Some Corrections on the paper of Y. Du and Z.G. Lin [SIAM J. Math. Anal., 42(2010), 377-405]

In Proposition 4.1 of [1] it is stated that, for any given constants $a > 0$, $b > 0$, $d > 0$ and $k \geq 0$, the problem

$$-dU'' + kU' = aU - bU^2 \quad \text{in } (0, \infty), \quad U(0) = 0 \tag{1}$$

admits a unique positive solution $U = U_k$, and it satisfies $U(r) \to \frac{a}{b}$ as $r \to +\infty$. Moreover, $U_k'(r) > 0$ for $r \geq 0$, $U_{k_1}'(0) > U_{k_2}'(0)$, $U_{k_1}(r) > U_{k_2}(r)$ for $r > 0$ and $k_1 < k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu) > 0$ such that $\mu U_{k_0}'(0) = k_0$. The number k_0 is the asymptotic spreading speed for the invasive species determined by the free boundary model of [1] and [2].

This statement is incorrect as stated, since the conclusions only hold for k in the range $0 \leq k < 2\sqrt{ad}$. The correct version is the following:

Proposition 1. *For any given constants $a > 0$, $b > 0$, $d > 0$ and $k \in [0, 2\sqrt{ad})$, problem (1) admits a unique positive solution $U = U_k$, and it satisfies $U(r) \to \frac{a}{b}$ as $r \to +\infty$. Moreover, $U_k'(r) > 0$ for $r \geq 0$, $U_{k_1}'(0) > U_{k_2}'(0)$, $U_{k_1}(r) > U_{k_2}(r)$ for $r > 0$ and $k_1 < k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu) \in (0, 2\sqrt{ad})$ such that $\mu U_{k_0}'(0) = k_0$.*

Proof. For large $l > 0$ and $k \in [0, 2\sqrt{ad})$, we consider the problem

$$-dU'' + kU' = aU - bU^2, \quad 0 < x < l, \quad U(0) = U(l) = 0. \tag{2}$$

Define

$$\lambda = \frac{k}{\sqrt{ad}} \quad \text{and } W(x) = \frac{b}{a} e^{-\frac{\lambda}{2}x} U\Big(\sqrt{\frac{d}{a}}\, x\Big).$$

Then (2) is changed to the equivalent problem

$$-W'' = \Big(1 - \frac{\lambda^2}{4}\Big) W - e^{\frac{\lambda}{2}x} W^2 \text{ in } (0, \tilde{l}),\ W(0) = W(\tilde{l}) = 0, \tag{3}$$

where

$$\tilde{l} := \sqrt{\frac{a}{d}}\, l.$$

By our assumption on k, we find that $1 - \frac{\lambda^2}{4} > 0$, and hence for all large l, the logistic type problem (3) has a unique positive solution W^l, which in turn defines a unique positive solution U^l for (2).

The proof now proceeds as in the proof of Proposition 4.1 in [1] until the discussion for the function

$$\sigma(k) := k - \mu U_k'(0)$$

near the end of the proof there. The function $\sigma(k)$ now is defined for $k \in [0, 2\sqrt{ad})$, and by the monotonicity of $U_k'(0)$ on k, we know that $\sigma(k)$ is strictly increasing with $\sigma(0) < 0$. Therefore there exists a unique $k_0 \in (0, 2\sqrt{ad})$ such that $\sigma(k_0) = 0$ provided that we can show

$$\lim_{k \nearrow 2\sqrt{ad}} \sigma(k) > 0. \tag{4}$$

To complete the proof, it remains to prove (4). To this end, we choose an arbitrary sequence $\{k_n\}$ of positive numbers that increases to $2\sqrt{ad}$ as $n \to \infty$ and consider the corresponding sequence of solutions $\{U_{k_n}\}$. Since $0 < U_{k_n}(x) < a/b$ for $x \in (0,\infty)$, and $U_{k_n}(x)$ is decreasing in n, the limit

$$U_*(x) := \lim_{n\to\infty} U_{k_n}(x)$$

exists and satisfies $0 \leq U_*(x) < a/b$ for $x \in (0,\infty)$. Moreover, applying standard L^p estimates to the equation for U_{k_n} and then the Sobolev embedding theorem, we easily see that $U_{k_n} \to U_*$ in $C^1_{loc}([0,\infty))$. Hence U_* satisfies in the week sense (and also classical sense)

$$-dU_*'' + 2\sqrt{ad}U_*' = aU_* - bU_*^2 \text{ in } (0,\infty),\ U_*(0) = 0.$$

Define

$$W_*(x) = \frac{b}{a}e^{-x}U_*\Big(\sqrt{\frac{d}{a}}\,x\Big).$$

Then one readily checks that

$$W_*'' = e^x W_*^2,\ 0 \leq W_* < 1 \text{ in } (0,\infty),\ W_*(0) = 0.$$

We claim that $U_* \equiv 0$. Otherwise by the strong maximum principle we have $U_*(x) > 0$ for $x > 0$ and $U_*'(0) > 0$. It follows that

$$W_*''(x) = e^x W_*^2(x) > 0 \text{ for } x > 0, \text{ and } W_*'(0) > 0.$$

This implies that $W_*(x) \to \infty$ as $x \to \infty$, a contradiction to the fact that $W_*(x) < 1$ for $x > 0$. Thus we must have $U_* \equiv 0$.

Since $\{k_n\}$ is an arbitrary sequence increasing to $2\sqrt{ad}$, the above discussion shows that $U_k \to 0$ as $k \to 2\sqrt{ad}$ in $C^1_{loc}([0,\infty))$. In particular, $U_k'(0) \to 0$ as $k \to 2\sqrt{ad}$. Therefore

$$\lim_{k\nearrow 2\sqrt{ad}} \sigma(k) = 2\sqrt{ad} > 0,$$

as we wanted. □

We next correct the other results of [1] affected by the mistake of Proposition 4.1 there. These are only concerned with some conclusions on the asymptotic behavior of the spreading speed k_0 when some of the parameters are large or small; for example, the second half of Proposition 4.3 in [1] needs to be changed. The rest of [1] remain valid as they are not affected by the mistake in Proposition 4.1 there.

By Proposition 1 above, for any $\lambda \in [0,2)$, the problem

$$-V'' + \lambda V' = V - V^2 \text{ in } (0,\infty),\ V(0) = 0$$

has a unique positive solution V_λ, and for each $\alpha > 0$, the equation

$$\lambda = \alpha V_\lambda'(0)$$

has a unique solution $\lambda = \lambda_0(\alpha) \in (0,2)$.

From the proof of Proposition 1 we know that the function

$$\eta(\lambda) := V_\lambda'(0)$$

is strictly decreasing (and continuous) for $\lambda \in [0,2)$, and

$$\eta(0) > 0,\ \eta(2-0) = 0.$$

Hence for each fixed $\alpha > 0$, $(\lambda_0(\alpha), \lambda_0(\alpha)/\alpha)$ is the unique intersection point of the increasing line $\eta = \frac{1}{\alpha}\lambda$ with the decreasing curve $\eta = \eta(\lambda)$ in the $\eta-\lambda$ plane. Clearly

$$\lim_{\alpha\to 0}(\lambda_0(\alpha), \lambda_0(\alpha)/\alpha) = (0, \eta(0)),\ \lim_{\alpha\to\infty}(\lambda_0(\alpha), \lambda_0(\alpha)/\alpha) = (2, 0). \tag{5}$$

A simple calculation confirms that for each $k > 0$,

$$V_{\frac{k}{\sqrt{ad}}}(x) = \frac{b}{a}U_k\Big(\sqrt{\frac{d}{a}}\,x\Big).$$

Hence

$$V'_{\frac{k}{\sqrt{ad}}}(0) = \frac{b}{a}\sqrt{\frac{d}{a}}\,U_k'(0),$$

and $\mu U_k'(0) = k$ is equivalent to

$$\frac{a\mu}{bd}V'_{\frac{k}{\sqrt{ad}}}(0) = \frac{k}{\sqrt{ad}}.$$

It follows that

$$\frac{k_0}{\sqrt{ad}} = \lambda_0\big(\frac{a\mu}{bd}\big).$$

We can now use (5) to obtain

$$\lim_{\frac{a\mu}{bd}\to\infty}\frac{k_0}{\sqrt{ad}} = 2,\ \lim_{\frac{a\mu}{bd}\to 0}\frac{k_0}{\sqrt{ad}}\frac{bd}{a\mu} = \eta(0).$$

We show next that $\eta(0) = 1/\sqrt{3}$. Indeed, by definition, $\eta(0) = V_0'(0)$ and V_0 satisfies

$$-V_0'' = V_0 - V_0^2,\ V_0 > 0 \text{ in } (0,\infty),\ V_0(0) = 0,\ V_0(\infty) = 1.$$

Hence

$$\int_0^\infty(-V_0'')V_0'dx = \int_0^\infty(V_0 - V_0^2)V_0'dx.$$

We have

$$\int_0^\infty(-V_0'')V_0'dx = V_0'(0)^2/2,$$

and

$$\int_0^\infty(V_0 - V_0^2)V_0'dx = \int_0^1(v - v^2)dv = 1/6.$$

Therefore

$$V_0'(0) = 1/\sqrt{3}.$$

Summarizing, we have proved the following result.

Proposition 2. *Let k_0 be the spreading speed determined by Proposition 1. Then*

$$\lim_{\frac{a\mu}{bd}\to\infty}\frac{k_0}{\sqrt{ad}} = 2,\ \lim_{\frac{a\mu}{bd}\to 0}\frac{k_0}{\sqrt{ad}}\frac{bd}{a\mu} = 1/\sqrt{3}.$$

Let us note that Proposition 2 indicates that when the quantity $\frac{a\mu}{bd}$ is large, then the spreading speed k_0 is well approximated by the formula

$$k_0 \approx 2\sqrt{ad},$$

while when this quantity is small, k_0 is well approximated by the formula

$$k_0 \approx \frac{\mu a}{\sqrt{3}b}\sqrt{\frac{a}{d}}.$$

In particular, for fixed $a, b, \mu > 0$, if we regard k_0 as a function of d, then

$$\lim_{d\to 0} \frac{k_0}{\sqrt{d}} = 2\sqrt{a}, \ \lim_{d\to\infty} k_0\sqrt{d} = \frac{\mu a^{3/2}}{\sqrt{3}\,b}.$$

These should be compared with Proposition 4.3 in [1].

Finally we remark that from Proposition 1 above we always have $0 < k_0 < 2\sqrt{ad}$. That is the spreading speed determined by the free boundary model is always smaller than that determined by the traveling wave solution approach.

References

[1] Y. Du and Z.G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal., 42(2010), 377-405.

[2] Y. Du and Z. Guo, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, II, Preprint, 2009.