).$$

5.1. Dichotomy for Ω_∞.

In this subsection we prove the following dichotomy.

Theorem 5.1. *Suppose that* (5.1) *hold and* $g\in C^{1,\alpha}([0,\delta_0])$ *for some small* $\delta_0>0$. *Then either* $\Omega_\infty=\mathbb{R}^n$ *or it is a bounded set. Moreover, if* $\Omega_\infty=\mathbb{R}^n$, *then for all large* t, $\Gamma(t)$ *is a smooth closed hypersurface, and there exists an increasing function* $M(t)$ *such that*

$$\Gamma(t)\subset\{x\in\mathbb{R}^n : M(t)-\frac{d_0}{2}\pi\leq|x|\leq M(t)\};$$

if Ω_∞ *is bounded, then* $u(t,x)\to 0$ *uniformly in* x *as* $t\to\infty$. *Here* d_0 *denotes the diameter of* Ω_0.

It is natural to ask: When $\Omega_\infty = \mathbb{R}^n$, what is the asymptotic behavior of u as $t \to \infty$? Without further restrictions on g, this cannot be answered. When g takes the logistic nonlinearity, this question is answered in the next subsection. In one space dimension with bistable or combustion nonlinearities, it is shown in [10] that the limit of u is usually the stable positive steady-state except in the transition case, where the limit is a ground state (for the bistable case) or the ignition constant (for combustion nonlinearity).

Theorem 5.1 is a consequence of some stronger results below. The proofs are based on the following simple geometric result, which is an analogue of Theorem 2 in [20] but we do not have the restriction that $n \geq 3$.

Theorem 5.2. *Suppose that* (4.1) *holds and* $g \in C^{1,\alpha}([0,\delta_0])$. *Then at any point* $x_0 \in \Gamma(t) \setminus \overline{\text{co}}(\Omega_0)$, *the inward normal line to* $\Gamma(t)$ *at* x_0 *intersects* $\overline{\text{co}}(\Omega_0)$.

Proof. Fix $t > 0$ and $x_0 \in \Gamma(t) \setminus \overline{\text{co}}(\Omega_0)$. Then choose $r > 0$ small so that $\overline{B_r(x_0)} \cap \overline{\text{co}}(\Omega_0) = \emptyset$. To simplify notations, we will write $W_0 = \overline{\text{co}}(\Omega_0)$.

Since u is smooth in $\overline{\Omega}(t) \cap B_r(x_0)$, we can use the Hopf boundary lemma to conclude that $|\nabla_x u(t,x)| \neq 0$ on $\Gamma(t) \cap B_r(x_0)$, where $\nabla_x u(t,x)|_{\Gamma(t)\cap B_r(x_0)}$ is understood as its limit when $x \in \Omega(t)$ goes to $\Gamma(t) \cap B_r(x_0)$. It follows that for all small $\varepsilon > 0$, the level set

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n : u(t,x) = \varepsilon\}$$

is close to $\Gamma(t)$ in $B_r(x_0)$, and $\Gamma_\varepsilon \cap B_r(x_0)$ is a smooth hypersurface. We will show that any ray inward normal to $\Gamma_\varepsilon \cap B_r(x_0)$ intersects W_0. The conclusion of the first part of the theorem then follows by letting $\varepsilon \to 0$ because u is $C^{2,\alpha}$ up to $\Gamma(t) \cap B_r(x_0)$ and $|\nabla_x u(t,x)| \neq 0$ on $\Gamma(t) \cap B_r(x_0)$.

Let x_1 be any point on Γ_ε and l the ray inward normal to Γ_ε at x_1. Assuming that

$$l \cap W_0 = \emptyset, \tag{5.2}$$

we will derive a contradiction.

By the definition of S_{x_1} we easily see that (5.2) implies the existence of a $\nu \in S_{x_1}$ satisfying $\nu \perp l$. By Lemma 4.2, we have $\partial_\nu u(t,x_1) < 0$. On the other hand, since ν is tangent to the level surface Γ_ε of u, we must have $\partial_\nu u(t,x_1) = 0$. This contradiction completes the proof. □

Let x_* be any point in Ω_0 and define

$$m(t) = \min_{x\in\Gamma(t)\setminus\overline{\text{co}}(\Omega_0)} |x - x_*|,\ M(t) = \max_{x\in\Gamma(t)} |x - x_*| = \max_{x\in\overline{\Omega}(t)} |x - x_*|.$$

Theorem 5.3. *Suppose that* (4.1) *holds,* $g \in C^{1,\alpha}([0,\delta_0])$, $B_{R_0}(x_*) \supset \overline{\text{co}}(\Omega_0)$, *and there exists* $t_0 > 0$ *such that* $M(t_0) > (\pi+1)R_0$. *Then for* $t \geq t_0$, $\tilde{\Gamma}(t) := \Gamma(t) \setminus \overline{\text{co}}(\Omega_0)$ *is a* $C^{2,\alpha}$ *closed hypersurface in* $\mathbb{R}^n$, *with* $m(t) > M(t) - \pi R_0$. *Thus*

$$\tilde{\Gamma}(t) \subset \{x \in \mathbb{R}^n :\ M(t) - \pi R_0 < |x - x_*| \leq M(t)\},\ \forall t \geq t_0.$$

Proof. Without loss of generality, assume $x_* = 0$ is the origin. Fix $t \geq t_0$ and let $x_0 \in \Gamma(t)$ satisfy $|x_0| = M(t)$. Since $\Omega(t)$ is expanding, $M(t) \geq M(t_0)$.

We claim that $\tilde{\Gamma}(t)$ is a closed hypersurface in $\mathbb{R}^n$ and $\tilde{\Gamma}(t) \cap \overline{B}_R(0) = \emptyset$, with $R = M(t) - \pi R_0$. Clearly the conclusions of the theorem will follow from this claim.

Let Π_0 be an arbitrary two dimensional hyperplane in $\mathbb{R}^n$ that passes through the origin and x_0. We may rotate the coordinate system so that Π_0 is the x_1x_2-plane with x_0 having coordinates $(M(t),0)$ on Π_0. In view of Theorem 5.2, $\Pi_0 \cap \tilde{\Gamma}(t)$ contains a curve l_0 with

$x_0 \in \ell_0$, and at each point on l_0 the normal line intersects the disc $\{\rho < R_0\}$ on Π_0, where the polar coordinates (ρ, θ) on Π_0 are used. This implies that l_0 can be expressed as

$$\rho = r(\theta),\ \theta^- \le \theta \le \theta^+,$$

with $-\pi \le \theta^- < 0 < \theta^+ \le \pi$. The normal line property implies that $R_0 > r'(\theta) > -R_0$ for all $\theta \in (\theta^-, \theta^+)$. We may assume that ℓ_0 is the maximal connected component of $\Pi_0 \cap \tilde{\Gamma}(t)$ that contains x_0.

We thus obtain, for any $P_0 = (r(\theta_0), \theta_0) \in \ell_0$,

$$|P_0| = r(\theta_0) = M(t) + \int_0^{\theta_0} r'(\theta) d\theta > M(t) - R_0|\theta_0| \ge M(t_0) - \pi R_0.$$

Since $M(t_0) - \pi R_0 > R_0$, clearly $B_\epsilon(P_0) \cap \overline{\text{co}}(\Omega_0) = \emptyset$, where $\epsilon = M(t_0) - (\pi + 1)R_0$. Hence $B_\epsilon(P_0) \cap \Pi_0 \cap \tilde{\Gamma}(t)$ is a $C^{2,\alpha}$ curve, which necessarily forms part of ℓ_0. This implies that $\theta^- = -\pi$, $\theta^+ = \pi$ and ℓ_0 is a closed curve in Π_0, and $\ell_0 \cap \overline{B}_R(0) = \emptyset$.

Since Π_0 is arbitrary, the above conclusion implies that $\tilde{\Gamma}(t)$ is a closed hypersurface in $\mathbb{R}^n$, with $\tilde{\Gamma}(t) \cap \overline{B}_R(0) = \emptyset$, as we claimed. □

5.1.1. **[Ω_∞ unbounded implies $\Omega_\infty = \mathbb{R}^n$].** Now we come to the proof of Theorem 5.1 for the case that Ω_∞ is unbounded. In such a case, we necessarily have

$$\lim_{t \to +\infty} M(t) = +\infty. \tag{5.3}$$

By Theorem 5.3, this implies

$$\lim_{t \to +\infty} m(t) = +\infty,$$

and hence $\Gamma(t) \setminus \overline{\text{co}}(\Omega_0)$ goes to infinity in every direction. However, this says nothing about the part $\Gamma(t) \cap \overline{\text{co}}(\Omega_0)$, which is nonempty for small $t > 0$ if Ω_0 is not convex.

If Ω_0 is convex, this set is empty and the proof of Theorem 5.1 for unbounded Ω_∞ is thus complete. The case that Ω_∞ is unbounded and that Ω_0 is not convex is covered in the following theorem.

Theorem 5.4. *Suppose that* (5.1) *holds and* $g \in C^{1,\alpha}([0, \delta_0])$*. If* Ω_∞ *is unbounded and not convex, then there is a* $T_0 > 0$*, such that for all* $t \ge T_0$*,*

$$\overline{\text{co}}(\Omega_0) \subset \Omega(t).$$

Proof. Without loss of generality, we assume $0 \in \Omega_0$. Suppose by way of contradiction that the conclusion of the theorem is false. Then we can define

$$\rho(t) := \max_{x \in \overline{\text{co}}(\Omega_0) \setminus \Omega(t)} |x|, \qquad \forall t > 0.$$

Since $\Omega(t)$ is expanding as t increases, $\rho(t)$ is non-increasing for $t \in (0, \infty)$. Take $R > 0$ such that $\overline{\text{co}}(\Omega_0) \subset B_R = B_R(0)$. By Theorem 5.3, there is a $T > 0$, such that for all $t \ge T$,

$$\overline{B_{5R}} \setminus \overline{\text{co}}(\Omega_0) \subset \Omega(t),$$

and hence $u > 0$ on $[T, +\infty) \times [\overline{B_{5R}} \setminus \overline{B_{\rho(t)}}]$, and $\rho(t) < R$ for $t > 0$.

Since $0 \in \Omega_0$ and Ω_0 is open, there exists $r_0 > 0$ such that $B_{r_0} \subset \Omega_0$. Thus $B_{r_0} \subset \Omega(t)$ and $\rho(t) \ge r_0$ for all $t > 0$. It follows that

$$\rho_\infty := \lim_{t \to \infty} \rho(t) \in [r_0, R). \tag{5.4}$$

Because $0 \le u(t,x) \le M$, we can write $g(u) = c(t,x)u$ with $\|c\|_{L^\infty} \le C_0$. Since $B_{5R} \setminus \overline{\text{co}}(\Omega_0) \subset \Omega(t)$ for $t \ge T$, by the Harnack inequality (see, e.g., Theorem 6.27 in [19]), we can find a constant C such that, for any $t \ge T+3$,

$$\sigma(t) := \frac{1}{2} \inf_{B_{3R}\setminus B_{2R}} u(t,\cdot) \ge C \sup_{[t-5/2,t-1]\times(B_{4R}\setminus B_R)} u. \tag{5.5}$$

The arguments below are divided into four steps.

Step 1. There exists $C > 0$ such that

$$\sigma(t) \ge C \sup_{y\in\tilde{\Gamma}(\tau)} |\nabla u(\tau,y)| \ \ \forall \tau \in [t-2,t-1],\ \forall t \ge T+3. \tag{5.6}$$

The normal line property of $\tilde{\Gamma}(t)$ in Theorem 5.2 implies that $\tilde{\Gamma}(t)$ is uniformly Lipschitz continuous for all $t \ge T$. The proof of the regularity of $\partial\{u > 0\}$ indicates that the $C^{2,\alpha}$-norm of the local expression of the free boundary given in Theorem 3.15, $y_1 = f(s, y_2, ..., y_n)$, is determined by the modulus of Lipschitz continuity of $\tilde{\Gamma}(t)$ and the L^∞ bound of u. Therefore, near each point $(t_0, x_0) \in \tilde{\Gamma}_T := \{(t,x) : x \in \tilde{\Gamma}(t), t \ge T\}$, after a suitable rotation of the x-coordinate system, $\tilde{\Gamma}_T$ can be expressed as $x_1 = f(t, x_2, ..., x_n)$, with $f \in C^{2,\alpha}$ and its $C^{2,\alpha}$-norm bounded by a constant independent of (t_0, x_0). Therefore there exists $r \in (0, R/2)$ and $\eta > 1$ such that for any $t_0 > T+1$ and $x_0 \in \tilde{\Gamma}(t_0)$, one can find a parabolic half ball

$$\mathcal{B}_r = \{(t,x) : |x-y_0|^2 + \eta(t_0-t) < r^2, t < t_0\}$$

that touches $\{u > 0\}$ at (t_0, x_0) from outside:

$$\mathcal{B}_r \cap \overline{\{u > 0\}} = \emptyset, \quad \overline{\mathcal{B}}_r \cap \tilde{\Gamma}_T = \{(t_0, x_0)\}.$$

We now define

$$\mathcal{A}_r := \{(t,x) \in \mathcal{B}_{2r} \setminus \overline{\mathcal{B}_r} : \eta(t_0-t) < r^2/2\}.$$

Clearly $\partial_p \mathcal{A}_r = S_r^1 \cup S_r^2 \cup S_r^3$ with

$$S_r^1 = \{(t,x) \in \partial_p \mathcal{B}_r : \eta(t_0-t) < r^2/2\},$$
$$S_r^2 = \{(t,x) \in \partial_p \mathcal{B}_{2r} : \eta(t_0-t) < r^2/2\},$$
$$S_r^3 = \{(t,x) : r^2/2 \le |x-y_0|^2 \le 3r^2/2,\ \eta(t_0-t) = r^2/2\}.$$

For $\beta > 0$ to be determined, we define

$$v(t,x) = e^{\beta\rho^2} - e^{\beta r^2} \text{ with } \rho^2 = |x-y_0|^2 + \eta(t_0-t).$$

A direct calculation gives

$$v_t - d\Delta v = \beta\big[-\eta + d(4\beta|x-y_0|^2 + 2N)\big]e^{\beta\rho^2}.$$

In $\mathcal{A}_r$, $|x-y_0|^2 = \rho^2 - \eta(t_0-t) \ge r^2/2$. Hence

$$v_t - d\Delta v \ge \beta\big[-\eta + 2d\beta r^2 + 2Nd\big]e^{\beta\rho^2} \ge k\, v \text{ in } \mathcal{A}_r,$$

provided that β is chosen large enough. Here $k > 0$ is chosen such that $g(\xi) \le k\xi$ for all $\xi \ge 0$.

Clearly

$$\Big[(S_r^2 \cup S_r^3) \cap \overline{\{u > 0\}}\Big] \cap \overline{\mathcal{B}_r} = (S_r^2 \cup S_r^3) \cap \{(t_0, x_0)\} = \emptyset.$$

Therefore we can find $\epsilon_0 > 0$ depending only on r and η such that

$$|x-y_0|^2 + \eta(t_0-t) \ge (1+\epsilon_0)r^2 \text{ for } (t,x) \in (S_r^2 \cup S_r^3) \cap \{u > 0\}.$$

It follows that

$$v \geq m_0 := e^{\beta(1+\epsilon_0)r^2} - e^{\beta r^2} \text{ in } (S_r^2 \cup S_r^3) \cap \{u > 0\}.$$

We may write

$$\partial_p(\mathcal{A}_r \cap \{u > 0\}) = \tilde{S}_r^1 \cup \tilde{S}_r^2$$

with

$$\tilde{S}_r^1 = \mathcal{A}_r \cap \partial\{u > 0\},\ \tilde{S}_r^2 = (S_r^2 \cup S_r^3) \cap \{u > 0\}. \qquad [\text{ Recall } S_r^1 \cap \{u > 0\} = \emptyset.]$$

Denote $M_0 = \sup_{\tilde{S}_r^2} u$, and define $v_0 = \frac{M_0}{m_0} v$. Then

$$u = 0 \leq v_0 \text{ on } \tilde{S}_r^1,\ u \leq M_0 \leq v_0 \text{ on } \tilde{S}_r^2,$$

and

$$(v_0)_t - d\Delta v_0 \geq k\, v_0 \geq g(v_0) \text{ in } \mathcal{A}_r \cap \{u > 0\}.$$

Therefore we can apply the maximum principle to conclude that $v_0 \geq u$ in $\mathcal{A}_r \cap \{u > 0\}$. It follows that, with $\nu_0 = (x_0 - y_0)/|x_0 - y_0|$,

$$\partial_{\nu_0} u(t_0, x_0) \leq \partial_{\nu_0} v_0(t_0, x_0) = \frac{M_0}{m_0} 2\beta r e^{\beta r^2} = CM_0,$$

with C independent of (t_0, x_0). Since the sphere $\{|x - y_0| = r\}$ is tangent to $\tilde{\Gamma}(t_0)$ at x_0, we have $\partial_{\nu_0} u(t_0, x_0) = |\nabla u(t_0, x_0)|$. Therefore

$$|\nabla u(t_0, x_0)| \leq CM_0.$$

Here and in what follows, we will use C to denote a generic positive constant which does not depend on t or t_0, but its value may change from place to place.

If we denote

$$\tilde{\Gamma}_\delta(t) = \{x \in \Omega(t) : \text{dist}(x, \tilde{\Gamma}(t)) \leq \delta\},\quad \forall t \geq T,$$

then by shrinking r if necessary we can guarantee

$$\tilde{S}_r^2 \subset N_r(t_0) := \{(t, x) : x \in \tilde{\Gamma}_{2r}(t), t_0 - 1/2 \leq t \leq t_0\}.$$

By Lemma 4.2, for all $t \geq T$,

$$\sup_{B_{3R} \setminus B_{2R}} u(t, \cdot) \geq \sup_{\tilde{\Gamma}_{2r}(t)} u(t, \cdot).$$

Therefore

$$\sup_{[t_0 - 1/2, t_0] \times (B_{3R} \setminus B_{2R})} u \geq \sup_{N_r(t_0)} u \geq M_0 \geq C|\nabla u(t_0, x_0)|.$$

Since $x_0 \in \tilde{\Gamma}(t_0)$ is arbitrary, this implies that

$$\sup_{[t_0 - 1/2, t_0] \times (B_{3R} \setminus B_{2R})} u \geq C \sup_{y \in \tilde{\Gamma}(t_0)} |\nabla u(t_0, y)|.$$

Taking $t_0 \in [t - 2, t - 1]$ and using (5.5), we obtain (5.6).

Step 2. An upper bound for $\sigma(t)$.

For any fixed $s \geq T + 3$, we choose a smooth function v^s over $\overline{B}_{5R} \setminus B_{\rho(s)}$ with the following properties:

(i) v^s is radially symmetric,
(ii) $v^s \equiv \sigma(s) = \frac{1}{2} \inf_{x \in B_{3R} \setminus B_{2R}} u(s, x)$ in $B_{3R} \setminus B_{2R}$,
(iii) $0 < v^s \leq u(s, \cdot)$ in $B_{5R} \setminus \overline{B}_{\rho(s)}$,
(iv) $v^s = 0$ on $\partial B_{5R} \cup \partial B_{\rho(s)}$.

Since g is locally Lipschitz, there exists $k > 0$ such that $g(u) \geq -ku$ in $[0, M]$. We now consider the problem

$$(5.7)\qquad \begin{cases} v_t - d\Delta v = -kv, & t > s,\ \ h(t) < r < 5R, \\ v(t, 5R) = 0, \quad v(t, h(t)) = 0, & t > s, \\ h'(t) = -\mu v_r(t, h(t)), & t > s, \\ h(s) = \rho(s), \ \ v(s, r) = v^s(r), & \rho(s) \leq r \leq 5R. \end{cases}$$

Similar to Theorem 2.1 in [7], we know that (5.7) has a unique classical solution pair (v, h) defined on some maximal time interval $[s, s + T_1)$, with $T_1 \in (0, \infty]$, and the Hopf boundary lemma guarantees that $h'(t) < 0$ for all $t \in (s, s + T_1)$. By Theorem 4.3 of [8], we find that $v \leq u$ in $\{(t, x) : t \in (s, s + T_1), h(t) < |x| < 5R\}$, and

$$(5.8)\qquad \rho_\infty \leq \rho(t) \leq h(t) \text{ for all } t \in (s, s + T_1).$$

This implies that $T_1 = \infty$.

Let (λ_1, ϕ_1) denote the first eigenpair of

$$-\Delta\phi = \lambda\phi \text{ in } B_{3R} \setminus \overline{B_{2R}},\ \ \phi = 0 \text{ on } \partial(B_{3R} \setminus \overline{B_{2R}}),$$

with $\phi_1 > 0$ in $B_{3R} \setminus \overline{B_{2R}}$ and $\|\phi_1\|_\infty = 1$. Set

$$v_*(t, x) = \sigma(s) e^{-(d\lambda_1 + k)(t - s)} \phi_1(x).$$

We have

$$\partial_t v_* - d\Delta v_* = -kv_* \text{ for } t \geq s,\ x \in B_{3R} \setminus \overline{B_{2R}},$$
$$v_* = 0 < v \text{ for } t \geq s,\ x \in \partial(B_{3R} \setminus \overline{B_{2R}}),$$

and

$$v_*(s, x) = \sigma(s)\phi_1 \leq \sigma(s) = v^s(|x|) = v(s, |x|) \text{ for } x \in B_{3R} \setminus \overline{B_{2R}}.$$

Therefore we can apply the standard comparison principle to conclude that

$$v(t, |x|) \geq v_*(t, x) = \sigma(s) e^{-(d\lambda_1 + k)(t - s)} \phi_1(x) \text{ for } t > s \text{ and } x \in B_{3R} \setminus \overline{B_{2R}}.$$

In particular,

$$(5.9)\qquad v(t, |x|) \geq C\sigma(s) \text{ for } |x| = \frac{5}{2}R,\ t \in [s + 1/4, s + 1],$$

with C independent of s.

Step 3. Completion of the proof under an extra assumption.

We claim that

$$(5.10)\qquad v_r(t, h(t)) \geq Cv(t - 1/6, 5R/2) \text{ for } t \geq s + 1/4.$$

Assuming (5.10) we now continue with the proof. In view of (5.9), it follows from (5.10) that

$$\begin{aligned} h'(t) &= -\mu v_r(t, h(t)) \\ &\leq -Cv(t - 1/6, 5R/2) \\ &\leq -C\sigma(s) \text{ for } t \in [s + \frac{5}{12}, s + \frac{7}{6}]. \end{aligned}$$

Recalling $h(s) = \rho(s),\ h(s + 1) \geq \rho(s + 1)$ and $h'(t) < 0$, we obtain

$$\rho(s + 1) - \rho(s) \leq h(s + 1) - h(s) = \int_s^{s+1} h'(t)dt \leq \int_{s+1/2}^{s+1} h'(t)dt.$$

Hence
$$\rho(s+1)-\rho(s)\leq -C\sigma(s).$$
Making use of (5.6), we obtain
$$\rho(s+1)-\rho(s)\leq -C\sup_{\tau\in[s-2,s-1]}\Big(\sup_{y\in\tilde\Gamma(\tau)}|\nabla u(\tau,y)|\Big)\leq -C\int_{s-2}^{s-1}\sup_{y\in\tilde\Gamma(\tau)}|\nabla u(\tau,y)|d\tau.$$
Using the above inequality for $s_0=T+3, s_{j+1}=s_j+1$ successively, we obtain
$$\rho(s_{j+1})-\rho(s_j)\leq -C\int_{s_{j-2}}^{s_{j-1}}\sup_{y\in\tilde\Gamma(\tau)}|\nabla u(\tau,y)|d\tau,\ j=2,3,...,$$
and hence
$$\rho_\infty-\rho(s_2)\leq -C\int_{s_0}^{\infty}\sup_{y\in\tilde\Gamma(\tau)}|\nabla u(\tau,y)|d\tau,$$
which yields
$$\int_{T+3}^{\infty}\sup_{y\in\tilde\Gamma(\tau)}|\nabla u(\tau,y)|d\tau<\infty.$$
We show next that this leads to a contradiction.

Fix a unit vector $\nu\in\mathbb{R}^n$, and define $r(t)>0$ by
$$r(t)\nu\in\tilde\Gamma(t),\quad t\geq T.$$
Then $u(t,r(t)\nu)\equiv 0$ and it follows that
$$u_t(t,r(t)\nu)+r'(t)\nabla u(t,r(t)\nu)\cdot\nu\equiv 0.$$
By the free boundary condition,
$$u_t(t,r(t)\nu)=\mu|\nabla u(t,r(t)\nu)|^2.$$
Hence
$$r'(t)=-\mu\frac{|\nabla u(t,r(t)\nu)|^2}{\nabla u(t,r(t)\nu)\cdot\nu}.$$
Since the inward normal line of $\tilde\Gamma(t)$ at $r(t)\nu$ intersects $\overline{\mathrm{co}}(\Omega_0)\subset B_R$, and for $t\geq T$, $r(t)\geq 5R$, there exists a positive constant c_0 such that
$$-\nabla u(t,r(t)\nu)\cdot\nu\geq c_0|\nabla u(t,r(t)\nu)|\ \ \forall t\geq T.$$
It follows that
$$r'(t)\leq\frac{\mu}{c_0}|\nabla u(t,r(t)\nu)|\leq\frac{\mu}{c_0}\sup_{y\in\tilde\Gamma(t)}|\nabla u(t,y)|\ \ \forall t\geq T.$$
Therefore
$$\begin{aligned}\lim_{t\to\infty}r(t)&=r(T+3)+\int_{T+3}^{\infty}r'(t)dt\\&\leq r(T+3)+\frac{\mu}{c_0}\int_{T+3}^{\infty}\sup_{y\in\tilde\Gamma(t)}|\nabla u(t,y)|dt<\infty,\end{aligned}$$
a contradiction to the fact that $r(t)\geq m(t)\to\infty$ as $t\to\infty$. This finishes the proof under the assumption of (5.10).

Step 4. Proof of (5.10).

Since $h(t)\geq\rho_\infty>0$, the sphere $\{|x|=h(t)\}$ is uniformly smooth for all $t>s$. Thus as in Step 1, one can obtain a uniform bound of the $C^{2,\alpha}$-norm of the free boundary $\{(t,x):|x|=h(t),t\geq s+\epsilon\}$ for any $\epsilon>0$. In particular, $h'(t)$ is uniformly bounded for

$t \geq s + 1/4$. Hence for each $t_0 \geq s + 1/4$ and $x_0 \in \partial B_{\tilde{h}(t_0)}$, we can construct a parabolic half ball

$$\mathcal{B}_r := \{(t,x) : |x - y_0|^2 + \eta(t_0 - t) < r^2,\ t < t_0\}$$

such that

$$\mathcal{B}_r \subset \{v > 0\},\ \overline{\mathcal{B}}_r \cap \partial\{v > 0\} = \{(t_0, x_0)\}.$$

Moreover, $r \in (0, R/2)$ and $\eta > 1$ can be chosen independently of such (t_0, x_0).

Define

$$\tilde{\mathcal{A}}_r := \{(t,x) \in \mathcal{B}_r : |x - y_0|^2 > r^2/2\}.$$

Then $\partial_p \tilde{\mathcal{A}}_r = \Sigma_r^1 \cup \Sigma_r^2$, with

$$\Sigma_r^1 = \{(t,x) \in \partial_p \mathcal{B}_r : \eta(t_0 - t) < r^2/2\},$$
$$\Sigma_r^2 = \{(t,x) : |x - y_0|^2 = r^2/2, 0 < \eta(t_0 - t) \leq r^2/2\}.$$

For $\beta > 0$ to be determined, we define

$$z(t,x) = e^{-\beta\rho^2} - e^{-\beta r^2} \text{ with } \rho^2 = |x - y_0|^2 + \eta(t_0 - t).$$

Then

$$\begin{aligned} z_t - d\Delta z &= \beta[\eta - d(4\beta|x - y_0|^2 + 2N)]e^{-\beta\rho^2} \\ &\leq \beta(\eta - 2d\beta r^2 - 2dN)e^{-\beta\rho^2} \\ &\leq -kz \qquad \text{for } (t,x) \in \tilde{A}_r, \end{aligned}$$

provided that β is chosen large enough.

Clearly $z = 0$ on Σ_r^1 and

$$z \leq \tilde{M}_0 := e^{-\beta r^2/2} - e^{-\beta r^2} \text{ on } \Sigma_r^2.$$

We may assume that $\eta > 4r^2$ with r sufficiently small. Then there exists $\delta > 0$ independent of (t_0, x_0) such that $0 < \delta \leq r$ and

$$\Sigma_r^2 \subset \tilde{N}(t_0) := \{(t,x) : \delta + h(t) \leq |x| \leq 4R,\ 0 < t_0 - t \leq 1/8\}.$$

Set

$$\tilde{m}_0 := \inf_{\tilde{N}(t_0)} v,\ z_0 = \frac{\tilde{m}_0}{\tilde{M}_0} z.$$

Then

$$z_0 = 0 \leq v \text{ on } \Sigma_r^1,\ z_0 \leq \tilde{m}_0 \leq v \text{ on } \Sigma_r^2$$

and

$$(z_0)_t - d\Delta z_0 \leq -kz \text{ in } \tilde{\mathcal{A}}_r.$$

It follows from the comparison principle that $z_0 \leq \tilde{v}$ in $\tilde{\mathcal{A}}_r$. Hence, with $\nu = \frac{y_0 - x_0}{|y_0 - x_0|}$, we have

$$\partial_\nu v(t_0, |x_0|) \geq \partial_\nu z_0(t_0, x_0) = 2\beta r e^{-\beta r^2} \frac{\tilde{m}_0}{\tilde{M}_0} = C\tilde{m}_0,$$

that is, $v_r(t_0, h(t_0)) \geq C\tilde{m}_0$.

Since $\lim_{t\to\infty} h(t) =: h_\infty \in [\rho_\infty, R)$, by enlarging T if necessary, we may assume, without loss of generality, that

$$h(t) \in (h_\infty, h_\infty + \frac{\delta}{4})\ \ \forall t \geq T.$$

Hence

$$\tilde{N}(t_0) \subset [t_0, t_0 + \frac{1}{8}] \times N_R \subset \left\{(t,x) : \frac{\delta}{4} + h(t) \leq |x| \leq 4R,\ t \in [t_0, t_0 + \frac{1}{8}]\right\},$$

where $N_R = \{x : h_\infty + \frac{\delta}{2} \le |x| \le 4R\}$.

By Harnack's inequality,

$$\tilde{m}_0 = \inf_{\tilde{N}(t_0)} v \ge \inf_{[t_0, t_0+\frac{1}{8}]\times N_R} v \ge Cv(t_0 - \frac{1}{6}, \frac{5}{2}R).$$

Therefore

$$v_r(t_0, h(t_0)) \ge Cv(t_0 - 1/6, 5R/2) \text{ for } t_0 \ge s + 1/4,$$

which is (5.10) with $t = t_0$. The proof of the theorem is now complete. □

5.1.2. **[Ω_∞ bounded implies $u \to 0$].** The following result shows that vanishing happens when Ω_∞ is bounded.

Theorem 5.5. *If Ω_∞ is bounded and* (5.1) *holds, then as $t \to +\infty$, $u(t,x)$ converges to* 0 *uniformly.*

We will prove this theorem by the following three lemmas. Note that when Ω_∞ is bounded, then in the approximate problem (4.2), we can choose any $B_R(0) \supset \overline{\Omega}_\infty$ and it works for all $T > 0$. Moreover, if we extend $u(t,x)$ by 0 outside $\Omega(t)$, then it satisfies (2.14) for all $B_R \supset \Omega_\infty$.

In the discussions below, we always assume that the conditions of Theorem 5.5 are satisfied. We first prove an energy inequality. Let u be the weak solution of (2.1). Define

$$E(u)(t) := \int_{\Omega(t)} \Big\{ -\frac{1}{2}\big[u_t - g(u)\big]u - G(u)\Big\}dx, \quad \text{with } G(u) = \int_0^u g(t)dt.$$

Lemma 5.6. *For $0 < T_1 < T_2 < +\infty$, we have*

$$E(u)(T_2) - E(u)(T_1) \le -\int_{T_1}^{T_2}\int_{\Omega(t)} |\partial_t u(t,x)|^2 dxdt, \tag{5.11}$$

where u is the unique weak solution of (2.1)*. Moreover, there exists $C_0 > 0$ such that*

$$E(u)(t) \ge -C_0 \ \ \forall t > 0.$$

Proof. Choose any $B_R \supset \overline{\Omega}_\infty$ and let u_m be the solution of (4.2). Define

$$E(u_m)(t) := \int_{B_R} \Big[\frac{d}{2}|\nabla u_m(t,x)|^2 - G(u_m(t,x))\Big]dx.$$

From (4.2) we can calculate directly to get, for $t_2 > t_1 + \delta$,

$$\begin{aligned} E(u_m)(t_2) - E(u_m)(t_1) &= \int_{t_1}^{t_2}\int_{B_R} -\alpha_m'(u_m)|\partial_t u_m|^2 dxdt \\ &\le \int_{t_1}^{t_2}\int_{B_R} -|\partial_t u_m|^2 dxdt. \end{aligned}$$

Here we have used the fact $\alpha_m' \ge 1$. Integrating the above inequality for t_1 over $[T_1, T_1+\delta]$ and then for t_2 over $[T_2, T_2+\delta]$, we obtain

$$\int_{T_2}^{T_2+\delta} E(u_m)(t)dt - \int_{T_1}^{T_1+\delta} E(u_m)(t)dt \le -\delta \int_{T_1+\delta}^{T_2}\int_{B_R} |\partial_t u_m|^2 dxdt. \tag{5.12}$$

A simple comparison consideration shows that $0 \le u_m \le C := \max\{M, \|u_0\|_\infty\}$. Since $u_m \to u$ weakly in $H^1((0,T)\times B_R)$ and strongly in $L^2((0,T)\times B_R)$ for any $T>0$, and since $u_m \le C$ for all m, we have

$$\liminf_{m\to\infty}\int_{T_1+\delta}^{T_2}\int_{B_R}|\partial_t u_m|^2 dxdt \ge \int_{T_1+\delta}^{T_2}\int_{B_R}|\partial_t u|^2 dxdt$$

and

$$\lim_{m\to\infty}\int_{T_1+\delta}^{T_2}\int_{B_R} G(u_m)dxdt = \int_{T_1+\delta}^{T_2}\int_{B_R} G(u)dxdt.$$

We show next that for $T>0$,

$$E_\delta(T) := \lim_{m\to\infty}\int_T^{T+\delta} E(u_m)(t)dt \text{ exists.}$$

Define, for $\xi \ge 0$,

$$A_m(\xi) = \int_0^\xi \alpha_m(s)ds,\ A(\xi) = \int_0^\xi \alpha(s)ds = \xi^2/2,$$

and

$$B_m(\xi) = \alpha_m(\xi)\xi,\ B(\xi) = \alpha(\xi)\xi = \xi^2.$$

From the definitions of α_m and α, we easily see that

$$A_m \to A \text{ and } B_m \to B \text{ uniformly over any bounded subset of } [0,\infty).$$

We now multiply (4.2) by u_m and integrate over $[T, T+\delta]\times B_R$ for an arbitrary $T>0$, and use integration by parts. It results

$$\int_T^{T+\delta}\int_{B_R}\left(\partial_t[\alpha_m(u_m)]u_m + d|\nabla u_m|^2\right)dxdt = \int_T^{T+\delta}\int_{B_R} g(u_m)u_m dxdt. \tag{5.13}$$

Since u_m is uniformly bounded and $u_m \to u$ in L^2, we have

$$\lim_{m\to\infty}\int_T^{T+\delta}\int_{B_R} g(u_m)u_m dxdt = \int_T^{T+\delta}\int_{B_R} g(u)u dxdt.$$

Since

$$\begin{aligned}\int_T^{T+\delta}\partial_t[\alpha_m(u_m)]u_m dt =& B_m(u_m(T+\delta,x)) - B_m(u_m(T,x))\\ &- A_m(u_m(T+\delta,x)) + A_m(u_m(T,x)),\end{aligned}$$

we find that

$$\begin{aligned}&\lim_{m\to\infty}\int_T^{T+\delta}\int_{B_R}\partial_t[\alpha_m(u_m)]u_m dxdt\\ &= \int_{B_R}\left[B(u(T+\delta,x)) - B(u(T,x)) - A(u(T+\delta,x)) + A(u(T,x))\right]dx\\ &= \int_{B_R}\frac{1}{2}\Big[u^2(T+\delta,x) - u^2(T,x)\Big]dx.\end{aligned}$$

Here we have used the fact that $u_m(t,\cdot) \to u(t,\cdot)$ a.e. in B_R, which is guaranteed by the strong convergence of $u_m \to u$ in $L^2((0,T)\times B_R)$ and the fact that $u_m(t,\cdot)$ are uniformly bounded in $H^1(B_R)$ (see the energy inequality of u_m in Lemma 3.3 of [8]).

It now follows from (5.13) that

$$\begin{aligned}&\lim_{m\to\infty}\int_T^{T+\delta}\int_{B_R} d|\nabla u_m|^2dxdt\\ &=-\int_{B_R}\Big\{\frac{1}{2}\big[u^2(T+\delta,x)-u^2(T,x)\big]-\int_T^{T+\delta}g(u)udt\Big\}dx.\end{aligned}\tag{5.14}$$

Therefore

$$\begin{aligned}E_\delta(T):=&\lim_{m\to\infty}\int_T^{T+\delta}E(u_m)(t)dt\\ =&-\frac{1}{2}\int_{B_R}\Big\{\frac{1}{2}\big[u^2(T+\delta,x)-u^2(T,x))\big]\\ &-\int_T^{T+\delta}\big[g(u(t,x))u(t,x)-2G(u(t,x))\big]dt\Big\}dx,\end{aligned}\tag{5.15}$$

and for $T>0$,

$$\lim_{\delta\to 0}\delta^{-1}E_\delta(T)=E(T),$$

with

$$E(T):=\int_{B_R}\Big\{-\frac{1}{2}\big[u_t(T,x)-g(u(T,x))\big]u(T,x)-G(u(T,x))\Big\}dx.$$

Since $u(t,\cdot)=0$ in $B_R\setminus\Omega(t)$, clearly $E(T)=E(u)(T)$. Letting $m\to\infty$ in (5.12), we obtain

$$E_\delta(T_2)-E_\delta(T_1)\le-\delta\int_{T_1+\delta}^{T_2}\int_{B_R}|\partial_t u|^2dxdt.\tag{5.16}$$

Diving this inequality by δ and letting $\delta\to 0$, we obtain (5.11).

Since we have a uniform bound for all u_m, for arbitrary $T>0$ and $\delta>0$,

$$\begin{aligned}0&\le\lim_{m\to\infty}\int_T^{T+\delta}\int_{B_R}\frac{d}{2}|\nabla u_m|^2dxdt\\ &=\lim_{m\to\infty}\int_T^{T+\delta}E(u_m)dt+\lim_{m\to\infty}\int_T^{T+\delta}\int_{B_R}G(u_m)dtdx\\ &\le E_\delta(T)+\delta C_0,\end{aligned}$$

where $C_0>0$ is independent of T and δ. It follows that

$$E(T)=\lim_{\delta\to 0}E_\delta(T)\delta^{-1}\ge -C_0,\ \ \forall T>0.$$

The proof is complete. □

The above energy inequality plays a key role in the proof of the following result.

Lemma 5.7. *For any sequence $t_i\to+\infty$, $v_i(t,x):=u(t_i+t,x)$ converges to 0 in $L^2((-1,1)\times B_R)$.*

Proof. Let $B_R\supset\overline{\Omega}_\infty$. By Lemma 5.6 we have, for $T\ge T_0>0$ and $\delta>0$,

$$\int_T^{T+\delta}\int_{B_R}|\partial_t u|^2dxdt\le E(u)(T)-E(u)(T+\delta)\le E(u)(T_0)+C_0.$$

By (5.14) and the fact that $0 \le u \le M$, there exists $C_1(\delta) > 0$ such that

$$\int_T^{T+\delta}\int_{B_R} |\nabla u|^2 dxdt \le \lim_{m\to\infty}\int_T^{T+\delta}\int_{B_R} |\nabla u_m|^2 dxdt \le C_1(\delta), \ \ \forall T > 0, \ \forall \delta > 0.$$

Thus $\exists C > 0$, such that for all i,

$$\iint_{(-1,1)\times B_R} \big(|\nabla v_i|^2 + |\partial_t v_i|^2\big) dxdt \le C.$$

Because of the uniform bound for all v_i, by the compactness embedding theorem for Sobolev spaces, we find a v such that, subject to passing to a subsequence, v_i converges to v weakly in $H^1((-1,1)\times B_R)$ and strongly in $L^2((-1,1)\times B_R)$.

Since $E(T) \ge -C_0$, by Lemma 5.6, $\lim_{t\to+\infty} E(t)$ exists. Moreover,

$$\iint_{(-1,1)\times B_R} |\partial_t v_i|^2 \le E(t_i - 1) - E(t_i + 1)),$$

which converges to 0 as $i \to +\infty$. By the weak convergence of $\partial_t v_i$ to v_t in $L^2((-1,1)\times B_R)$, we get

$$\iint_{(-1,1)\times B_R} |v_t|^2 = 0,$$

that is, v is independent of t.

The remaining part is to prove $v \equiv 0$. By definition of the weak solutions, $\forall \varphi \in C_0^\infty((-1,1)\times B_R)$,

$$\iint_{(-1,1)\times B_R} \alpha(v_i)\varphi_t + v_i\Delta\varphi + g(v_i)\varphi = 0. \tag{5.17}$$

We claim that

$$\lim_{i\to+\infty}\iint_{(-1,1)\times B_R} \alpha(v_i)\varphi_t = 0.$$

This can be seen by decomposing the region of integration into three parts:

$$\Delta_i^1 := (-1,1)\times\Omega(t_i - 1),\ \Delta_i^2 := (-1,1)\times[B_R\setminus\Omega_\infty]$$

and

$$\Delta_i^3 := (-1,1)\times[\Omega_\infty\setminus\Omega(t_i-1)].$$

Over Δ_i^1, $v_i > 0$ and hence $\alpha(v_i) = v_i$. Over Δ_i^2, $v_i = 0$ and hence $\alpha(v_i) = -d\mu^{-1}$; thus the integral is 0. Since $\Omega(t)$ expands to Ω_∞ as $t\to\infty$, we find that $|\Delta_i^3| \to 0$ as $i\to\infty$. Therefore we have

$$\iint_{(-1,1)\times B_R} \alpha(v_i)\varphi_t dtdx = \iint_{(-1,1)\times\Omega_\infty} v_i\varphi_t dtdx + \iint_{\Delta_i^3}[\alpha(v_i) - v_i]\phi_t dtdx.$$

Our claim now follows by letting $i\to\infty$, since v is independent of t.

By passing to the limit in (5.17) and choosing suitable test functions of the form $\varphi(t,x) = \xi(t)\phi(x)$, we obtain

$$\int_{B_R}\big[v\Delta\phi + g(v)\phi\big]dx = 0 \quad \forall\phi\in C_0^\infty(B_R).$$

That is, $v \in H^1(B_R)$ is a solution of

$$-\Delta v = g(v) \text{ in } B_R.$$

By our construction, $v \geq 0$ in B_R and $v = 0$ in $B_R \setminus \Omega_\infty$. Then by the strong maximum principle, $v \equiv 0$. This implies that the entire sequence $v_i \to 0$ in $L^2((-1,1) \times B_R)$. □

The convergence of $v_i \to 0$ can be improved.

Lemma 5.8. *Let v_i be defined as in Lemma 5.7; then v_i converges to 0 uniformly in any compact subset of $(-1,1) \times B_R$.*

Proof. For any $T > 0$, by (2.5) we easily deduce that

$$\int_0^T \int_{B_R} [-u\phi_t + d\nabla u \cdot \nabla \phi] dxdt - \int_{B_R} \tilde{u}_0(x)\phi(0,x)dx \leq \int_0^T \int_{B_R} g(u)\phi dxdt$$

for every nonnegative $\phi \in C^1((0,T) \times B_R)$ satisfying $\phi = 0$ near $([0,T] \times \partial B_R) \cup \{T\} \times B_R$. Thus u satisfies (in the weak sense)

$$\begin{cases} u_t - d\Delta u \leq g(u) & \text{in } (0,\infty) \times B_R, \\ u = 0 & \text{on } (0,\infty) \times \partial B_R, \\ u = \tilde{u}_0 & \text{on } \{0\} \times B_R. \end{cases}$$

It follows that v_j satisfies (in the weak sense)

$$v_t - d\Delta v \leq g(v) \text{ in } (-1,1) \times B_R, \; v = 0 \text{ on } (-1,1) \times \partial B_R.$$

Let K be any compact subset of $(-1,1) \times B_R$. We now choose $R_j \in (0,R)$ and $s_j \in (1/2,1)$ such that as $j \to \infty$, R_j decreases to some $R_0 > 0$ and s_j decreases to some s_0, such that $B_{R_0} \supset \Omega_\infty$ and $K \subset (-s_0,s_0) \times B_{R_0}$. For $j = 1,2,...$, denote $Q_j := (-s_j,s_j) \times B_{R_j}$, and define $\{p_j\}$ by

$$p_1 = 2, \; \frac{1}{p_{j+1}} = \begin{cases} \frac{1}{p_j} - \frac{2}{n+2} & \text{if } p_j < \frac{n+2}{2}, \\ \frac{2}{n+3} & \text{if } p_j \geq \frac{n+2}{2}. \end{cases}$$

Clearly there exists $k > 0$ such that for $j = k$, $p_j = p_k > \frac{n+2}{2}$. Let V_i^j be the unique solution of

$$\begin{cases} V_t - d\Delta V = g(v_i) & \text{in } Q_j, \\ V = 0 & \text{on } (-s_j,s_j) \times \partial B_{R_j}, \\ V = v_i & \text{on } \{-s_j\} \times B_{R_j}. \end{cases}$$

Then by the maximum principle we deduce $v_i \leq V_i^j$ in Q_j. Moreover, by the interior L^p estimates, we have

$$\|V_i^1\|_{W^{1,2}_{p_1}(Q_2)} \leq C_1\|v_i\|_{L^2(Q_1)},$$

and by the Sobolev embedding theorem (see Lemma 3.3 in Chapter II of [18]),

$$\|V_i^1\|_{L^{p_2}(Q_2)} \leq C\|V_i^1\|_{W^{1,2}_{p_1}(Q_2)} \leq C_2\|v_i\|_{L^2(Q_1)}.$$

Thus

$$\|v_i\|_{L^{p_2}(Q_2)} \to 0 \text{ as } i \to \infty.$$

By a simple induction argument we deduce

$$\|V_i^j\|_{W^{1,2}_{p_j}(Q_{j+1})} \leq C_j\|v_i\|_{L^{p_j}(Q_j)} \to 0 \text{ as } i \to \infty$$

for every $j \geq 1$. Then by Lemma 3.3 in Chapter II of [18],

$$\|V_i^k\|_{L^\infty(Q_{k+1})} \leq C\|V_i^k\|_{W^{1,2}_{p_k}(Q_{k+1})} \to 0 \text{ as } i \to \infty.$$

It follows that

$$\|v_i\|_{L^\infty(Q_{k+1})} \to 0 \text{ as } i \to \infty.$$

Since $K \subset Q_{k+1}$, we thus deduce $v_i \to 0$ uniformly in K. □

Clearly Theorem 5.5 is a consequence of Lemma 5.8. The proof of Theorem 5.1 is thus completed.

5.2. **The spreading-vanishing dichotomy with logistic nonlinearity.** In this subsection we use Theorem 5.1 combined with results of [7] and [8] to obtain the spreading-vanishing dichotomy described in Theorem 1.3.

Theorem 5.9. *Suppose that $\Omega_\infty = \mathbb{R}^n$, $g(x,u) = au - bu^2$, and $M(t)$ is given in Theorem 5.1. Then there exists a constant $k_0(\mu) \in (0, 2\sqrt{ad})$ such that*

$$\lim_{t\to\infty} \frac{M(t)}{t} = k_0(\mu),$$

and for every $c \in (0, k_0(\mu))$,

$$\lim_{t\to\infty} \max_{|x|\le ct} \left| u(t,x) - \frac{a}{b} \right| = 0. \tag{5.18}$$

Proof. Since $\Omega_\infty = \mathbb{R}^n$, from Theorem 5.1 we see that $M(t) \to \infty$ as $t \to \infty$. For $T > 0$ to be determined later, we consider the auxiliary radially symmetric problem

$$\begin{cases} v_t - d\Delta v = av - bv^2 & t > 0,\ \ 0 < r < h(t), \\ v_r(t,0) = 0, \quad v(t,h(t)) = 0 & t > 0, \\ h'(t) = -\mu v_r(t,h(t)) & t > 0, \\ h(0) = R_0,\ \ v(0,r) = \underline{u}_T(r) & 0 \le r \le R_0, \end{cases} \tag{5.19}$$

where $R_0 = M(T) - \frac{d_0}{2}\pi$ and $\underline{u}_T(r)$ is a C^1 function that satisfies $\underline{u}_T(R_0) = 0$ and

$$0 < \underline{u}_T(|x|) \le u(T,x) \text{ for } |x| \le R_0.$$

By [7], there exists $R^* > 0$ (determined by a, d and the dimension n) such that when $R_0 \ge R^*$, the unique positive solution (v,h) of (5.19) satisfies

$$\lim_{t\to\infty} \frac{h(t)}{t} = k_0(\mu).$$

We now choose $T > 0$ such that $R_0 = M(T) - \frac{d_0}{2}\pi \ge R^*$.

We then consider the problem

$$\begin{cases} V_t - d\Delta V = aV - bV^2 & t > 0,\ \ 0 < r < k(t), \\ V_r(t,0) = 0, \quad V(t,k(t)) = 0 & t > 0, \\ k'(t) = -\mu V_r(t,k(t)) & t > 0, \\ k(0) = M(T),\ \ V(0,r) = \overline{u}_T(r) & 0 \le r \le M(T), \end{cases} \tag{5.20}$$

where $\overline{u}_T(r)$ is a C^1 function that satisfies $\overline{u}_T(M(T)) = 0$ and

$$\overline{u}_T(|x|) \ge u(T,x) \text{ for } |x| \le M(T).$$

Denote $\mathcal{O}(t) = B_{h(t)}(0)$ and $\mathcal{G}(t) = B_{k(t)}(0)$. Then by Theorem 6.1 of [8], we have

$$\mathcal{O}(t) \subset \Omega(t+T) \subset \mathcal{G}(t)\ \forall t \ge 0.$$

Hence

$$h(t) \le M(t+T) - \frac{d_0}{2}\pi < M(t+T) \le k(t).$$

By [7], we also have

$$\lim_{t\to\infty}\frac{k(t)}{t}=k_0(\mu).$$

Therefore we necessarily have

$$\lim_{t\to\infty}\frac{M(t)}{t}=k_0(\mu).$$

Now (5.18) follows directly from Theorem 6.4 of [8]. □

Remark 5.10. *$k_0(\mu)$ is determined in Proposition 3.1 of* [7]. *It is an increasing function of μ and $k_0(\mu)\to 2\sqrt{ad}$ as $\mu\to\infty$. More analysis of $k_0(\mu)$ is given in* [3].

To complete the proof of Theorem 1.3, it remains to show the following result.

Theorem 5.11. *Suppose that $g(u)=au-bu^2$. Then there exists $\mu^*\geq 0$ such that $\Omega_\infty=\mathbb{R}^n$ for $\mu>\mu^*$, and Ω_∞ is bounded when $\mu\in(0,\mu^*]$. Moreover, $\mu^*=0$ if Ω_0 contains a ball of radius $R^*:=\sqrt{\frac{d}{a}\lambda_1}$, where λ_1 is the first eigenvalue of*

$$-\Delta\phi=\lambda\phi \text{ in } B_1(0),\ \phi=0 \text{ on } \partial B_1(0),$$

and $\mu^*>0$ if $\overline{\Omega}_0$ is contained in an open ball of radius R^*.

Proof. Choose a small ball $B_*\subset\Omega_0$ and consider problem (1.5) with Ω_0 replaced by B_*, and with u_0 replaced by a radially symmetric function $\underline{u}_0$ satisfying $0<\underline{u}_0\leq u_0$ in B_* and $\underline{u}_0=0$ on ∂B_*. This is a radially symmetric problem with a unique radially symmetric solution u_* and we can use the result of [7] to conclude that there exists $\underline{\mu}^*>0$ such that spreading happens when $\mu>\underline{\mu}^*$. By Theorem 4.3 of [8], we have $u\geq u_*$ and hence we necessarily have $\Omega_\infty=\mathbb{R}^n$ for $\mu>\underline{\mu}^*$.

Define

$$\mu^*:=\inf\{\mu_0>0:\Omega_\infty=\mathbb{R}^n \text{ for } \mu>\mu_0\}.$$

Clearly $\mu^*\leq\underline{\mu}^*$. If $\mu^*=0$, then there is nothing to prove.

Suppose next that $\mu^*>0$. We claim that for any $\mu\leq\mu^*$, Ω_∞ is bounded. To show this we need to consider the continuous dependence of the solution of (2.11) on the parameter μ. So we denote the unique solution by w_μ to stress this dependence. For fixed $T>0$, and $\mu_n\to\mu_0>0$, from (2.11) we find that w_{μ_n} is bounded in $W_p^{1,2}(\Omega_{T,R})$ for any $p>1$. Therefore by passing to a subsequence w_{μ_n} converges weakly in $W_p^{1,2}(\Omega_{T,R})$ to some w_0 which satisfies (2.11) with $\mu=\mu_0$. By uniqueness, $w_0=w_{\mu_0}$. Hence the entire sequence converges to w_{μ_0}. By Sobolev embedding, the convergence hold in $H_{1+\gamma}(\Omega_{T,R})$, $\forall\gamma\in(0,1)$. Therefore $w_\mu\to w_{\mu_0}$ uniformly in compact subsets of $(0,\infty)\times\mathbb{R}^n$ as $\mu\to\mu_0>0$. (We assume that $w_\mu(t,\cdot)$ and $w_{\mu_0}(t,\cdot)$ are extended by zero outside their supports.) This proves the continuous dependence of the solution on μ.

Let us also observe that $\Omega_\mu(t)\supset\Omega_{\mu_0}(t)$ for $\mu\geq\mu_0>0$, where $\Omega_\mu(t)=\{x:u_\mu(t,x)>0\}$ and u_μ is the unique solution of (1.5). Indeed, by Theorem 3.5 of [8], $u_\mu\geq u_{\mu_0}$, which implies $\Omega_\mu(t)\supset\Omega_{\mu_0}(t)$ for all $t>0$.

We now come back to the proof of the claim. Suppose by way of contradiction that it is not true. Then there exists $\mu_0\in(0,\mu^*]$ such that $\Omega_\infty=\mathbb{R}^n$ when $\mu=\mu_0$. By Theorem 5.3, $m(t)\to\infty$ as $t\to\infty$, and therefore for any $R>0$, there exists $T>0$ such that we can put a ball B_{2R} of radius $2R$ inside $\Omega_{\mu_0}(T)$. By the continuity of w_μ in μ, there exists $\epsilon>0$ depending on T and R such that $B_R\subset\Omega_{\mu_0-\epsilon}(T)$. By Theorems 2.1 and 2.5 of [7], if $R\geq R^*$, problem (1.5) with Ω_0 replaced by B_R, and with u_0 replaced by any smooth radially symmetric function $\underline{u}_0$ positive in B_R and vanishing on ∂B_R, has a unique radial solution v_μ and spreading happens for all $\mu > 0$. We now fix $R > R^*$ and choose the radially symmetric $\underline{u}_0$ such that $\underline{u}_0 \le u_{\mu_0-\epsilon}(T,\cdot)$ in B_R. By Theorem 4.3 of [8], we have $u_{\mu_0-\epsilon}(T+t,\cdot) \ge v_{\mu_0-\epsilon}(t,\cdot)$, which implies that $\Omega_\mu(t) \to \mathbb{R}^n$ as $t \to \infty$ for $\mu = \mu_0 - \epsilon$ and hence for all $\mu \ge \mu_0 - \epsilon$ due to the monotonicity of $\Omega_\mu(t)$ in μ. But this contradicts the definition of μ^* since $\mu_0 - \epsilon < \mu^*$. The claim is proved.

Let B^* be a ball such that $\Omega_0 \subset B^*$. We want to show that in this case $\mu^* > 0$. Consider (1.5) with Ω_0 replaced by B^* and u_0 replaced by a radially symmetric $\overline{u}_0$ satisfying $\overline{u}_0 \ge u_0$ in Ω_0, $\overline{u}_0$ is positive in B^* and vanishes on ∂B^*. This new problem has a radially symmetric solution u^* and $u \le u^*$. By [7], if the radius of B^*, denoted by R, is less than $R^* = \sqrt{\frac{d}{a}\lambda_1}$, then there exists a unique $\overline{\mu}^* > 0$ such that vanishing happens for the new problem when $\mu \in (0, \overline{\mu}^*]$. Therefore Ω_∞ must be bounded when $\mu \le \overline{\mu}^*$, which implies that $\mu^* \ge \overline{\mu}^* > 0$.

On the other hand, if Ω_0 contains a ball of radius R^*, then we denote this ball by B_* and argue as at the beginning of the proof; we obtain that $\mu^* \le \underline{\mu}^* = 0$. Therefore $\mu^* = 0$. This completes the proof. □

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