SPREADING-VANISHING DICHOTOMY IN A DIFFUSIVE LOGISTIC MODEL WITH A FREE BOUNDARY, II

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ABSTRACT. We study the diffusive logistic equation with a free boundary in higher space dimensions and heterogeneous environment. Such a model may be used to describe the spreading of a new or invasive species, with the free boundary representing the expanding front. For simplicity, we assume that the environment and the solution are radially symmetric. In the special case of one space dimension and homogeneous environment, this free boundary problem was investigated in [10]. We prove that the spreading-vanishing dichotomy established in [10] still holds in the more general and ecologically realistic setting considered here. Moreover, when spreading occurs, we obtain best possible upper and lower bounds for the spreading speed of the expanding front. When the environment is asymptotically homogeneous at infinity, these two bounds coincide. Our results indicate that the asymptotic spreading speed determined by this model does not depend on the spatial dimension.

1. INTRODUCTION

An important problem in invasion ecology is to understand the nature of spreading of the invasive species. It is well known that many animal species spread to their new environment in a linear fashion, namely the spreading radius eventually exhibits a linear growth curve against time ([27, 22]). This phenomenon seems first observed by Skellam [28] in examining the spreading of muskrat in Europe in the early 1900s. He calculated the area of the muskrat range from a map obtained from field data, took the square root (which gives the spreading radius) and plotted it against years, and found that the data points lay on a straight line. Several mathematical models have been proposed to describe this phenomenon and one may find many in [27].

One of the most successful mathematical approaches to this problem is based on the investigation of front propagation governed by the following diffusive logistic equation over the entire space $\mathbb{R}^N$:

$$u_t - d\Delta u = u(a - bu),\ t > 0,\ x \in \mathbb{R}^N. \tag{1.1}$$

Date: February 16, 2011.

1991 *Mathematics Subject Classification.* 35K20, 35R35, 35J60, 92B05.

Key words and phrases. Diffusive logistic equation, free boundary, spreading-vanishing dichotomy, invasive population, spreading speed.

We thank the referee for a very careful reading of the manuscript and for useful suggestions and comments. This work was supported by the Australian Research Council and also by NSFC 10871060 and NSFC 10671172 of China.

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Here $u = u(t,x)$ may be regarded as the population density of a spreading species with diffusion rate d, intrinsic growth rate a and habitat carrying capacity a/b. In the pioneering works of Fisher [13] and Kolmogorov et al [18], for space dimension $N = 1$, traveling wave solutions have been found for (1.1): For any $c \geq c^* := 2\sqrt{ad}$, there exists a solution $u(t,x) := W(x - ct)$ with the property that

$$W'(y) < 0 \text{ for } y \in \mathbb{R}^1,\ W(-\infty) = a/b,\ \ W(+\infty) = 0;$$

no such solution exists if $c < c^*$. The number c^* is called the minimal speed of the traveling waves. Fisher [13] claims that c^* is the spreading speed for the advantageous gene in his research, and used a probabilistic argument to support his claim. Skellam [28] was able to use a linear model (i.e., (1.1) with $b = 0$) and a similar probabilistic argument to show that c^* should be the speed of spreading. A clearer description and rigorous proof of this fact were given by Aronson and Weinberger (see Section 4 in [1]), who showed that for a new population $u(t,x)$ (governed by the above logistic equation) with initial distribution $u(0,x)$ confined to a compact set of x (i.e., $u(0,x) = 0$ outside a compact set), one has

$$\lim_{t\to\infty,\ |x|\leq(c^*-\epsilon)t} u(t,x) = a/b, \qquad \lim_{t\to\infty,\ |x|\geq(c^*+\epsilon)t} u(t,x) = 0$$

for any small $\epsilon > 0$. These results have been extended to higher dimensions in [2], and further related research may be found, for example, in [3, 4, 5, 14, 20, 29, 30] and the references therein.

A shortcoming of this approach is that it predicts successful spreading and establishment of the new species with any nontrivial initial population $u(0,x)$ (namely $u(t,x) \to a/b$ as $t \to \infty$), regardless of its initial size and supporting area. This is in sharp contrast to numerous empirical evidences; for example, the introduction of several bird species from Europe to North America in the 1900s was successful only after many initial attempts.

The phenomenon that a species starting with small initial size may fail to establish is often explained by the "Allee effect", which states that populations shrink at very low densities because, on average, individuals cannot replace themselves in such a situation. In mathematics, to include the Allee effect, one usually replaces the logistic reaction term $u(a - bu)$ in (1.1) by a bistable function $f(u)$ such as

$$f_0(u) = au(1-u)(u-\theta),\ \theta \in (0, 1/2).$$

It is well known that for a bistable nonlinear term $f(u)$ behaving like $f_0(u)$, there is a unique $c_0 > 0$ such that the equation

$$u_t - du_{xx} = f(u),\ x \in (-\infty, +\infty)$$

has a unique traveling wave solution (up to translation in x) with speed c_0, and no traveling wave solution exists for any other speed c (see, e.g., [1]). The constant c_0 is also the spreading speed for the model, and when the special form $f_0(u)$ is used, then (see [15, 16])

$$c_0 = (1/2 - \theta)\sqrt{2ad}.$$

Recently, Du and Lin [10] used a free boundary model to study the spreading of species in one space dimension with the same logistic nonlinearity as in (1.1), and showed that, depending on the initial size, both spreading and vanishing can happen. The model in [10] has the following form:

$$
(1.2)\qquad \begin{cases} u_t - du_{xx} = u(a-bu), & t>0,\ 0<x<h(t),\\ u_x(t,0)=0,\ u(t,h(t))=0, & t>0,\\ h'(t) = -\mu u_x(t,h(t)), & t>0,\\ h(0)=h_0, u(0,x)=u_0(x), & 0\le x\le h_0, \end{cases}
$$

where $x = h(t)$ is the moving boundary to be determined, h_0, μ, d, a and b are given positive constants, and the initial function $u_0(x)$ satisfies

$$
(1.3)\qquad u_0 \in C^2([0,h_0]),\ u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \text{ in } [0,h_0).
$$

Here $u(t,x)$ stands for the population density of a new or invasive species over a one dimensional habitat, and the initial function $u_0(x)$ stands for the population of the species in the very early stage of its introduction, which occupies an initial region $[0,h_0]$. It is assumed that the species can only invade further into the environment from the right end of the initial region, and the spreading front expands at a speed that is proportional to the population gradient at the front, which gives rise to the Stefan condition $h'(t) = -\mu u_x(t,h(t))$. It was shown in [10] that (1.2) has a unique solution $(u(t,x),h(t))$ defined for all $t>0$, with $u(t,x)>0$ and $h'(t)>0$. Moreover, a spreading-vanishing dichotomy holds for (1.2), namely, as time $t\to\infty$, the population $u(t,x)$ either successfully establishes itself in the new environment (called spreading), in the sense that $h(t)\to\infty$ and $u(t,x)\to a/b$, or the population fails to establish and vanishes eventually (called vanishing), namely $h(t)\to h_\infty \le \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and $u(t,x)\to 0$. Furthermore, when spreading occurs, for large time, the spreading speed approaches a positive constant k_0, i.e., $h(t) = [k_0+o(1)]t$ as $t\to\infty$. The asymptotic spreading speed k_0 is uniquely determined by an auxiliary elliptic problem induced from (1.2), and is independent of the initial population size u_0. The criteria for spreading or vanishing are as follows. If the initial occupying area $[0,h_0]$ is beyond a critical size, namely $h_0 \ge \frac{\pi}{2}\sqrt{\frac{d}{a}}$, then regardless of the initial population size $u_0(x)$ (satisfying (1.3)), spreading always happens. On the other hand, if $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$, then whether spreading or vanishing occurs is determined by the initial population size u_0 and the coefficient μ in the Stefan condition (assuming the other parameters are fixed). It was shown that for such h_0, with each given u_0, there exists a critical $\mu^*>0$ depending on u_0, such that spreading occurs if $\mu>\mu^*$ and vanishing happens when $\mu\le\mu^*$.

If the left boundary $x=0$ in (1.2) is replaced by a free boundary $x=g(t)$ governed by $g'(t) = -\mu u_x(t,g(t))$, it was proved in [10] that a similar spreading-vanishing dichotomy holds, and in the case of spreading, both the left front $x=g(t)$ and the right front $x=h(t)$ expand to infinity at the same asymptotic speed k_0.

The main purpose of this paper is to show that most of these results of [10] continue to hold in the more realistic situation of higher space dimensions and heterogeneous environment. For simplicity, we assume that the environment and the solution are radially symmetric. (The general case is considered in [DG].) So we will study the behavior of the positive solution $u(t,r)$, $r=|x|$, $x\in\mathbb{R}^N$ $(N\ge 2)$, to the following problem,

$$
\text{(1.4)} \qquad \begin{cases} u_t - d\Delta u = u(\alpha(r) - \beta(r)u), & t > 0,\ 0 < r < h(t), \\ u_r(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0, u(0,r) = u_0(r), & 0 \le r \le h_0, \end{cases}
$$

where $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, $r = h(t)$ is the moving boundary to be determined, h_0, μ and d are given positive constants, $\alpha, \beta \in C^{\nu_0}([0,\infty))$ for some $\nu_0 \in (0,1)$, and there are positive constants $\kappa_1 \le \kappa_2$ such that

$$
\text{(1.5)} \qquad \kappa_1 \le \alpha(r) \le \kappa_2, \quad \kappa_1 \le \beta(r) \le \kappa_2 \quad \text{for } r \in [0,\infty).
$$

The initial function $u_0(r)$ satisfies

$$
\text{(1.6)} \qquad u_0 \in C^2([0,h_0]),\ u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \text{ in } [0,h_0).
$$

Thus problem (1.4) describes the spreading of a new or invasive species with population density $u(t,|x|)$ over an N-dimensional habitat, which is radially symmetric but heterogeneous. The initial function $u_0(|x|)$ stands for the population in the very early stage of its introduction, which occupies an initial region B_{h_0}. Here and in what follows we use B_R to stand for the ball with center at 0 and radius R. The spreading front is represented by the free boundary $|x| = h(t)$, which is the sphere $\partial B_{h(t)}$ whose radius $h(t)$ grows at a speed that is proportional to the population gradient at the front: $h'(t) = -\mu u_r(t,h(t))$. The coefficient function $\alpha(|x|)$ represents the intrinsic growth rate of the species, $\beta(|x|)$ measures its intra-specific competition, and d is the diffusion rate.

In Section 2 below, we first state the global existence and uniqueness result for (1.4) (Theorem 2.1), then we prove the spreading-vanishing dichotomy (Theorem 2.4) and obtain sharp thresholds that govern the alternatives in the dichotomy (Theorems 2.5 and 2.10). The proof of Theorem 2.1 is postponed to Section 4, since it is rather long and is only a modification of the proof in section 2 of [10], which in turn follows the approach in [7]. In Section 3, we obtain estimates for the spreading speed, namely best possible bounds for $\underline{\lim}_{t\to+\infty}\frac{h(t)}{t}$ and $\overline{\lim}_{t\to+\infty}\frac{h(t)}{t}$ (see Theorem 3.6). These bounds are determined by an auxiliary elliptic equation over the half line $[0,\infty)$ (see Proposition 3.1), which arises naturally from the original problem, and was first introduced in [10]. If the environment is asymptotically homogeneous at infinity, namely, $\alpha(r) \to \alpha_*$ and $\beta(r) \to \beta_*$ as $r \to +\infty$, these bounds coincide and hence the limit of $\frac{h(t)}{t}$ exists as $t \to +\infty$.

Though the outline of the approach in this paper largely follows that of [10], most of the technical proofs here are different from and much more involved than the corresponding ones in [10], and some of the results here are proved by completely different methods.

We now briefly compare (1.4) with (1.1). Firstly, the spreading-vanishing dichotomy for (1.4) appears more realistic than the persistent spreading predicted by (1.1). Secondly, for any finite $t > 0$, our density function $u(t,x)$ is supported on a finite domain of x, which expands as t increases. This more closely resembles the spreading processes in the real world than (1.1), whose solution is positive for all x as long as $t > 0$. Finally we notice that, while (1.1) gives an asymptotic spreading speed of $2\sqrt{ad}$ (for large time), which is independent of b and is increasing with the diffusion rate d, the asymptotic spreading speed of (1.4) (with asymptotically homogeneous environment) depends on all the parameters and on $\alpha(r), \beta(r)$ in (1.4), and in sharp contrast, it is not increasing with respect to d (at least for large d); moreover, the bounds of the spreading speed determined by (1.4) are always smaller than $2\sqrt{ad}$, and both upper and lower bounds converge to $2\sqrt{ad}$ as $\mu \to \infty$ (see (3.3)).

Similar free boundary conditions to the one in (1.2) have been used in ecological models over *bounded* spatial domains in several earlier papers; see for example, [23, 24, 25], [17], [21]. But the purposes of these papers are very different from ours.

Our results can be easily extended to cover a more general reaction term $f(r,u)$ which behaves like $\alpha(r)u - \beta(r)u^2$. We leave this to the interested reader.

2. The spreading-vanishing dichotomy

In this section we prove the spreading-vanishing dichotomy. Though our approach here mainly follows the lines of [10], considerable changes in the proofs are needed, since the situation here is more general and difficult.

The following existence uniqueness result can be proved by adequately modifying the arguments in section 2 of [10]. So we state the result here but postpone its proof to section 4 below.

Theorem 2.1. *Problem* (1.4) *has a unique solution* $(u(t,r), h(t))$*, which is defined for all* $t > 0$*. Moreover,* $u(t,r) > 0$*,* $h'(t) > 0$ *for* $t > 0$ *and* $0 \le r < h(t)$*, and* $h \in C^1([0,\infty))$*,* $u \in C^{1,2}(D)$*, with* $D = \{(t,r) : t > 0, 0 \le r \le h(t)\}$*.*

It follows from Theorem 2.1 that $r = h(t)$ is monotonic increasing and therefore there exists $h_\infty \in (0, +\infty]$ such that $\lim_{t\to+\infty} h(t) = h_\infty$.

Let $\lambda_1(d, \alpha, R)$ be the principal eigenvalue of the problem

$$(2.1) \qquad \begin{cases} -d\Delta\phi &= \lambda\alpha(|x|)\phi \quad \text{in } B_R \\ \phi &= 0 \qquad\qquad \text{on } \partial B_R. \end{cases}$$

It is well-known that $\lambda(d, \alpha, \cdot)$ is a strictly decreasing continuous function and

$$\lim_{R\to 0^+} \lambda_1(d,\alpha,R) = +\infty, \quad \lim_{R\to+\infty} \lambda_1(d,\alpha,R) = 0.$$

Therefore, for fixed $d > 0$ and $\alpha \in C^{\nu_0}([0,\infty))$, there is a unique $R^* := R^*(d,\alpha)$ such that

$$(2.2) \qquad \lambda_1(d,\alpha,R^*) = 1$$

and

$$1 > \lambda_1(d,\alpha,R) \text{ for } R > R^*; \quad 1 < \lambda_1(d,\alpha,R) \text{ for } R < R^*.$$

The spreading-vanishing dichotomy is a consequence of the following two lemmas.

Lemma 2.2. *If* $h_\infty < +\infty$*, then* $h_\infty \le R^*$*, and* $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([0,h(t)])} = 0$*.*

Proof: We first prove that $h_\infty \le R^*$. Otherwise $h_\infty > R^*$ and there exists $T > 0$ such that $h(t) > R^*$ for all $t \ge T$. Thus,

$$1 > \lambda_1(d,\alpha,h(t)) \text{ for all } t \ge T.$$

Moreover, for any sufficiently small $\varepsilon > 0$, there is $T_0 := T_0(\varepsilon) > T$ such that for $t \ge T_0$,

$$R^* < h_\infty - \varepsilon < h(t) < h_\infty.$$

Consider the problem

$$\begin{cases} w_t - d\Delta w = w(\alpha(r) - \beta(r)w), \ \ t \geq T_0, \ \ r \in [0, h_\infty - \varepsilon], \\ w_r(t,0) = 0 \ \ \ w(t, h_\infty - \varepsilon) = 0, \ \ t \geq T_0, \\ w(T_0, r) = u(T_0, r), \ \ r \in [0, h_\infty - \varepsilon]. \end{cases} \tag{2.3}$$

This is a logistic problem with $1 > \lambda_1(d, \alpha, h_\infty - \varepsilon)$. It is well-known (see, for example, Proposition 3.3 in [6]) that (2.3) admits a unique positive solution $\underline{w} = \underline{w}_\varepsilon(t, r)$. Moreover,

$$\underline{w}(t, \cdot) \to V_{h_\infty - \varepsilon} \ \text{ in } C^2([0, h_\infty - \varepsilon]) \text{ as } t \to \infty, \tag{2.4}$$

where $V_{h_\infty - \varepsilon}(r)$ is the unique positive (radial) solution of the problem

$$\begin{cases} -d\Delta V &= V(\alpha(r) - \beta(r)V) \quad \text{in } B_{h_\infty - \varepsilon}, \\ V &= 0 \quad \text{on } \partial B_{h_\infty - \varepsilon}. \end{cases} \tag{2.5}$$

By the comparison principle

$$u(t, r) \geq \underline{w}(t, r) \quad \text{for } t > T_0, \ r \in [0, h_\infty - \varepsilon]. \tag{2.6}$$

This implies that

$$\underline{\lim}_{t \to +\infty} u(t, r) \geq V_{h_\infty - \epsilon}(r) \ \text{ for } r \in [0, h_\infty - \varepsilon]. \tag{2.7}$$

On the other hand, consider the problem

$$\begin{cases} w_t - d\Delta w = w(\alpha(r) - \beta(r)w), \ \ t \geq T_0, \ \ r \in [0, h_\infty], \\ w_r(t,0) = 0 \ \ \ w(t, h_\infty) = 0, \ \ t \geq T_0, \\ w(T_0, r) = \tilde{u}(T_0, r), \ \ r \in [0, h_\infty], \end{cases} \tag{2.8}$$

where

$$\tilde{u}(T_0, r) = \begin{cases} u(T_0, r) & \text{for } r \in [0, h(T_0)], \\ 0 & \text{for } r \in (h(T_0), h_\infty]. \end{cases}$$

Similarly (2.8) admits a unique positive solution $\overline{w}(t, r)$ with

$$\overline{w}(t, \cdot) \to V_{h_\infty} \ \text{ in } C^2([0, h_\infty]) \text{ as } t \to +\infty, \tag{2.9}$$

where V_{h_∞} is the unique positive (radial) solution of the problem

$$\begin{cases} -d\Delta V &= V(\alpha(r) - \beta(r)V) \quad \text{in } B_{h_\infty}, \\ V &= 0 \quad \text{on } \partial B_{h_\infty}. \end{cases} \tag{2.10}$$

Meanwhile, the comparison principle implies that

$$u(t, r) \leq \overline{w}(t, r) \ \text{ for } t > T_0, \ r \in [0, h(t)] \tag{2.11}$$

and hence

$$\overline{\lim}_{t \to +\infty} u(t, r) \leq V_{h_\infty}(r) \ \text{ for } r \in [0, h_\infty]. \tag{2.12}$$

By a standard compactness and uniqueness argument, we can easily show that

$$V_{h_\infty - \varepsilon} \to V_{h_\infty} \ \text{ in } C^2_{loc}([0, h_\infty)) \text{ as } \epsilon \to 0^+.$$

Thus, (2.7), (2.12) and the arbitrariness of ϵ imply

$$\lim_{t \to \infty} u(t, r) = V_{h_\infty}(r) \ \text{ for } r \in [0, h_\infty). \tag{2.13}$$

We claim that

$$\|u(t, \cdot) - V_{h_\infty}\|_{C^2([0, h(t)])} \to 0 \ \text{ as } t \to +\infty.$$

Indeed, if we straighten the free boundary as in the proof of Theorem 2.1, so that $[0,h(t)]$ is changed to $[0,h_0]$, $u(t,r)$ is changed to $\tilde{u}(t,r)$ and V_{h_∞} is changed to $\tilde{V}_{h_\infty}$, then by standard regularity theory it is easily seen that $\tilde{u}(t,.)$ has a common bound in $C^{2,\nu}([0,h_0])$ for all $t \geq 1$. Thus for each sequence $t_n \to +\infty$ we can extract a subsequence, still denoted by t_n, such that $\tilde{u}(t_n,r)$ converges to some $\tilde{V}$ in $C^2([0,h_0])$. By (2.13), we necessarily have $\tilde{V} = \tilde{V}_{h_\infty}$. This implies that

$$\|\tilde{u}(t,\cdot) - \tilde{V}_{h_\infty}\|_{C^2([0,h_0])} \to 0 \text{ as } t \to \infty,$$

which is equivalent to the claimed convergence on $u(t,x) \to V_{h_\infty}(x)$.

Thus, as $t \to \infty$,

$$u_r(t,h(t)) \to V'_{h_\infty}(h_\infty) < 0.$$

It follows that

$$h'(t) = -\mu u_r(t,h(t)) \to -\mu V'_{h_\infty}(h_\infty) > 0 \text{ as } t \to \infty,$$

which implies $h_\infty = \infty$, a contradiction to our assumption that $h_\infty < \infty$. Therefore we must have $h_\infty \leq R^*$.

We are now ready to show that $\|u(t,\cdot)\|_{C([0,h(t)])} \to 0$ as $t \to \infty$. Let $\overline{u}(t,r)$ denote the unique positive solution of the problem

$$\begin{cases} \overline{u}_t - d\Delta\overline{u} = \overline{u}[\alpha(r) - \beta(r)\overline{u}], & t > 0,\ 0 < r < h_\infty, \\ \overline{u}_r(t,0) = 0, \quad \overline{u}(t,h_\infty) = 0, & t > 0, \\ \overline{u}(0,r) = \tilde{u}_0(r), & 0 < r < h_\infty, \end{cases} \tag{2.14}$$

where

$$\tilde{u}_0(r) = \begin{cases} u_0(r), & 0 \leq r \leq h_0, \\ 0, & r \geq h_0. \end{cases}$$

The comparison principle gives $0 \leq u(t,r) \leq \overline{u}(t,r)$ for $t > 0$ and $r \in [0,h(t)]$. Since $h_\infty \leq R^*$, we have $1 \leq \lambda_1(d,\alpha,h_\infty)$ and it follows from a well-known conclusion on the logistic problem (2.14) that $\overline{u}(t,r) \to 0$ uniformly for $r \in [0,h_\infty]$ as $t \to +\infty$ (see, for example, Corollary 3.4 in [6]). Thus $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([0,h(t)])} = 0$. □

Lemma 2.3. *If $h_\infty = +\infty$, then*

$$\lim_{t\to+\infty} u(t,r) = \hat{U}(r) \text{ locally uniformly for } r \in [0,+\infty), \tag{2.15}$$

where $\hat{U}(|x|)$ is the unique positive (radial) solution of the equation

$$-d\Delta u = u[\alpha(|x|) - \beta(|x|)u] \quad \text{in } \mathbb{R}^N. \tag{2.16}$$

Proof: The existence and uniqueness of a positive solution of (2.16) follows from Theorem 2.3 of [11] (by choosing both γ and τ there to be 0). It must be radially symmetric since (2.16) is invariant under rotations around the origin of $\mathbb{R}^N$.

To show (2.15), we use a squeezing argument introduced in [12]. We first consider the Dirichlet problem

$$-d\Delta v = v[\alpha(r) - \beta(r)v], \quad v(R) = 0,$$

and the boundary blow-up problem

$$-d\Delta w = w[\alpha(r) - \beta(r)w], \quad w(R) = +\infty.$$

When R is large, it is well-known that these problems have positive radial solutions v_R and w_R, respectively. By the comparison principle given in [12], as $R \to +\infty$, v_R increases to the unique positive solution $\hat{U}$ of (2.16) and w_R decreases to $\hat{U}$.

Choose an increasing sequence of positive number R_n such that $R_n \to +\infty$ as $n \to \infty$, and $1 > \lambda_1(d, \alpha, R_n)$ for all n. Then, as $n \to \infty$, both v_{R_n} and w_{R_n} converge to $\hat{U}$. For each n, we can find $T_n > 0$ such that $h(t) \geq R_n$ for $t \geq T_n$. The problem

$$\begin{cases} w_t - d\Delta w = w(\alpha(r) - \beta(r)w), \ \ t \geq T_n, \ \ r \in [0, R_n], \\ w_r(t, 0) = 0 \quad w(t, R_n) = 0, \ \ t \geq T_n, \\ w(T_n, r) = u(T_n, r), \ \ r \in [0, R_n], \end{cases} \tag{2.17}$$

admits a unique positive solution $w_n(t, r)$ and

$$w_n(t, r) \to v_{R_n}(r) \ \text{ uniformly for } r \in [0, R_n] \text{ as } t \to +\infty. \tag{2.18}$$

By the comparison principle, we have

$$w_n(t, r) \leq u(t, r) \ \text{ for } t \geq T_n \text{ and } r \in [0, R_n].$$

Therefore,

$$\underline{\lim}_{t \to +\infty} u(t, r) \geq v_{R_n}(r) \ \text{ uniformly in } r \in [0, R_n].$$

Sending $n \to \infty$, we obtain

$$\underline{\lim}_{t \to +\infty} u(t, r) \geq \hat{U}(r) \ \text{ locally uniformly for } r \in [0, +\infty). \tag{2.19}$$

Analogously, by arguments similar to those in the proof of Theorem 4.1 of [12], we see that

$$\overline{\lim}_{t \to +\infty} u(t, r) \leq w_{R_n}(r) \ \text{ uniformly for } r \in [0, R_n],$$

which implies (by sending $n \to \infty$)

$$\overline{\lim}_{t \to +\infty} u(t, r) \leq \hat{U}(r) \ \text{ locally uniformly for } r \in [0, +\infty). \tag{2.20}$$

From (2.19) and (2.20) we see that (2.15) holds. □

Combing Lemmas 2.2 and 2.3, we immediately obtain the following spreading-vanishing dichotomy:

Theorem 2.4. *Let $(u(t, r), h(t))$ be the solution of the free boundary problem* (1.4). *Then the following alternative holds:*

Either

(i) Spreading: $h_\infty = +\infty$ *and*

$$\lim_{t \to +\infty} u(t, r) = \hat{U}(r) \ \textit{ locally uniformly for } r \in [0, \infty),$$

or

(ii) Vanishing: $h_\infty \leq R^*$ *and* $\lim_{t \to +\infty} \ ||u(t, \cdot)||_{C([0, h(t)])} = 0$.

We next decide exactly when each of the two alternatives occurs. We need to divide our discussion into two cases:

$$(a) \ \ h_0 \geq R^*, \quad (b) \ \ h_0 < R^*.$$

In case (a), due to $h'(t) > 0$ for $t > 0$, we must have $h_\infty > R^*$. Hence Lemma 2.2 implies the following result.

Theorem 2.5. *If $h_0 \geq R^*$, then $h_\infty = +\infty$.*

As in [10], in order to study case (b), and also for later applications, we need a comparison principle which can be used to estimate both $u(t,r)$ and the free boundary $r=h(t)$.

Lemma 2.6. *Suppose that* $T\in(0,\infty)$, $\overline{h}\in C^1([0,T])$, $\overline{u}\in C^{1,2}(D_T^*)$ *with* $D_T^*=\{(t,r)\in\mathbb{R}^2: 0\le t\le T, 0\le r\le \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t - d\Delta \overline{u}\ge \overline{u}[\alpha(r)-\beta(r)\overline{u}], & 0<t\le T,\ 0<r<\overline{h}(t),\\ \overline{u}=0, \quad \overline{h}'(t)\ge -\mu\overline{u}_r, & 0<t\le T,\ r=\overline{h}(t),\\ \overline{u}_r(t,0)\le 0, & 0<t\le T. \end{cases}$$

If

$$h_0\le \overline{h}(0) \text{ and } u_0(r)\le \overline{u}(0,r) \text{ in } [0,h_0],$$

then the solution (u,h) *of the free boundary problem* (1.4) *satisfies*

$$h(t)\le \overline{h}(t) \text{ in } (0,T],\ u(r,t)\le\overline{u}(r,t) \text{ for } t\in(0,T] \text{ and } r\in(0,h(t)).$$

Proof: This is almost identical to the proof of Lemma 3.5 in [10]. So we omit the details. □

Remark 2.7. *The pair* $(\overline{u},\overline{h})$ *in Lemma 2.6 is usually called an upper solution of the problem* (1.4)*. We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 2.6 for lower solutions.*

Let us now consider case (b), where $h_0<R^*$. We first examine the case that μ is large, then we look at the case $\mu>0$ is small, and finally we use Lemma 2.6 and Remark 2.7 to prove the existence of a critical μ^* so that spreading occurs when $\mu>\mu^*$ and vanishing happens if $\mu\in(0,\mu^*]$.

Lemma 2.8. *Suppose* $h_0<R^*$. *Then there exists* $\mu^0>0$ *depending on* u_0 *such that spreading occurs if* $\mu\ge\mu^0$.

Proof: We argue indirectly. Suppose that there is an increasing sequence $\{\mu_n\}$ satisfying $\mu_n\to+\infty$ as $n\to\infty$ such that the unique solution (u^n,h^n) of (1.4) with $\mu=\mu_n$ satisfies $h^n_\infty:=\lim_{t\to\infty}h^n(t)<+\infty$ for all n. Then it follows from Lemma 2.2 that $h^n_\infty\le R^*$ and hence

$$u^n(t,r)\le w^*(t,r) \text{ for } t>0 \text{ and } r\in[0,h^n(t)], \tag{2.21}$$

where $w^*(t,r)$ is the unique positive solution of the problem

$$\begin{cases} w_t-d\Delta w = w(\alpha(r)-\beta(r)w), & t>0,\ r\in[0,R^*],\\ w_r(t,0)=0 \quad w(t,R^*)=0, & t>0,\\ w(0,r)=\hat{u}_0(r), & r\in[0,R^*], \end{cases}$$

with

$$\hat{u}_0(r)=\begin{cases} u_0(r) & r\in[0,h_0]\\ 0 & r\in(h_0,R^*]\end{cases}$$

The fact that $1=\lambda_1(d,\alpha,R^*)$ implies that

$$\lim_{t\to+\infty}\|w^*(t,\cdot)\|_{C([0,R^*])}\to 0. \tag{2.22}$$

This and (2.21) imply that there is $T>0$ independent of n such that

$$u^n(t,r) \leq \frac{\kappa_1}{\kappa_2} \ \text{ for } t \geq T \text{ and } r \in [0, h^n(t)].$$

For convenience, we will omit n from u^n, h^n, h^n_∞ and μ_n in the following argument.

Direct calculation gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{h(t)} r^{N-1}u(t,r)\mathrm{d}r &= \int_0^{h(t)} r^{N-1}u_t(t,r)\mathrm{d}r + h^{N-1}(t)h'(t)u(t,h(t)) \\
&= d\int_0^{h(t)} r^{N-1}\Delta u \mathrm{d}r + \int_0^{h(t)} u[\alpha(r)-\beta(r)u]r^{N-1}\mathrm{d}r \\
&= d\int_0^{h(t)} (r^{N-1}u_r(r))_r \mathrm{d}r + \int_0^{h(t)} u[\alpha(r)-\beta(r)u]r^{N-1}\mathrm{d}r \\
&= -\frac{d}{\mu}h^{N-1}(t)h'(t) + \int_0^{h(t)} u[\alpha(r)-\beta(r)u]r^{N-1}\mathrm{d}r.
\end{aligned}$$

Integrating from T to $t>T$ yields

$$\begin{aligned}
\int_0^{h(t)} r^{N-1}u(t,r)\mathrm{d}x &= \int_0^{h(T)} r^{N-1}u(T,r)\mathrm{d}r + \frac{d}{N\mu}(h(T)^N - h(t)^N) \\
&\quad + \int_T^t \int_0^{h(s)} u[\alpha(r)-\beta(r)u]r^{N-1}\mathrm{d}r\mathrm{d}s \\
&\geq \frac{d}{N\mu}(h(T)^N - h(t)^N) + \int_0^{h(T)} r^{N-1}u(T,r)\mathrm{d}r,
\end{aligned}$$

since the fact that $0<u(t,r)\leq \frac{\kappa_1}{\kappa_2}$ for $t\geq T$ and $r\in[0,h(t)]$ implies

$$\alpha(r)-\beta(r)u(t,r) \geq \kappa_1 - \kappa_2 u(t,r) \geq 0 \ \text{ for } t\geq T \text{ and } r\in[0,h(t)].$$

Sending $t\to+\infty$ we obtain, in view of (2.21) and (2.22),

$$\frac{d}{N\mu}(h(T)^N - h_\infty^N) + \int_0^{h(T)} r^{N-1}u(T,r)\mathrm{d}r \leq 0$$

and hence

$$\mu \leq \frac{d\,[(R^*)^N - h(T)^N]}{N\int_0^{h(T)} r^{N-1}u(T,r)\mathrm{d}r}. \tag{2.23}$$

By Lemma 2.6, $u^n(t,x)$ and $h^n(t)$ are increasing in n. Therefore

$$u^n(t,x) \geq u^1(t,x) \ \text{ and } h^n(t) \geq h^1(t).$$

Thus from (2.23) we deduce

$$\mu_n \leq \frac{d\,[(R^*)^N - h^1(T)^N]}{N\int_0^{h^1(T)} r^{N-1}u^1(T,r)\mathrm{d}r}.$$

This contradicts our assumption that $\mu_n\to+\infty$ as $n\to\infty$. □

Lemma 2.9. *Suppose $h_0 < R^*$. Then there exists $\mu_0>0$ depending on u_0 such that vanishing happens if $\mu\leq\mu_0$.*

Proof: We are going to construct a suitable upper solution to (1.4) and then apply Lemma 2.6. For $t > 0$ and $r \in [0, \sigma(t)]$, we define

$$\sigma(t) = h_0(1 + \delta - \frac{\delta}{2} e^{-\gamma t}), \quad w(t, r) = M e^{-\gamma t} V\left(\frac{h_0}{\sigma(t)} r\right),$$

where M, δ, γ are positive constants to be chosen later and $V(|x|)$ is the first eigenfunction of the problem

$$\begin{cases} -d\Delta V = \lambda_1(d, \alpha, h_0)\alpha(|x|)V & \text{in } B_{h_0} \\ V = 0 & \text{on } \partial B_{h_0} \end{cases}$$

with $V \geq 0$ and $\|V\|_\infty = 1$. Since $h_0 < R^*$, we have

$$1 < \lambda_1(d, \alpha, h_0).$$

We also observe that $V'(0) = 0$ and

$$-d(r^{N-1}V')' = r^{N-1}\lambda_1(d, \alpha, h_0)\alpha(r)V > 0 \ \text{ for } 0 < r < h_0$$

imply

$$V'(r) < 0 \text{ for } 0 < r \leq h_0.$$

Set $\tau(t) = 1 + \delta - \frac{\delta}{2}e^{-\gamma t}$ so that $\sigma(t) = h_0\tau(t)$. Direct calculations yield

$$\begin{aligned}
& w_t - d\Delta w - w[\alpha(r) - \beta(r)w] \\
&= M e^{-\gamma t}[-\gamma V - r\tau^{-2}\tau'(t)V' - d\tau^{-2}V'' - d(N-1)r^{-1}\tau^{-1}V' \\
&\qquad -V(\alpha(r) - \beta(r)Me^{-\gamma t}V)] \\
&= M e^{-\gamma t}[-\gamma V - r\tau'(t)\tau^{-2}V' + \tau^{-2}\lambda_1(d, \alpha, h_0)\alpha(\frac{r}{\tau})V \\
&\qquad -V(\alpha(r) - \beta(r)Me^{-\gamma t}V)] \\
&\geq M e^{-\gamma t}V[-\gamma + \tau^{-2}\lambda_1(d, \alpha, h_0)\alpha(\frac{r}{\tau}) - \alpha(r) + \beta(r)Me^{-\gamma t}V] \\
&\geq M e^{-\gamma t}V[-\gamma + \frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2}\alpha(\frac{r}{\tau}) - \alpha(r) + \beta(r)Me^{-\gamma t}V] \\
&= M e^{-\gamma t}V\Big[-\gamma + \Big(\frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2}\frac{\alpha(\frac{r}{\tau})}{\alpha(r)} - 1\Big)\alpha(r) + \beta(r)Me^{-\gamma t}V\Big].
\end{aligned}$$

Clearly

$$1 + \frac{\delta}{2} \leq \tau(t) \leq 1 + \delta, \quad h_0(1 + \frac{\delta}{2}) \leq \sigma(t) \leq h_0(1 + \delta).$$

Hence, due to $1 < \lambda_1(d, \alpha, h_0)$, we can choose $\delta > 0$ sufficiently small such that

$$\varrho := \min_{t > 0, r \in [0, \sigma(t)]} \frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2}\frac{\alpha(\frac{r}{\tau})}{\alpha(r)} - 1 > 0. \tag{2.24}$$

Setting $\gamma = \varrho\kappa_1$, we deduce

$$w_t - d\Delta w - w[\alpha(r) - \beta(r)w] \geq 0 \ \text{ for } t > 0,\ r \in [0, \sigma(t)].$$

We now choose $M > 0$ sufficiently large such that

$$u_0(r) \leq MV\Big(\frac{r}{(1 + \delta/2)}\Big) = w(0, r) \ \text{ for } r \in [0, h_0].$$

We calculate

$$\sigma'(t) = \frac{1}{2} h_0 \gamma \delta e^{-\gamma t},$$

$$-\mu w_r(t, \sigma(t)) = \mu M e^{-\gamma t} \frac{h_0}{\sigma(t)} |V_r(h_0)| \le \mu M e^{-\gamma t} \frac{|V_r(h_0)|}{1 + \delta/2}.$$

Hence if we take

$$\mu_0 = \frac{\delta(1 + \delta/2)\gamma h_0}{2M|V_r(h_0)|},$$

then for any $0 < \mu \le \mu_0$,

$$\sigma'(t) \ge -\mu w_r(t, \sigma(t)),$$

and thus (w, σ) satisfies

$$\begin{cases} w_t - d\Delta w \ge w[\alpha(r) - \beta(r)w], & t > 0,\ 0 < r < \sigma(t), \\ w = 0, \quad \sigma'(t) \ge -\mu w_r, & t > 0,\ r = \sigma(t), \\ w_r(t, 0) = 0, & t > 0, \\ \sigma(0) = (1 + \frac{\delta}{2}) h_0 > h_0. & \end{cases}$$

Hence we can apply Lemma 2.6 to conclude that $h(t) \le \sigma(t)$ and $u(t, r) \le w(t, r)$ for $0 \le r \le h(t)$ and $t > 0$. It follows that $h_\infty \le \lim_{t\to\infty} \sigma(t) = h_0(1 + \delta) < \infty$. □

We are now ready to apply Lemma 2.6 to prove the existence of a threshold value μ^* of μ that governs the alternatives in the spreading-vanishing dichotomy for the case $h_0 < R^*$.

Theorem 2.10. *If $h_0 < R^*$, then there exists $\mu^* > 0$ depending on u_0 such that vanishing occurs if $\mu \le \mu^*$, and spreading happens if $\mu > \mu^*$.*

Proof: This is similar to the proof of Theorem 3.9 in [10]. We give the details below for completeness.

Define $\Sigma := \{\mu > 0 : h_\infty \le R^*\}$. By Lemmas 2.9 and 2.2 we have $\Sigma \supset (0, \mu_0]$. Using Lemma 2.8 we find on the other hand $\Sigma \cap [\mu^0, \infty) = \emptyset$. Therefore $\mu^* := \sup \Sigma \in [\mu_0, \mu^0]$. By this definition and Lemma 2.2, we find that $h_\infty = +\infty$ when $\mu > \mu^*$.

We claim that $\mu^* \in \Sigma$. Otherwise $h_\infty = \infty$ for $\mu = \mu^*$. Hence we can find $T > 0$ such that $h(T) > R^*$. To stress the dependence of the solution (u, h) of (1.4) on μ, we now write (u_μ, h_μ) instead of (u, h). So we have $h_{\mu^*}(T) > R^*$. By the continuous dependence of (u_μ, h_μ) on μ, we can find $\epsilon > 0$ small so that $h_\mu(T) > R^*$ for all $\mu \in [\mu^* - \epsilon, \mu^* + \epsilon]$. It follows that for all such μ,

$$\lim_{t \to +\infty} h_\mu(t) > h_\mu(T) > R^*.$$

This implies that $[\mu^* - \epsilon, \mu^* + \epsilon] \cap \Sigma = \emptyset$, and $\sup \Sigma \le \mu^* - \epsilon$, contradicting the definition of μ^*. This proves our claim that $\mu^* \in \Sigma$.

For $\mu \in (0, \mu^*)$, (u_{μ^*}, h_{μ^*}) is an upper solution of (1.4). Hence we can use Lemma 2.6 to deduce that $h_\mu(t) \le h_{\mu^*}(t)$ for $t > 0$. It follows that

$$\lim_{t\to\infty} h_\mu(t) \le \lim_{t\to\infty} h_{\mu^*}(t) \le R^*.$$

Hence $\mu \in \Sigma$. Thus we have proved that $\Sigma = (0, \mu^*]$. The proof is complete. □

3. Estimates of spreading speed

In this section we estimate the spreading speed of the expanding front $r = h(t)$ when spreading occurs. We will find $0 < k_* \le k^* < \infty$ such that

$$k_* \le \underline{\lim}_{t\to+\infty} \frac{h(t)}{t} \le \overline{\lim}_{t\to+\infty} \frac{h(t)}{t} \le k^*. \tag{3.1}$$

Moreover, if $\alpha(r) \to \alpha_*$ and $\beta(r) \to \beta_*$ as $r \to \infty$, we show that $k_* = k^*$ and hence

$$\lim_{t\to+\infty} \frac{h(t)}{t} = k_*.$$

The constants k_* and k^* are determined through the following result, which is a corrected version of Propositions 4.1 and 4.3 of [10].

Proposition 3.1. *Let $d > 0$ be given in (1.4). For any given constants $a > 0$, $b > 0$ and $k \in [0, 2\sqrt{ad})$, the problem*

$$-dU'' + kU' = aU - bU^2 \quad \text{in } (0,\infty), \quad U(0) = 0 \tag{3.2}$$

admits a unique positive solution $U = U_k = U_{a,b,k}$, and it satisfies $U(r) \to \frac{a}{b}$ as $r \to +\infty$. Moreover, $U_k'(r) > 0$ for $r \ge 0$, $U_{k_1}'(0) > U_{k_2}'(0)$, $U_{k_1}(r) > U_{k_2}(r)$ for $r > 0$ and $k_1 < k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu, a, b) \in (0, 2\sqrt{ad})$ such that $\mu U_{k_0}'(0) = k_0$. Furthermore,

$$\lim_{\frac{a\mu}{bd}\to\infty} \frac{k_0}{\sqrt{ad}} = 2, \quad \lim_{\frac{a\mu}{bd}\to 0} \frac{k_0}{\sqrt{ad}} \frac{bd}{a\mu} = 1/\sqrt{3}. \tag{3.3}$$

It was shown in [10] that $k_0(\mu, a, b)$ is increasing in μ and a, and is decreasing in b. More precisely,

$$\mu_1 \ge \mu_2, \ a_1 \ge a_2 \text{ and } b_1 \le b_2 \ \text{ imply } \ k_0(\mu_1, a_1, b_1) \ge k_0(\mu_2, a_2, b_2),$$

with strict inequality holding when $(\mu_1, a_1, b_1) \ne (\mu_2, a_2, b_2)$. It can also be easily shown that $k_0(\mu, a, b)$ is a continuous function.

By (1.5), we have

$$\alpha^\infty := \overline{\lim}_{r\to+\infty} \alpha(r) \le \kappa_2, \quad \alpha_\infty := \underline{\lim}_{r\to+\infty} \alpha(r) \ge \kappa_1;$$

$$\beta^\infty := \overline{\lim}_{r\to+\infty} \beta(r) \le \kappa_2, \quad \beta_\infty := \underline{\lim}_{r\to+\infty} \beta(r) \ge \kappa_1.$$

We will show that in (3.1), one can take

$$k_* = k_0(\mu, \alpha_\infty, \beta^\infty), \ k^* = k_0(\mu, \alpha^\infty, \beta_\infty).$$

To prove this, we need some preparations. Firstly we need some simple variants of Lemma 2.6 and Remark 2.7.

Lemma 3.2. *Let $d_1(s), d_2(s), a(s), b(s)$ and $l(s)$ be Hölder continuous functions for $s \ge 0$, all positive except possibly $d_2(s)$. Suppose that $T \in (0,\infty)$, $\overline{h} \in C^1([0,T])$, $\overline{u} \in C^{1,2}(D_T^*)$ with $D_T^* = \{(t,r) \in \mathbb{R}^2 : 0 \le t \le T, 0 \le r \le \overline{h}(t)\}$, and*

$$\begin{cases} \overline{u}_t - d_1(r)\overline{u}_{rr} - d_2(r)\overline{u}_r \ge \overline{u}[a(r) - b(r)\overline{u}], & 0 < t \le T, \ 0 < r < \overline{h}(t), \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu \overline{u}_r, & 0 < t \le T, \ r = \overline{h}(t), \\ \overline{u}(t,0) \ge l(t), & 0 < t \le T. \end{cases}$$

If $h \in C^1([0,T])$ *and* $u \in C^{1,2}(D_T)$ *with* $D_T = \{(t,r) \in \mathbb{R}^2 : 0 \le t \le T, 0 \le r \le h(t)\}$ *satisfy*

$$0 < h(0) \le \overline{h}(0),\ 0 < u(0,r) \le \overline{u}(0,r) \text{ for } 0 \le r \le h(0),$$

and

$$(3.4)\qquad \begin{cases} u_t - d_1(r)u_{rr} - d_2(r)u_r = u[a(r) - b(r)u], & t > 0,\ 0 < r < h(t),\\ u = 0, \quad h'(t) = -\mu u_r, & t > 0,\ r = h(t),\\ u(t,0) = l(t), & t > 0, \end{cases}$$

then

$$h(t) \le \overline{h}(t) \text{ in } (0,T],\ u(r,t) \le \overline{u}(r,t) \text{ for } t \in (0,T] \text{ and } r \in (0,h(t)).$$

Proof: This is similar to the proof of Lemma 3.5 in [10]. For small $\epsilon > 0$, let (u_ϵ, h_ϵ) denote the unique solution of (3.4) with $h_0 := h(0)$ replaced by $h_0^\epsilon := h_0(1-\epsilon)$, with μ replaced by $\mu_\epsilon := \mu(1-\epsilon)$, and with $u_\epsilon(0,r) = u_0^\epsilon(r)$ for some $u_0^\epsilon \in C^2([0,h_0^\epsilon])$ satisfying

$$0 < u_0^\epsilon(r) \le u(0,r) \text{ in } [0,h_0^\epsilon],\ u_0^\epsilon(h_0^\epsilon) = 0,$$

and as $\epsilon \to 0$,

$$u_0^\epsilon\left(\frac{h_0}{h_0^\epsilon} r\right) \to u(0,r)$$

in the $C^2([0,h_0])$ norm. The fact that such a unique solution exists can be proved in the same way as for (1.4).

We claim that $h_\epsilon(t) < \overline{h}(t)$ for all $t \in (0,T]$. Clearly this is true for small $t > 0$. If our claim does not hold, then we can find a first $t^* \le T$ such that $h_\epsilon(t) < \overline{h}(t)$ for $t \in (0,t^*)$ and $h_\epsilon(t^*) = \overline{h}(t^*)$. It follows that

$$(3.5)\qquad h_\epsilon'(t^*) \ge \overline{h}'(t^*).$$

We now compare u_ϵ and $\overline{u}$ over the region

$$\Omega_{t^*} := \{(t,r) \in \mathbb{R}^2 : 0 < t \le t^*, 0 \le r < h_\epsilon(t)\}.$$

The strong maximum principle yields $u_\epsilon(t,r) < \overline{u}(t,r)$ in Ω_{t^*}. Hence $w(t,r) := \overline{u}(t,r) - u_\epsilon(t,r) > 0$ in Ω_{t^*} with $w(t^*, h_\epsilon(t^*)) = 0$. It follows that $w_r(t^*, h_\epsilon(t^*)) \le 0$, from which we deduce, in view of $(u_\epsilon)_r(t^*, h(t^*)) < 0$ and $\mu_\epsilon < \mu$, that $h_\epsilon'(t^*) < \overline{h}'(t^*)$. But this contradicts (3.5). This proves our claim that $h_\epsilon(t) < \overline{h}(t)$ for all $t \in (0,T]$. We may now apply the usual comparison principle over Ω_T to conclude that $u_\epsilon < \overline{u}$ in Ω_T.

Since the unique solution (u_ϵ, h_ϵ) depends continuously on the parameter ϵ, as $\epsilon \to 0$, (u_ϵ, h_ϵ) converges to (u,h). The desired result then follows by letting $\epsilon \to 0$ in the inequalities $u_\epsilon < \overline{u}$ and $h_\epsilon < \overline{h}$. □

Similar to Remark 2.7, we have the following analogue of Lemma 3.2:

Lemma 3.3. *Let* $d_1(s), d_2(s), a(s), b(s)$ *and* $l(s)$ *be as in Lemma 3.2. Suppose that* $T \in (0,\infty)$, $\underline{h} \in C^1([0,T])$, $\underline{u} \in C^{1,2}(D_T^\dagger)$ *with* $D_T^\dagger = \{(t,r) \in \mathbb{R}^2 : 0 \le t \le T, 0 \le r \le \underline{h}(t)\}$, *and*

$$\begin{cases} \underline{u}_t - d_1(r)\underline{u}_{rr} - d_2(r)\underline{u}_r \le \underline{u}[a(r) - b(r)\underline{u}], & 0 < t \le T,\ 0 < r < \underline{h}(t),\\ \underline{u} = 0, \quad \underline{h}'(t) \le -\mu \underline{u}_r, & 0 < t \le T,\ r = \underline{h}(t),\\ \underline{u}(t,0) \le l(t), & 0 < t \le T. \end{cases}$$

If $h \in C^1([0,T])$ *and* $u \in C^{1,2}(D_T)$ *satisfy* (3.4) *and*

$$h(0) \ge \underline{h}(0),\ u(0,r) \ge \underline{u}(0,r) \text{ for } 0 \le r \le \underline{h}(0),\ u(0,r) \ge 0 \text{ for } \underline{h}(0) \le r \le h(0),$$

then

$$h(t) \geq \underline{h}(t) \text{ in } (0,T],\ u(r,t) \geq \underline{u}(r,t) \text{ for } t \in (0,T] \text{ and } r \in (0, \underline{h}(t)).$$

We also need the following result:

Lemma 3.4. *Suppose that d, a and b are given positive constants, that $c(s)$ and $l(s)$ are Hölder continuous functions for $s \geq 0$ with $l(s)$ positive, and that $\sigma(t)$ is a continuous positive function for $t \geq 0$. Let $v \in C^{1,2}(D)$ ($D = \{(t,r) : 0 \leq r \leq \sigma(t), t \geq 0\}$) be a solution of*

$$\begin{cases} v_t - dv_{rr} + c(r)v_r = v(a - bv), & t > 0, 0 < r < \sigma(t), \\ v(t,0) = l(t),\ v(t,\sigma(t)) = 0, & t > 0, \\ v(0,r) = v_0(r) \geq 0, & 0 < r < \sigma(0). \end{cases} \tag{3.6}$$

Suppose that

$$\lim_{r\to\infty} c(r) = 0,\ \lim_{t\to\infty} l(t) = l_\infty \in \Big[\frac{a}{b}, \infty\Big),\ \lim_{t\to\infty} \sigma(t) = \infty.$$

Then

$$\underline{\lim}_{t\to\infty} v(t,r) \geq \frac{a}{b} \text{ locally uniformly for } r \in [0,\infty).$$

Proof: By the maximum principle, $v(t,r) > 0$ for $t > 0$ and $0 \leq r < \sigma(t)$. For any given $R > 0$ and small $\epsilon > 0$, we can find $T_R > 0$ such that $\sigma(t) > R$ and $l(t) \geq l_\infty - \epsilon$ for all $t \geq T_R$. We now consider the auxiliary problem

$$\begin{cases} w_t - dw_{rr} + c(r)w_r = w(a - bw), & t > T_R, 0 < r < R, \\ w(t,0) = l_\infty - \epsilon,\ w(t,R) = 0, & t > T_R, \\ w(T_R, r) = v(T_R, r), & 0 < r < R. \end{cases} \tag{3.7}$$

By the comparison principle we have $w(t,r) \leq v(t,r)$ for $t > T_R$ and $0 \leq r \leq R$. By [26], problem (3.7) with initial function $v(T_R, r)$ replaced by 0 has a unique solution $w_*(t,r)$ which is increasing in t, and the same problem with initial function $v(T_R, r)$ replaced by a large constant M has a unique solution $w^*(t,r)$ which is decreasing in t, and moreover, $w_*(t,r) < w^*(t,r)$ for all $t > T_R$ and $0 \leq r \leq R$, and

$$\underline{w}(r) := \lim_{t\to\infty} w_*(t,r),\ \overline{w}(r) := \lim_{t\to\infty} w^*(t,r)$$

are positive solutions of the elliptic problem

$$-dw_{rr} + c(r)w_r = w(a - bw) \text{ for } 0 < r < R,\ w(0) = l_\infty - \epsilon,\ w(R) = 0. \tag{3.8}$$

By Lemma 2.1 of [12], we deduce $\underline{w} = \overline{w} = w_R$, the unique positive solution of (3.8). Hence

$$\lim_{t\to\infty} w_*(t,r) = \lim_{t\to\infty} w^*(t,r) = w_R(r).$$

By the comparison principle, the solution of (3.7) satisfies $w_*(t,r) \leq w(t,r) \leq w^*(t,r)$ for $t \geq T_R$ and $0 \leq r \leq R$. It follows that

$$\lim_{t\to\infty} w(t,r) = w_R(r).$$

From [26] we also know that the above convergence is uniform in r. It follows that

$$\underline{\lim}_{t\to\infty} v(t,r) \geq w_R(r)$$

uniformly in $r \in [0,R]$.

As in [12], one can easily show that as R increases to infinity, $w_R(r)$ increases to the minimal positive solution W of

$$-dw_{rr} + c(r)w_r = w(a - bw) \text{ for } r > 0, \ w(0) = l_\infty - \epsilon. \tag{3.9}$$

It follows that

$$\underline{\lim}_{t\to\infty} v(t,r) \geq W(r) \tag{3.10}$$

locally uniformly in $r \in [0, \infty)$.

We show next that $W(r) \geq \min\{l_\infty - \epsilon, a/b\}$ for $r \geq 0$. We first prove that $W(r) \to a/b$ as $r \to \infty$. Indeed, for any increasing positive sequence $r_n \to \infty$, we define $c_n(r) = c(r_n + r)$ for $-r_n/2 \leq r \leq r_n/2$. Clearly $\|c_n(r)\|_{L^\infty([-r_n/2, r_n/2])} \to 0$ as $n \to \infty$.

Since $a > 0$ and $r_n \to \infty$, for all large n, the logistic problem

$$-dw'' + c_n(r)w' = w(a - bw) \text{ in } (-r_n/2, r_n/2), \ w(-r_n/2) = w(r_n/2) = 0$$

has a unique positive solution w_n. By the comparison principle (see Lemma 2.1 of [12]), we have

$$W(r) \geq w_n(r - r_n) \ \text{ for } r_n/2 < r < 3r_n/2.$$

Hence $W(r_n) \geq w_n(0)$.

On the other hand, by a standard elliptic regularity argument, one finds that, by passing to a subsequence, w_n converges in $C^1_{loc}(\mathbb{R}^1)$ to a positive solution w_∞ of

$$-dw'' = w(a - bw) \text{ in } \mathbb{R}^1.$$

By Theorem 1.1 of [12], we find that $w_\infty \equiv a/b$. Hence the entire sequence w_n converges to a/b and

$$\underline{\lim}_{n\to\infty} W(r_n) \geq \lim_{n\to\infty} w_n(0) = a/b.$$

If we use w^n to denote the unique positive solution of the boundary blow-up problem

$$-dw'' + c_n(r)w' = w(a - bw) \text{ in } (-r_n/2, r_n/2), \ w(-r_n/2) = w(r_n/2) = \infty,$$

we similarly deduce $W(r_n) \leq w^n(0)$ and $w^n(0) \to a/b$ as $n \to \infty$. Thus

$$\overline{\lim}_{n\to\infty} W(r_n) \leq a/b.$$

Hence we must have $\lim_{r\to\infty} W(r) = a/b$.

For any small $\delta > 0$, define $v_\delta = \min\{l_\infty - \epsilon, a/b - \delta\}$. Then for all large R, $v_\delta \leq W(r)$ for $r \in \{0, R\}$, and

$$0 = -dv''_\delta < v_\delta(a - bv_\delta).$$

Hence we can apply Lemma 2.1 of [12] to conclude that $W(r) \geq v_\delta$ in $[0, R]$. Since R can be arbitrarily large, this implies that $W(r) \geq \min\{l_\infty - \epsilon, a/b - \delta\}$ for all $r \geq 0$. Since $\delta > 0$ can be arbitrarily small, we finally deduce $W(r) \geq \min\{l_\infty - \epsilon, a/b\}$ for $r \geq 0$.

We may now use (3.10) to obtain

$$\underline{\lim}_{t\to\infty} v(t,r) \geq \min\{l_\infty - \epsilon, a/b\}.$$

Since $\epsilon > 0$ can be arbitrarily small, this implies

$$\underline{\lim}_{t\to\infty} v(t,r) \geq \min\{l_\infty, a/b\} = a/b.$$

□

Remark 3.5. *Under the assumptions of Lemma 3.4, it can be shown that*

$$\lim_{t\to\infty} v(t,r) = V(r) \text{ locally uniformy for } r \in [0,\infty),$$

where V is the unique positive solution of

$$-dV'' = V(a-bV) \text{ in } (0,\infty),\ V(0) = l_\infty.$$

This conclusion is not needed in this paper though.

We are now ready to prove the main result of this section.

Theorem 3.6. *If $h_\infty = +\infty$, then*

$$\overline{\lim}_{t\to+\infty}\frac{h(t)}{t} \le k_0(\mu,\alpha^\infty,\beta_\infty),\ \underline{\lim}_{t\to+\infty}\frac{h(t)}{t} \ge k_0(\mu,\alpha_\infty,\beta^\infty). \tag{3.11}$$

Proof: By Theorem 1.1 of [8], the unique positive solution $\hat{U}$ of (2.16) satisfies

$$\overline{\lim}_{r\to\infty}\hat{U}(r) \le \frac{\alpha^\infty}{\beta_\infty},\ \underline{\lim}_{r\to\infty}\hat{U}(r) \ge \frac{\alpha_\infty}{\beta^\infty}.$$

For any $\varepsilon > 0$, there is $R := R(\varepsilon) > 1$ such that for $r \ge R$,

$$\alpha(r) \le \alpha^\infty_\varepsilon := \alpha^\infty + \varepsilon, \quad \alpha(r) \ge \alpha^\varepsilon_\infty := \alpha_\infty - \varepsilon,$$

$$\beta(r) \le \beta^\infty_\varepsilon := \beta^\infty + \varepsilon, \quad \beta(r) \ge \beta^\varepsilon_\infty := \beta_\infty - \varepsilon,$$

and

$$\frac{\alpha_\infty^{\varepsilon/2}}{\beta^\infty_{\varepsilon/2}} < \hat{U}(r) < \frac{\alpha^\infty_{\varepsilon/2}}{\beta_\infty^{\varepsilon/2}}.$$

Since $h_\infty = +\infty$ and $\lim_{t\to\infty} u(t,r) = \hat{U}(r)$, there exists $T := T(R) > 0$ such that

$$h(T) > 3R \text{ and } u(t+T, 2R) < \frac{\alpha^\infty_\varepsilon}{\beta^\varepsilon_\infty} \text{ for all } t \ge 0.$$

Setting

$$\tilde{u}(t,r) = u(t+T, r+2R) \text{ and } \tilde{h}(t) = h(t+T) - 2R,$$

and denoting

$$\tilde{\Delta}u = u_{rr} + \frac{N-1}{r+2R}u_r,$$

we obtain

$$\begin{cases} \tilde{u}_t - d\tilde{\Delta}\tilde{u} = \tilde{u}\left[\alpha(r+2R) - \beta(r+2R)\tilde{u}\right], & t>0,\ 0<r<\tilde{h}(t),\\ \tilde{u}(t,0) = u(t+T,2R), \quad \tilde{u}(t,\tilde{h}(t)) = 0, & t>0,\\ \tilde{h}'(t) = -\mu\tilde{u}_r(\tilde{h}(t)), & t>0,\\ \tilde{u}(0,r) = u(T, r+2R), & 0<r<\tilde{h}(0). \end{cases} \tag{3.12}$$

By our choice of R, for $r \ge 0$,

$$\alpha(r+2R) \le \alpha^\infty_\varepsilon, \quad \beta(r+2R) \ge \beta^\varepsilon_\infty.$$

Let $u^*(t)$ be the unique solution of the problem

$$\frac{du^*}{dt} = u^*(\alpha^\infty_\varepsilon - \beta^\varepsilon_\infty u^*) \quad \text{for } t>0;\ u^*(0) = \max\{\alpha^\infty_\varepsilon/\beta^\varepsilon_\infty, ||\tilde{u}(0,\cdot)||_\infty\}. \tag{3.13}$$

Then

$$u^*(t) \geq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon} \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} u^*(t) = \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}.$$

Now we have

$$u^*(0) \geq \tilde{u}(0,r),\ \tilde{u}(t,0) \leq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon} \leq u^*(t),\ \tilde{u}(t,\tilde{h}(t)) = 0 \leq u^*(t),$$

and

$$u_t^* - d\tilde{\Delta}u^* = u^*(\alpha_\varepsilon^\infty - \beta_\infty^\varepsilon u^*) \geq u^*[\alpha(r+2R) - \beta(r+2R)u^*].$$

Hence we can apply the comparison principle to deduce

$$\tilde{u}(t,r) \leq u^*(t) \text{ for } 0 < r < \tilde{h}(t),\ t > 0. \tag{3.14}$$

As a consequence, there exists $\tilde{T} = \tilde{T}_\varepsilon > 0$ such that

$$\tilde{u}(t,r) \leq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}(1-\varepsilon)^{-1}, \quad \forall t \geq \tilde{T},\ r \in [0, \tilde{h}(t)].$$

Let $U_\varepsilon := U_{\alpha_\varepsilon^\infty, \beta_\infty^\varepsilon, k^\varepsilon}$ denote the unique positive solution of (3.2) with $a = \alpha_\varepsilon^\infty$, $b = \beta_\infty^\varepsilon$ and $k = k^\varepsilon := k_0(\mu, \alpha_\varepsilon^\infty, \beta_\infty^\varepsilon)$. Since $U_\varepsilon(r) \to \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}$ as $r \to +\infty$, there exists $R_0 := R_0(\varepsilon) > 2R$ such that

$$U_\varepsilon(r) > \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}(1-\varepsilon) \text{ for } r \geq R_0.$$

We now define

$$\xi(t) = (1-\varepsilon)^{-2}k^\varepsilon t + R_0 + \tilde{h}(\tilde{T}) \quad \text{for } t \geq 0,$$
$$w(t,r) = (1-\varepsilon)^{-2}U_\varepsilon(\xi(t) - r) \text{ for } t \geq 0,\ 0 \leq r \leq \xi(t).$$

Then

$$\xi'(t) = (1-\varepsilon)^{-2}k^\varepsilon,$$

$$-\mu w_r(t, \xi(t)) = \mu(1-\varepsilon)^{-2}U_\varepsilon'(0) = (1-\varepsilon)^{-2}k^\varepsilon,$$

and so we have

$$\xi'(t) = -\mu w_r(t, \xi(t)).$$

Clearly

$$w(t, \xi(t)) = 0.$$

Moreover, for $0 \leq r \leq \tilde{h}(\tilde{T})$,

$$w(0,r) = (1-\varepsilon)^{-2}U_\varepsilon(\xi(0) - r) \geq (1-\varepsilon)^{-2}U_\varepsilon(R_0) \geq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}(1-\varepsilon)^{-1} \geq \tilde{u}(\tilde{T}, r)$$

and $w(0,r) > 0$ for $\tilde{h}(\tilde{T}) < r < \xi(0)$. It is also easily seen that for $t > 0$,

$$w(t,0) = (1-\varepsilon)^{-2}U_\varepsilon(\xi(t)) \geq (1-\varepsilon)^{-2}U_\varepsilon(R_0) \geq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}(1-\varepsilon)^{-1} \geq \tilde{u}(t+\tilde{T}, 0).$$

Direct calculations show that, for $t > 0$ and $0 < r < \xi(t)$,

$$\begin{aligned} w_t - d\tilde{\Delta} w &= (1-\varepsilon)^{-2}\left[U_\varepsilon'\xi' - dU_\varepsilon'' + d(N-1)(r+2R)^{-1}U_\varepsilon'\right] \\ &= (1-\varepsilon)^{-2}[(1-\varepsilon)^{-2}k^\varepsilon U_\varepsilon' - dU_\varepsilon'' + d(N-1)(r+2R)^{-1}U_\varepsilon'] \\ &\geq (1-\varepsilon)^{-2}(k^\varepsilon U_\varepsilon' - dU_\varepsilon'') \quad \text{(due to } U_\varepsilon' \geq 0) \\ &= (1-\varepsilon)^{-2}(\alpha_\varepsilon^\infty U_\varepsilon - \beta_\infty^\varepsilon U_\varepsilon^2) \\ &= \alpha_\varepsilon^\infty w - (1-\varepsilon)^2\beta_\infty^\varepsilon w^2 \\ &\geq \alpha_\varepsilon^\infty w - \beta_\infty^\varepsilon w^2. \end{aligned}$$

Hence we can use Lemma 3.2 to conclude that

$$\tilde{u}(t+\tilde{T}, r) \leq w(t,r), \quad \tilde{h}(t+\tilde{T}) \leq \xi(t) \;\text{ for }\; t \geq 0,\; 0 \leq r \leq \tilde{h}(t+\tilde{T}).$$

It follows that

$$\overline{\lim}_{t\to+\infty}\frac{h(t)}{t} = \overline{\lim}_{t\to+\infty}\frac{\tilde{h}(t)}{t} \leq \lim_{t\to+\infty}\frac{\xi(t-\tilde{T})}{t} = k^\varepsilon(1-\varepsilon)^{-2}.$$

Since $\varepsilon > 0$ can be arbitrarily small, and $k^\varepsilon \to k_0(\mu, \alpha^\infty, \beta_\infty)$ as $\varepsilon \to 0$, we deduce

$$\overline{\lim}_{t\to+\infty}\frac{h(t)}{t} \leq k_0(\mu, \alpha^\infty, \beta_\infty).$$

Next we show

$$\underline{\lim}_{t\to+\infty}\frac{h(t)}{t} \geq k_0(\mu, \alpha_\infty, \beta^\infty)$$

by constructing a suitable lower solution. To this end, we denote

$$k_\varepsilon = k_0(\mu, \alpha_\infty^\varepsilon, \beta_\varepsilon^\infty) \;\text{ and }\; V_\varepsilon = U_{\alpha_\infty^\varepsilon, \beta_\varepsilon^\infty, k_\varepsilon}.$$

Then consider the auxiliary problem

$$\left\{\begin{array}{ll} v_t - d\tilde{\Delta} v = v(\alpha_\infty^\varepsilon - \beta_\varepsilon^\infty v), & t > 0,\; 0 < r < \tilde{h}(t), \\ v(t,0) = \tilde{u}(t,0), \quad v(t,\tilde{h}(t)) = 0, & t > 0, \\ v(0,r) = \tilde{u}(0,r), & r \in [0, \tilde{h}(0)], \end{array}\right. \tag{3.15}$$

where $\tilde{u}$ and $\tilde{h}$ are defined as before. Since

$$\lim_{t\to\infty}\tilde{u}(t,0) \to \hat{U}(2R) > \frac{\alpha_\infty^{\varepsilon/2}}{\beta_{\varepsilon/2}^\infty},$$

we can apply Lemma 3.4 to (3.15) to conclude that

$$\underline{\lim}_{t\to+\infty} v(t,r) \geq \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty} \;\text{ locally uniformly for } r \in [0,\infty). \tag{3.16}$$

Since

$$\alpha(r+2R) \geq \alpha_\infty^\varepsilon, \quad \beta(r+2R) \leq \beta_\varepsilon^\infty,$$

from the comparison principle we deduce

$$\tilde{u}(t,r) \geq v(t,r) \;\text{ for } t > 0,\, r \in [0, \tilde{h}(t)],$$

and hence, in view of (3.16), we have

$$\underline{\lim}_{t\to+\infty}\tilde{u}(t,r) \geq \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty} \;\text{ locally uniformly for } r \in [0,\infty). \tag{3.17}$$

Define

$$\eta(t) = (1-\varepsilon)^2 k_\varepsilon t + \tilde{h}(0) \quad \text{for } t \geq 0,$$

and

$$w(t,r) = (1-\varepsilon)^2 V_\varepsilon(\eta(t) - r) \quad \text{for } \ t \geq 0, \ \ 0 \leq r \leq \eta(t).$$

Then

$$\eta'(t) = (1-\varepsilon)^2 k_\varepsilon,$$
$$-\mu w_r(t, \eta(t)) = \mu(1-\varepsilon)^2 V_\varepsilon'(0) = (1-\varepsilon)^2 k_\varepsilon,$$

and so we have

$$\eta'(t) = -\mu w_r(t, \eta(t)).$$

Clearly, $w(t, \eta(t)) = 0$. Since $V_\varepsilon'(r) > 0$ for $r > 0$ and

$$\lim_{r\to+\infty} V_\varepsilon(r) = \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty},$$

we must have

$$V_\varepsilon(r) < \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty} \text{ for } r > 0.$$

Therefore, due to (3.17) we can find some $\hat{T} = \hat{T}(\varepsilon) > 0$ such that

$$\tilde{u}(t + \hat{T}, 0) \geq w(t, 0) \ \text{ for } t \geq 0 \tag{3.18}$$

and

$$\tilde{u}(\hat{T}, r) \geq w(0, r) \quad \text{for } r \in [0, \eta(0)]. \tag{3.19}$$

Direct calculations yield

$$\begin{aligned}
w_t - d\tilde{\Delta} w &= (1-\varepsilon)^2 V_\varepsilon' \eta' - d(1-\varepsilon)^2 \Big[V_\varepsilon'' - \frac{N-1}{r+2R} V_\varepsilon'\Big] \\
&= (1-\epsilon)^2 \Big[(1-\varepsilon)^2 V_\varepsilon' k_\varepsilon - dV_\varepsilon'' + \frac{d(N-1)}{r+2R} V_\varepsilon'\Big] \\
&= (1-\varepsilon)^2 \Big[\Big((1-\varepsilon)^2 k_\varepsilon + \frac{d(N-1)}{r+2R}\Big) V_\varepsilon' - dV_\varepsilon''\Big] \\
&\leq (1-\varepsilon)^2 [k_\varepsilon V_\varepsilon' - dV_\varepsilon''] \quad\ \ (\text{since } V_\varepsilon' \geq 0) \\
&\leq w[\alpha_\infty^\varepsilon - \beta_\varepsilon^\infty w] \ \ \text{ for } t \geq 0,\, 0 \leq r \leq \eta(t),
\end{aligned}$$

where we have used the fact that for large R,

$$(1-\varepsilon)^2 k_\varepsilon + \frac{d(N-1)}{r+2R} \leq k_\varepsilon.$$

Hence, we can use Lemma 3.3 to conclude that

$$\tilde{u}(t+\hat{T}, r) \geq w(t,r), \ \ \tilde{h}(t+\hat{T}) \geq \eta(t) \quad \text{for } t \geq 0,\, 0 \leq r \leq \eta(t).$$

It follows that

$$\underline{\lim}_{t\to+\infty} \frac{h(t)}{t} = \underline{\lim}_{t\to+\infty} \frac{\tilde{h}(t)}{t} \geq \lim_{t\to+\infty} \frac{\eta(t-\hat{T})}{t} = (1-\varepsilon)^2 k_\varepsilon.$$

Since $\varepsilon > 0$ can be arbitrarily small, this implies

$$\underline{\lim}_{t\to+\infty} \frac{h(t)}{t} \geq k_0(\mu, \alpha_\infty, \beta^\infty).$$

The proof of the theorem is now complete. □

The result below follows trivially from Theorem 3.6.

Corollary 3.7. *Assume that* $h_\infty = +\infty$ *and*

$$\alpha(r) \to \alpha_*, \quad \beta(r) \to \beta_* \ \text{ as } r \to +\infty, \tag{3.20}$$

then

$$\lim_{t\to+\infty} \frac{h(t)}{t} = k_0(\mu, \alpha_*, \beta_*).$$

4. Proof of Theorem 2.1

In this section, we prove the existence and uniqueness of a global solution to (1.4). The proof follows that of [10] with suitable modifications. Firstly we prove the local existence and uniqueness result by the contraction mapping theorem, then we use suitable estimates to show that the solution is defined for all $t > 0$.

Theorem 4.1. *For any given* u_0 *satisfying* (1.6) *and any constant* $\nu \in (0, 1)$, *there is a* $T > 0$ *such that problem* (1.4) *admits a unique solution*

$$(u, h) \in C^{(1+\nu)/2,1+\nu}(D_T) \times C^{1+\nu/2}([0, T]);$$

moreover,

$$\|u\|_{C^{(1+\nu)/2,1+\nu}(D_T)} + \|h\|_{C^{1+\nu/2}([0,T])} \le C, \tag{4.1}$$

where $D_T = \{(t, r) \in \mathbb{R}^2 : t \in [0, T],\ r \in [0, h(t)]\}$, C *and* T *only depend on* h_0, ν *and* $\|u_0\|_{C^2([0,h_0])}$.

Proof: As in [10], we first follow [7] and straighten the free boundary. Let $\zeta(s)$ be a function in $C^3[0, \infty)$ satisfying

$$\zeta(s) = 1 \ \text{ if } |s - h_0| < \frac{h_0}{8},\ \zeta(s) = 0 \ \text{ if } |s - h_0| > \frac{h_0}{2}, \quad |\zeta'(s)| < \frac{5}{h_0} \text{ for all } s.$$

Consider the transformation

$$(t, y) \to (t, x) \ \text{ where } x = y + \zeta(|y|)(h(t) - h_0)y/|y|,\ y \in \mathbb{R}^N,$$

which induces the transformation

$$(t, s) \to (t, r) \ \text{ with } r = s + \zeta(s)(h(t) - h_0), \quad 0 \le s < \infty.$$

For fixed $t \ge 0$, as long as

$$|h(t) - h_0| \le \frac{h_0}{8},$$

the transformation $x \to y$ determined above is a diffeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$, and the induced transformation $s \to r$ is a diffeomotphism from $[0, \infty)$ onto $[0, \infty)$.

Moreover, it changes the free boundary $|x| = h(t)$ to the fixed sphere $|y| = h_0$. Now, direct calculations show that

$$\begin{aligned}
\frac{\partial s}{\partial r} = \frac{1}{1+\zeta'(s)(h(t)-h_0)} &:= \sqrt{A(h(t),s)},\\
\frac{\partial^2 s}{\partial r^2} = -\frac{\zeta''(s)(h(t)-h_0)}{[1+\zeta'(s)(h(t)-h_0)]^3} &:= B(h(t),s),\\
-\frac{1}{h'(t)}\frac{\partial s}{\partial t} = \frac{\zeta(s)}{1+\zeta'(s)(h(t)-h_0)} &:= C(h(t),s).
\end{aligned}$$

Let us denote

$$\begin{aligned}
\frac{(N-1)\sqrt{A}}{s+\zeta(s)(h(t)-h_0)} &:= D(h(t),s),\\
\alpha(s+\zeta(s)(h(t)-h_0)) &:= \tilde{\alpha}(h(t),s),\\
\beta(s+\zeta(s)(h(t)-h_0)) &:= \tilde{\beta}(h(t),s).
\end{aligned}$$

If we set

$$u(t,r) = u(t, s+\zeta(s)(h(t)-h_0)) = w(t,s),$$

then

$$u_t = w_t - h'(t)C(h(t),s)w_s, \quad u_r = \sqrt{A(h(t),s)}w_s,$$
$$u_{rr} = A(h(t),s)w_{ss} + B(h(t),s)w_s$$

and the free boundary problem (1.4) becomes

$$\tag{4.2} \begin{cases} w_t - Adw_{ss} - (Bd + h'C + Dd)w_s = w(\tilde{\alpha} - \tilde{\beta}w), & t>0,\ 0<s<h_0,\\ w=0, \quad h'(t) = -\mu w_s, & t>0,\ s=h_0,\\ w_s(t,0)=0, & t>0,\\ h(0)=h_0, \quad w(0,s)=u_0(s), & 0\le s\le h_0, \end{cases}$$

where $A = A(h(t),s)$, $B = B(h(t),s)$, $C = C(h(t),s)$, $D = D(h(t),s)$, $\tilde{\alpha} = \tilde{\alpha}(h(t),s)$ and $\tilde{\beta} = \tilde{\beta}(h(t),s)$.

Denote $\tilde{h}_0 = -\mu u_0'(h_0)$, and for $0 < T \le \frac{h_0}{8(1+\tilde{h}_0)}$, define $\Delta_T = [0,T]\times[0,h_0]$,

$$\begin{aligned}
\mathcal{D}_{1T} &= \{w \in C(\Delta_T) : w(0,s) = u_0(s),\ ||w-u_0||_{C(\Delta_T)} \le 1\},\\
\mathcal{D}_{2T} &= \{h \in C^1([0,T]) :\ h(0)=h_0,\ h'(0)=\tilde{h}_0,\ \|h'-\tilde{h}_0\|_{C([0,T])} \le 1\}.
\end{aligned}$$

It is easily seen that $\mathcal{D} := \mathcal{D}_{1T}\times\mathcal{D}_{2T}$ is a complete metric space with the metric

$$d((w_1,h_1),(w_2,h_2)) = ||w_1-w_2||_{C(\Delta_T)} + ||h_1'-h_2'||_{C([0,T])}.$$

Let us note that for $h_1, h_2 \in \mathcal{D}_{2T}$, due to $h_1(0) = h_2(0) = h_0$,

$$\tag{4.3} \|h_1-h_2\|_{C([0,T])} \le T||h_1'-h_2'||_{C([0,T])}.$$

Next, we shall prove the existence and uniqueness result by using the contraction mapping theorem. Firstly we observe that due to our choice of T, for any given $(w,h) \in \mathcal{D}_{1T}\times\mathcal{D}_{2T}$, we have

$$|h(t)-h_0| \le T(1+\tilde{h}_0) \le \frac{h_0}{8}.$$

Therefore the transformation $(t,s)\to(t,r)$ introduced at the beginning of the proof is well-defined. Moreover, for $0\le s\le h_0/2$, we have $\zeta(s)\equiv 0$ and hence for such s,

$$A\equiv 1,\ B\equiv C\equiv 0,\ D\equiv (N-1)/s.$$

Therefore

$$-Adw_{ss}-(Bd+h'C+Dd)w_s=-d\Delta w \text{ in the ball } |y|\le h_0/2.$$

Thus although $D=D(h(t),s)$ is singular at $s=0$,

$$Adw_{ss}+(Bd+h'C+Dd)w_s$$

actually represents an elliptic operator acting on $w=w(t,y)(=w(t,|y|))$ over the ball $|y|\le h_0$, whose coefficients are continuous in (t,y) when $h\in\mathcal{D}_{2T}$.

Applying standard L^p theory and then the Sobolev imbedding theorem ([19]), we find that for any $(w,h)\in\mathcal{D}$, the following initial boundary value problem

$$\begin{cases} \overline{w}_t-Ad\overline{w}_{ss}-(Bd+h'C+Dd)\overline{w}_s=w(\tilde\alpha-\tilde\beta w), & t>0,\ 0\le s<h_0,\\ \overline{w}_s(t,0)=0,\quad \overline{w}(t,h_0)=0, & t>0,\\ \overline{w}(0,s)=u_0(s), & 0\le s\le h_0 \end{cases} \tag{4.4}$$

admits a unique solution $\overline{w}\in C^{(1+\nu)/2,1+\nu}(\Delta_T)$, and

$$||\overline{w}||_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}\le C_1, \tag{4.5}$$

where C_1 is a constant dependent on h_0, ν and $||u_0||_{C^2[0,h_0]}$.

Setting

$$\overline{h}(t):=h_0-\int_0^t\mu\overline{w}_s(\tau,h_0)d\tau, \tag{4.6}$$

we have

$$\overline{h}'(t)=-\mu\overline{w}_s(t,h_0),\ \overline{h}(0)=h_0,\ \overline{h}'(0)=-\mu\overline{w}_s(0,h_0)=\tilde h_0,$$

and hence $\overline{h}'\in C^{\nu/2}([0,T])$ with

$$||\overline{h}'||_{C^{\nu/2}([0,T])}\le C_2:=\mu C_1. \tag{4.7}$$

We now define $\mathcal{F}:\mathcal{D}\to C(\Delta_T)\times C^1([0,T])$ by

$$\mathcal{F}(w,h)=(\overline{w},\overline{h}).$$

Clearly $(w,h)\in\mathcal{D}$ is a fixed point of $\mathcal{F}$ if and only if it solves (4.2).

By (4.5) and (4.7), we have

$$||\overline{h}'-\tilde h_0||_{C([0,T])}\le||\overline{h}'||_{C^{\nu/2}([0,T])}T^{\nu/2}\le\mu C_1T^{\nu/2},$$

$$||\overline{w}-u_0||_{C(\Delta_T)}\le||\overline{w}-u_0||_{C^{(1+\nu)/2,0}(\Delta_T)}T^{(1+\nu)/2}\le C_1T^{(1+\nu)/2}.$$

Therefore if we take $T\le\min\{(\mu C_1)^{-2/\nu},\ C_1^{-2/(1+\nu)}\}$, then $\mathcal{F}$ maps $\mathcal{D}$ into itself.

Next we prove that $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$ for $T>0$ sufficiently small. Indeed, let $(w_i,h_i)\in\mathcal{D}$ $(i=1,2)$ and denote $(\overline{w}_i,\overline{h}_i)=\mathcal{F}(w_i,h_i)$. Then it follows from (4.5) and (4.7) that

$$||\overline{w}_i||_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}\le C_1,\quad ||\overline{h}_i'(t)||_{C^{\nu/2}([0,T])}\le C_2.$$

Setting $W=\overline{w}_1-\overline{w}_2$, we find that $W(t,s)$ satisfies

$$
\begin{aligned}
&W_t-A(h_2,s)dW_{ss}-[B(h_2,s)d+h_2'C(h_2,s)+D(h_2,s)d]W_s\\
&\quad=[A(h_1,s)-A(h_2,s)]d\overline{w}_{1,ss}+[B(h_1,s)-B(h_2,s)+D(h_1,s)-D(h_2,s)]d\overline{w}_{1,s}\\
&\qquad+[h_1'C(h_1,s)-h_2'C(h_2,s)]\overline{w}_{1,s}+(w_1-w_2)(\tilde{\alpha}(h_1,s)-\tilde{\beta}(h_1,s)(w_1+w_2))\\
&\qquad+w_2[(\tilde{\alpha}(h_1,s)-\tilde{\alpha}(h_2,s))-(\tilde{\beta}(h_1,s)-\tilde{\beta}(h_2,s))w_2],\ t>0,\ 0\le s<h_0,\\
&W_s(t,0)=0,\quad W(t,h_0)=0,\quad t>0,\\
&W(0,s)=0,\quad 0\le s\le h_0.
\end{aligned}
$$

Using the L^p estimates for parabolic equations and Sobolev's imbedding theorem, we obtain

$$
||\overline{w}_1-\overline{w}_2||_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}\le C_3(||w_1-w_2||_{C(\Delta_T)}+||h_1-h_2||_{C^1([0,T])}), \tag{4.8}
$$

where C_3 depends on C_1, C_2 and the functions A, B, C and D in the definition of the transformation $(t,s)\to(t,r)$. Taking the difference of the equations for $\overline{h}_1, \overline{h}_2$ results in

$$
||\overline{h}_1'-\overline{h}_2'||_{C^{\nu/2}([0,T])}\le \mu\Big(||\overline{w}_{1,s}-\overline{w}_{2,s}||_{C^{\nu/2,0}(\Delta_T)}\Big). \tag{4.9}
$$

Combining (4.3), (4.8) and (4.9), and assuming $T\le 1$, we obtain

$$
\begin{aligned}
&||\overline{w}_1-\overline{w}_2||_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}+||\overline{h}_1'-\overline{h}_2'||_{C^{\nu/2}([0,T])}\\
&\quad\le C_4(||w_1-w_2||_{C(\Delta_T)}+||h_1'-h_2'||_{C[0,T]}),
\end{aligned}
$$

with C_4 depending on C_3 and μ. Hence for

$$
T:=\min\left\{1,\left(\frac{1}{2C_4}\right)^{2/\nu},(\mu C_1)^{-2/\nu},C_1^{-2/(1+\nu)},\frac{h_0}{8(1+\tilde{h}_0)}\right\},
$$

we have

$$
\begin{aligned}
&||\overline{w}_1-\overline{w}_2||_{C(\Delta_T)}+||\overline{h}_1'-\overline{h}_2'||_{C([0,T])}\\
&\quad\le T^{(1+\nu)/2}||\overline{w}_1-\overline{w}_2||_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}+T^{\nu/2}||\overline{h}_1'-\overline{h}_2'||_{C^{\nu/2}([0,T])}\\
&\quad\le C_4T^{\nu/2}(||w_1-w_2||_{C(\Delta_T)}+||h_1'-h_2'||_{C([0,T])})\\
&\quad\le \frac{1}{2}(||w_1-w_2||_{C(\Delta_T)}+||h_1'-h_2'||_{C([0,T])}).
\end{aligned}
$$

This shows that for this T, $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$. It now follows from the contraction mapping theorem that $\mathcal{F}$ has a unique fixed point (w,h) in $\mathcal{D}$. Moreover, by the Schauder estimates, we have additional regularity for (w,h) as a solution of (4.2), namely, $h\in C^{1+\nu/2}(0,T]$ and $w\in C^{1+\nu/2,2+\nu}((0,T]\times[0,h_0])$, and (4.5), (4.7) hold. In other words, $(w(t,s),h(t))$ is a unique local classical solution of the problem (4.2). □

To show that the local solution obtained in Theorem 4.1 can be extended to all $t>0$, we need the following estimate.

Lemma 4.2. *Let (u,h) be a solution to problem (1.4) defined for $t\in(0,T_0)$ for some $T_0\in(0,+\infty]$. Then there exist constants C_1 and C_2 independent of T_0 such that*

$$
0<u(t,r)\le C_1,\quad 0<h'(t)\le C_2\ \text{ for } 0\le r<h(t),\ t\in(0,T_0).
$$

Proof: Using the strong maximum principle to the equation of u we immediately obtain

$$u(t,r)>0,\ \ u_r(t,h(t))<0 \quad \text{for } 0<t<T_0, 0\le r<h(t).$$

Hence $h'(t)>0$ for $t\in(0,T_0)$.

Since (1.5) holds, it follows from the comparison principle that $u(t,r)\le \overline{u}(t)$ for $t\in(0,T_0)$ and $r\in[0,h(t)]$, where

$$\overline{u}(t):=\frac{\kappa_2}{\kappa_1}e^{\frac{\kappa_2}{\kappa_1}t}\Big(e^{\frac{\kappa_2}{\kappa_1}t}-1+\frac{\kappa_2}{\kappa_1||u_0||_\infty}\Big)^{-1},$$

which is the solution of the problem

$$\frac{d\overline{u}}{dt}=\overline{u}(\kappa_2-\kappa_1\overline{u}) \quad t>0;\ \overline{u}(0)=||u_0||_\infty. \tag{4.10}$$

Thus we have

$$u(t,r)\le C_1:=\sup_{t\ge 0}\overline{u}(t).$$

It remains to show that $h'(t)\le C_2$ for all $t\in(0,T_0)$ and some C_2 independent of T_0. To this end, we define

$$\Omega=\Omega_M:=\{(t,r):0<t<T_0,\ h(t)-M^{-1}<r<h(t)\}$$

and construct an auxiliary function

$$w(t,r):=C_1[2M(h(t)-r)-M^2(h(t)-r)^2].$$

We will show that M can be chosen so that $w(t,r)\ge u(t,r)$ holds over Ω.

Direct calculations show that, for $(t,r)\in\Omega$,

$$\begin{aligned} w_t &= 2C_1Mh'(t)[1-M(h(t)-r)]\ge 0,\\ -w_r &= 2MC_1[1-M(h(t)-r)]\ge 0,\\ -w_{rr} &= 2C_1M^2,\quad w[\alpha(r)-\beta(r)w]\le \kappa_2C_1. \end{aligned}$$

It follows that

$$w_t-d(w_{rr}+\frac{N-1}{r}w_r)\ge 2dC_1M^2\ge \kappa_2 u \ \text{ in } \Omega$$

if $M^2\ge\frac{\kappa_2}{2d}$. On the other hand,

$$w(t,h(t)-M^{-1})=C_1\ge u(t,h(t)-M^{-1}),\quad w(t,h(t))=0=u(t,h(t)).$$

Thus, if we can choose M such that $u_0(r)\le w(0,r)$ for $r\in[h_0-M^{-1},h_0]$, then we can apply the maximum principle to $w-u$ over Ω to deduce that $u(t,r)\le w(t,r)$ for $(t,r)\in\Omega$. It would then follow that

$$u_r(t,h(t))\ge w_r(t,h(t))=-2MC_1,\ h'(t)=-\mu u_r(t,h(t))\le C_2:=2MC_1\mu.$$

To complete the proof, we only have to find some M independent of T_0 such that $u_0(r)\le w(0,r)$ for $r\in[h_0-M^{-1},h_0]$. We calculate

$$w_r(0,r)=-2C_1M[1-M(h_0-r)]\le -C_1M \text{ for } r\in[h_0-(2M)^{-1},h_0].$$

Therefore upon choosing

$$M:=\max\left\{\sqrt{\frac{\kappa_2}{2d}},\frac{4\|u_0\|_{C^1([0,h_0])}}{3C_1}\right\},$$

we will have

$$w_r(0,r) \le u_0'(r) \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Since $w(0,h_0) = u_0(h_0) = 0$, the above inequality implies

$$w(0,r) \ge u_0(r) \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Moreover, for $r \in [h_0 - M^{-1}, h_0 - (2M)^{-1}]$, we have

$$w(0,r) \ge \frac{3}{4}C_1,\ u_0(r) \le \|u_0\|_{C^1([0,h_0])} M^{-1} \le \frac{3}{4}C_1.$$

Therefore $u_0(r) \le w(0,r)$ for $r \in [h_0 - M^{-1}, h_0]$. This completes the proof. □

Theorem 4.3. *The solution of problem* (1.4) *exists and is unique for all* $t \in (0,\infty)$.

Proof: Let $[0, T_{max})$ be the maximal time interval in which the solution exists. By Theorem 4.1, $T_{max} > 0$. It remains to show that $T_{max} = \infty$. Arguing indirectly, we assume that $T_{max} < \infty$. By Lemma 4.2, there exist C_1 and C_2 independent of T_{max} such that for $t \in [0, T_{\max})$ and $r \in [0, h(t)]$,

$$0 \le u(t,r) \le C_1,\ h_0 \le h(t) \le h_0 + C_2 t, 0 \le h'(t) \le C_2.$$

We now fix $\delta_0 \in (0, T_{max})$ and $\tilde{T} > T_{max}$. By standard parabolic regularity, we can find $C_3 > 0$ depending only on δ_0, $\tilde{T}$, C_1 and C_2 such that $||u(t,\cdot)||_{C^2([0,h(t)])} \le C_3$ for $t \in [\delta_0, T_{\max})$. It then follows from the proof of Theorem 4.1 that there exists a $\tau > 0$ depending only on C_3, C_2 and C_1 such that the solution of problem (1.4) with initial time $T_{max} - \tau/2$ can be extended uniquely to the time $T_{max} - \tau/2 + \tau$. But this contradicts the assumption. The proof is complete. □

Remark 4.4. *It follows from the uniqueness of the solution to* (1.4) *and some standard compactness argument that the unique solution* (u,h) *depends continuously on* u_0 *and the parameters appearing in* (1.4).

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