

# SPREADING-VANISHING DICHOTOMY IN A DIFFUSIVE LOGISTIC MODEL WITH A FREE BOUNDARY, II

YIHONG DU<sup>†</sup> AND ZONGMING GUO<sup>‡</sup>

ABSTRACT. We study the diffusive logistic equation with a free boundary in higher space dimensions and heterogeneous environment. Such a model may be used to describe the spreading of a new or invasive species, with the free boundary representing the expanding front. For simplicity, we assume that the environment and the solution are radially symmetric. In the special case of one space dimension and homogeneous environment, this free boundary problem was investigated in [8]. We prove that the spreading-vanishing dichotomy established in [8] still holds in the more general and ecologically more realistic setting considered here. Moreover, when spreading occurs, we obtain upper and lower bounds for the spreading speed of the expanding front, with both bounds coinciding to give the asymptotic spreading speed if the environment is asymptotically homogeneous at infinity. Our results also indicate that the asymptotic spreading speed determined by this model does not depend on the spatial dimension.

## 1. INTRODUCTION

An important problem in invasion ecology is to understand the nature of spreading of the invasive species. It is well known that many animal species spread to their new environment in a linear fashion, namely the spreading radius eventually exhibits a linear growth curve against time ([23, 18]). This phenomenon seems first observed by Skellam [24] in examining the spreading of muskrat in Europe in the early 1900s: He calculated the area of the muskrat range from a map obtained from field data, took the square root (which gives the spreading radius) and plotted it against years, and found that the data points lay on a straight line. Several mathematical models have been proposed to describe this phenomenon and one may find many in [23].

One of the most successful mathematical approaches to this problem is based on the investigation of front propagation governed by the following diffusive logistic

---

*Date:* Jan. 2010.

1991 *Mathematics Subject Classification.* 35K20, 35R35, 35J60, 92B05.

*Key words and phrases.* Diffusive logistic equation, free boundary, spreading-vanishing dichotomy, invasive population, spreading speed.

This work was supported by the Australian Research Council and also by NSFC 10871060 and NSFC 10671172 of China.

<sup>†</sup> Department of Mathematics, School of Science and Technology, University of New England, Armidale, NSW 2351, Australia, and Department of Mathematics, Qufu Normal University, China.

<sup>‡</sup> Department of Mathematics, Henan Normal University, Xinxiang, 453007, China.

**Emails:** ydu@turing.une.edu.au (Y. Du), gzm@htu.cn (Z. Guo).

equation over the entire space  $\mathbb{R}^N$ :

$$(1.1) \quad u_t - d\Delta u = u(a - bu), \quad t > 0, \quad x \in \mathbb{R}^N.$$

Here  $u = u(t, x)$  may be regarded as the population density of a spreading species with dispersal rate  $d$ , intrinsic growth rate  $a$  and habitat carrying capacity  $1/b$ . In the pioneering works of Fisher [11] and Kolmogorov et al [14], for space dimension  $N = 1$ , traveling wave solutions have been found for (1.1): For any  $|c| \geq c^* := 2\sqrt{ad}$ , there exists a solution  $u(t, x) := W(x - ct)$  with the property that

$$W'(y) < 0 \text{ for } y \in \mathbb{R}^1, \quad W(-\infty) = a/b, \quad W(+\infty) = 0;$$

no such solution exists if  $|c| < c^*$ . The number  $c^*$  is called the minimal speed of the traveling waves. Fisher [11] claims that  $c^*$  is the spreading speed for the advantaged gene in his research, and used a probabilistic argument to support his claim. Skellam [24] was able to use a linear model (i.e., (1.1) with  $b = 0$ ) and a similar probabilistic argument to show that  $c^*$  should be the speed of spreading. A clearer description and rigorous proof of this fact were given by Aronson and Weinberger (see Section 4 in [1]), who showed that for a new population  $u(t, x)$  (governed by the above logistic equation) with initial distribution  $u(0, x)$  confined to a compact set of  $x$  (i.e.,  $u(0, x) = 0$  outside a compact set), one has

$$\lim_{t \rightarrow \infty, |x| \leq (c^* - \epsilon)t} u(t, x) = a/b, \quad \lim_{t \rightarrow \infty, |x| \geq (c^* + \epsilon)t} u(t, x) = 0$$

for any small  $\epsilon > 0$ . These results have been extended to higher dimensions in [2], and extensive further development on traveling wave solutions and the spreading speed has been achieved in several directions; we refer to [27, 12, 25, 3, 4, 16, 26] and the references therein for more details.

A shortcoming of this approach is that it predicts successful spreading and establishment of the new species with any nontrivial initial population  $u(0, x)$  (namely  $u(t, x) \rightarrow a/b$  as  $t \rightarrow \infty$ ), regardless of its initial size and supporting area. This is in sharp contrast to numerous empirical evidences; for example, the introduction of several bird species from Europe to North America in the 1900s was successful only after many initial attempts.

Recently, Du and Lin [8] used a free boundary model to study the spreading of species in one space dimension, which does not have this shortcoming. The model in [8] is given by the following diffusive logistic problem:

$$(1.2) \quad \begin{cases} u_t - du_{xx} = u(a - bu), & t > 0, \quad 0 < x < h(t), \\ u_x(t, 0) = 0, \quad u(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ h(0) = h_0, \quad u(0, x) = u_0(x), & 0 \leq x \leq h_0, \end{cases}$$

where  $x = h(t)$  is the moving boundary to be determined,  $h_0, \mu, d, a$  and  $b$  are given positive constants, and the initial function  $u_0(x)$  satisfies

$$(1.3) \quad u_0 \in C^2([0, h_0]), \quad u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \text{ in } [0, h_0].$$

Here  $u(t, x)$  stands for the population density of a new or invasive species over a one dimensional habitat, and the initial function  $u_0(x)$  stands for the population of the

species in the very early stage of its introduction, which occupies an initial region  $[0, h_0]$ . It is assumed that the species can only invade further into the environment from the right end of the initial region, and the spreading front expands at a speed that is proportional to the population gradient at the front, which gives rise to the Stefan condition  $h'(t) = -\mu u_x(t, h(t))$ . It was shown in [8] that (1.2) has a unique solution  $(u(t, x), h(t))$  defined for all  $t > 0$ , with  $u(t, x) > 0$  and  $h'(t) > 0$ . Moreover, a spreading-vanishing dichotomy holds for (1.2), namely, as time  $t \rightarrow \infty$ , the population  $u(t, x)$  either successfully establishes itself in the new environment (called spreading), in the sense that  $h(t) \rightarrow \infty$  and  $u(t, x) \rightarrow a/b$ , or the population fails to establish and vanishes eventually (called vanishing), namely  $h(t) \rightarrow h_\infty \leq \frac{\pi}{2} \sqrt{\frac{d}{a}}$  and  $u(t, x) \rightarrow 0$ . Furthermore, when spreading occurs, for large time, the spreading speed approaches a positive constant  $k_0$ , i.e.,  $h(t) = [k_0 + o(1)]t$  as  $t \rightarrow \infty$ . The asymptotic spreading speed  $k_0$  is uniquely determined by an auxiliary elliptic problem induced from (1.2) (given by Proposition 4.1 in [8]), and is independent of the initial population size  $u_0$ . The criteria for spreading or vanishing are the following: If the initial occupying area  $[0, h_0]$  is beyond a critical size, namely  $h_0 \geq \frac{\pi}{2} \sqrt{\frac{d}{a}}$ , then regardless of the initial population size  $u_0(x)$  (satisfying (1.3)), spreading always happens. On the other hand, if  $h_0 < \frac{\pi}{2} \sqrt{\frac{d}{a}}$ , then whether spreading or vanishing occurs is determined by the initial population size  $u_0$  and the coefficient  $\mu$  in the Stefan condition (assuming the other parameters are fixed). We showed that for such  $h_0$ , with each given  $u_0$ , there exists a critical  $\mu^* > 0$  depending on  $u_0$ , such that spreading occurs if  $\mu > \mu^*$  and vanishing happens when  $\mu \leq \mu^*$ .

If the left boundary  $x = 0$  in (1.2) is replaced by a free boundary  $x = g(t)$  governed by  $g'(t) = -\mu u_x(t, g(t))$ , it was proved in [8] that a similar spreading-vanishing dichotomy holds, and in the case of spreading, both the left front  $x = g(t)$  and the right front  $x = h(t)$  expand to infinity at the same asymptotic speed  $k_0$ .

The main purpose of this paper is to show that most of these results of [8] continue to hold in the more realistic situation of higher space dimensions and heterogeneous environment. For simplicity, we assume that the environment and the solution are radially symmetric. (The general case will be considered in a future paper.) So we will study the behavior of the positive solution  $u(t, r)$ ,  $r = |x|$ ,  $x \in \mathbb{R}^N$  ( $N \geq 2$ ), to the following problem,

$$(1.4) \quad \begin{cases} u_t - d\Delta u = u(\alpha(r) - \beta(r)u), & t > 0, 0 < r < h(t), \\ u_r(t, 0) = 0, u(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t, h(t)), & t > 0, \\ h(0) = h_0, u(0, r) = u_0(r), & 0 \leq r \leq h_0, \end{cases}$$

where  $\Delta u = u_{rr} + \frac{N-1}{r}u_r$ ,  $r = h(t)$  is the moving boundary to be determined,  $h_0$ ,  $\mu$  and  $d$  are given positive constants,  $\alpha, \beta \in C^{\nu_0}([0, \infty))$  for some  $\nu_0 \in (0, 1)$ , and there are positive constants  $\kappa_1 \leq \kappa_2$  such that

$$(1.5) \quad \kappa_1 \leq \alpha(r) \leq \kappa_2, \quad \kappa_1 \leq \beta(r) \leq \kappa_2 \quad \text{for } r \in [0, \infty).$$

The initial function  $u_0(r)$  satisfies

$$(1.6) \quad u_0 \in C^2([0, h_0]), \quad u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \text{ in } [0, h_0).$$

Thus problem (1.4) describes the spreading of a new or invasive species with population density  $u(t, |x|)$  over an  $N$ -dimensional habitat, which is radially symmetric but heterogeneous. The initial function  $u_0(|x|)$  stands for the population in the very early stage of its introduction, which occupies an initial region  $B_{h_0}$ . Here and in what follows we use  $B_R$  to stand for the ball with center at 0 and radius  $R$ . The spreading front is represented by the free boundary  $|x| = h(t)$ , which is the  $N$ -dimensional sphere  $\partial B_{h(t)}$  whose radius  $h(t)$  grows at a speed that is proportional to the population gradient at the front:  $h'(t) = -\mu u_r(t, h(t))$ . The coefficient function  $\alpha(|x|)$  represents the intrinsic growth rate of the species,  $\beta(|x|)$  measures its intra-specific competition, and  $d$  is the dispersal rate.

In Section 2 below, we first state the global existence and uniqueness result for (1.4) (Theorem 2.1), then we prove the spreading-vanishing dichotomy (Theorem 2.4) and obtain sharp thresholds that govern the alternatives in the dichotomy (Theorems 2.5 and 2.10). The proof of Theorem 2.1 is postponed to Section 4, since it is rather long and is only a modification of the proof in section 2 of [8], which in turn follows the approach in [6]. In Section 3, we obtain sharp estimates for the spreading speed, namely sharp bounds for  $\underline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t}$  and  $\overline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t}$  (see Theorem 3.6). These bounds are determined by an auxiliary elliptic equation over the half line  $[0, \infty)$  (see Proposition 3.1), which arises naturally from the original problem, and was first introduced in [8]. If the environment is asymptotically homogeneous at infinity, namely,  $\alpha(r) \rightarrow \alpha_*$  and  $\beta(r) \rightarrow \beta_*$  as  $r \rightarrow +\infty$ , these bounds coincide and hence the limit of  $\frac{h(t)}{t}$  exists as  $t \rightarrow +\infty$ .

Though the outline of the approach in this paper largely follows that of [8], most of the technical proofs here are different from and much more involved than the corresponding ones in [8], and some of the results here are proved by completely different methods.

We now briefly compare (1.4) with (1.1). Firstly, the spreading-vanishing dichotomy for (1.4) appears more realistic than the persistent spreading predicted by (1.1). We refer to [8] for several ecological evidences in support of this conclusion. Secondly, while considerable success has been achieved in extending the theories of traveling wave solutions and spreading speed for (1.1) to the case of heterogeneous environment which has some kind of periodicity properties, it appears very difficult to extend these theories to the case of a general heterogeneous environment. In contrast, our investigation in this paper demonstrates that, such an extension for (1.2) is relatively easy, at least for the radially symmetric case. Finally we notice that, while (1.1) gives an asymptotic spreading speed of  $2\sqrt{ad}$  (for large time), which is independent of  $b$  and is increasing with the dispersal rate  $d$ , we will show that the asymptotic spreading speed of (1.4) depends on all the parameters and on  $\alpha(r), \beta(r)$  in (1.4), and in sharp contrast, it is not increasing with respect to  $d$  (at least for large  $d$ ). It is unclear whether or not the spreading speed obtained here is more

realistic than the one determined by (1.1), and we refer to section 6 of [8] for some discussions of this point.

Similar free boundary conditions to the one in (1.2) have been used in ecological models over *bounded* spatial domains in several earlier papers; see for example, [19, 20, 21], [13], [17]. But the purposes of these papers are very different from ours.

Our results can be easily extended to cover a more general reaction term  $f(r, u)$  which behaves like  $\alpha(r)u - \beta(r)u^2$ . We leave this to the interested reader.

## 2. THE SPREADING-VANISHING DICHOTOMY

In this section we prove the spreading-vanishing dichotomy. Though our approach here mainly follows the lines of [8], considerable changes in the proofs are needed, since the situation here is more general and difficult.

The following existence uniqueness result can be proved by adequately modifying the arguments in section 2 of [8]. So we state the result here but postpone its proof to section 4 below.

**Theorem 2.1.** *Problem (1.4) has a unique solution  $(u(t, r), h(t))$ , which is defined for all  $t > 0$ . Moreover,  $u(t, r) > 0$ ,  $h'(t) > 0$  for  $t > 0$  and  $0 \leq r < h(t)$ , and  $h \in C^1([0, \infty))$ ,  $u \in C^{1,2}(D)$ , with  $D = \{(t, r) : t > 0, 0 \leq r \leq h(t)\}$ .*

It follows from Theorem 2.1 that  $r = h(t)$  is monotonic increasing and therefore there exists  $h_\infty \in (0, +\infty]$  such that  $\lim_{t \rightarrow +\infty} h(t) = h_\infty$ .

Let  $\lambda_1(d, \alpha, R)$  be the principal eigenvalue of the problem

$$(2.1) \quad \begin{cases} -d\Delta\phi = \lambda\alpha(|x|)\phi & \text{in } B_R \\ \phi = 0 & \text{on } \partial B_R. \end{cases}$$

It is well-known that  $\lambda(d, \alpha, \cdot)$  is a strictly decreasing continuous function and

$$\lim_{R \rightarrow 0^+} \lambda_1(d, \alpha, R) = +\infty, \quad \lim_{R \rightarrow +\infty} \lambda_1(d, \alpha, R) = 0.$$

Therefore, for fixed  $d > 0$  and  $\alpha \in C^{\nu_0}([0, \infty))$ , there is a unique  $R^* := R^*(d, \alpha)$  such that

$$(2.2) \quad \lambda_1(d, \alpha, R^*) = 1$$

and

$$1 > \lambda_1(d, \alpha, R) \text{ for } R > R^*; \quad 1 < \lambda_1(d, \alpha, R) \text{ for } R < R^*.$$

The spreading-vanishing dichotomy is a consequence of the following two lemmas.

**Lemma 2.2.** *If  $h_\infty < +\infty$ , then  $h_\infty \leq R^*$ , and  $\lim_{t \rightarrow +\infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0$ .*

**Proof:** We first prove that  $h_\infty \leq R^*$ . Otherwise  $h_\infty > R^*$  and there exists  $T > 0$  such that  $h(t) > R^*$  for all  $t \geq T$ . Thus,

$$1 > \lambda_1(d, \alpha, h(t)) \text{ for all } t \geq T.$$

Moreover, for any sufficiently small  $\varepsilon > 0$ , there is  $T_0 := T_0(\varepsilon) > T$  such that for  $t \geq T_0$ ,

$$R^* < h_\infty - \varepsilon < h(t) < h_\infty.$$

Consider the problem

$$(2.3) \quad \begin{cases} w_t - d\Delta w = w(\alpha(r) - \beta(r)w), & t \geq T_0, \quad r \in [0, h_\infty - \varepsilon], \\ w_r(t, 0) = 0 \quad w(t, h_\infty - \varepsilon) = 0, & t \geq T_0, \\ w(T_0, r) = u(T_0, r), & r \in [0, h_\infty - \varepsilon]. \end{cases}$$

This is a logistic problem with  $1 > \lambda_1(d, \alpha, h_\infty - \varepsilon)$ . It is well-known (see, for example, Proposition 3.3 in [5]) that (2.3) admits a unique positive solution  $\underline{w} = \underline{w}_\varepsilon(t, r)$ . Moreover,

$$(2.4) \quad \underline{w}(t, \cdot) \rightarrow V_{h_\infty - \varepsilon} \text{ in } C^2([0, h_\infty - \varepsilon]) \text{ as } t \rightarrow \infty,$$

where  $V_{h_\infty - \varepsilon}(r)$  is the unique positive (radial) solution of the problem

$$(2.5) \quad \begin{cases} -d\Delta V = V(\alpha(r) - \beta(r)V) & \text{in } B_{h_\infty - \varepsilon}, \\ V = 0 & \text{on } \partial B_{h_\infty - \varepsilon}. \end{cases}$$

By the comparison principle

$$(2.6) \quad u(t, r) \geq \underline{w}(t, r) \quad \text{for } t > T_0, \quad r \in [0, h_\infty - \varepsilon].$$

This implies that

$$(2.7) \quad \underline{\lim}_{t \rightarrow +\infty} u(t, r) \geq V_{h_\infty - \varepsilon}(r) \quad \text{for } r \in [0, h_\infty - \varepsilon].$$

On the other hand, consider the problem

$$(2.8) \quad \begin{cases} w_t - d\Delta w = w(\alpha(r) - \beta(r)w), & t \geq T_0, \quad r \in [0, h_\infty], \\ w_r(t, 0) = 0 \quad w(t, h_\infty) = 0, & t \geq T_0, \\ w(T_0, r) = \tilde{u}(T_0, r), & r \in [0, h_\infty], \end{cases}$$

where

$$\tilde{u}(T_0, r) = \begin{cases} u(T_0, r) & \text{for } r \in [0, h(T_0)], \\ 0 & \text{for } r \in (h(T_0), h_\infty]. \end{cases}$$

Similarly (2.8) admits a unique positive solution  $\bar{w}(t, r)$  with

$$(2.9) \quad \bar{w}(t, \cdot) \rightarrow V_{h_\infty} \text{ in } C^2([0, h_\infty]) \text{ as } t \rightarrow +\infty,$$

where  $V_{h_\infty}$  is the unique positive (radial) solution of the problem

$$(2.10) \quad \begin{cases} -d\Delta V = V(\alpha(r) - \beta(r)V) & \text{in } B_{h_\infty}, \\ V = 0 & \text{on } \partial B_{h_\infty}. \end{cases}$$

Meanwhile, the comparison principle implies that

$$(2.11) \quad u(t, r) \leq \bar{w}(t, r) \quad \text{for } t > T_0, \quad r \in [0, h(t)]$$

and hence

$$(2.12) \quad \overline{\lim}_{t \rightarrow +\infty} u(t, r) \leq V_{h_\infty}(r) \quad \text{for } r \in [0, h_\infty].$$

By a standard compactness and uniqueness argument, we can easily show that

$$V_{h_\infty - \varepsilon} \rightarrow V_{h_\infty} \text{ in } C_{loc}^2([0, h_\infty]) \text{ as } \varepsilon \rightarrow 0^+.$$

Thus, (2.7), (2.12) and the arbitrariness of  $\varepsilon$  imply

$$(2.13) \quad \lim_{t \rightarrow \infty} u(t, r) = V_{h_\infty}(r) \quad \text{for } r \in [0, h_\infty].$$

This and the regularity of  $u(t, r)$  also imply that

$$\|u(t, \cdot) - V_{h_\infty}\|_{C^2([0, h(t)])} \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Therefore, as  $t \rightarrow \infty$ ,

$$u_r(t, h(t)) \rightarrow V'_{h_\infty}(h_\infty) < 0.$$

This contradicts the fact that

$$u_r(t, h(t)) = -\frac{1}{\mu}h'(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Therefore we must have  $h_\infty \leq R^*$ .

We are now ready to show that  $\|u(t, \cdot)\|_{C([0, h(t)])} \rightarrow 0$  as  $t \rightarrow \infty$ . Let  $\bar{u}(t, r)$  denote the unique positive solution of the problem

$$(2.14) \quad \begin{cases} \bar{u}_t - d\Delta\bar{u} = \bar{u}[\alpha(r) - \beta(r)\bar{u}], & t > 0, 0 < r < h_\infty, \\ \bar{u}_r(t, 0) = 0, \quad \bar{u}(t, h_\infty) = 0, & t > 0, \\ \bar{u}(0, r) = \tilde{u}_0(r), & 0 < r < h_\infty, \end{cases}$$

where

$$\tilde{u}_0(r) = \begin{cases} u_0(r), & 0 \leq r \leq h_0, \\ 0, & r \geq h_0. \end{cases}$$

The comparison principle gives  $0 \leq u(t, r) \leq \bar{u}(t, r)$  for  $t > 0$  and  $r \in [0, h(t)]$ . Since  $h_\infty \leq R^*$ , we have  $1 \leq \lambda_1(d, \alpha, h_\infty)$  and it follows from a well-known conclusion on the logistic problem (2.14) that  $\bar{u}(t, r) \rightarrow 0$  uniformly for  $r \in [0, h_\infty]$  as  $t \rightarrow +\infty$  (see, for example, Corollary 3.4 in [5]). Thus  $\lim_{t \rightarrow +\infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0$ .  $\square$

**Lemma 2.3.** *If  $h_\infty = +\infty$ , then*

$$(2.15) \quad \lim_{t \rightarrow +\infty} u(t, r) = \hat{U}(r) \text{ locally uniformly for } r \in [0, +\infty),$$

where  $\hat{U}(|x|)$  is the unique positive (radial) solution of the equation

$$(2.16) \quad -d\Delta u = u[\alpha(|x|) - \beta(|x|)u] \text{ in } \mathbb{R}^N.$$

**Proof:** The existence and uniqueness of a positive solution of (2.16) follows from Theorem 2.3 of [9] (by choosing both  $\gamma$  and  $\tau$  there to be 0). It must be radially symmetric since (2.16) is invariant under rotations around the origin of  $\mathbb{R}^N$ .

To show (2.15), we use a squeezing argument introduced in [10]. We first consider the Dirichlet problem

$$-d\Delta v = v[\alpha(r) - \beta(r)v], \quad v(R) = 0,$$

and the boundary blow-up problem

$$-d\Delta w = w[\alpha(r) - \beta(r)w], \quad w(R) = +\infty.$$

When  $R$  is large, it is well-known that these problems have positive radial solutions  $v_R$  and  $w_R$ , respectively. By the comparison principle given in [10], as  $R \rightarrow +\infty$ ,  $v_R$  increases to the unique positive solution  $\hat{U}$  of (2.16) and  $w_R$  decreases to  $\hat{U}$ .

Choose an increasing sequence of positive number  $R_n$  such that  $R_n \rightarrow +\infty$  as  $n \rightarrow \infty$ , and  $1 > \lambda_1(d, \alpha, R_n)$  for all  $n$ . Then, as  $n \rightarrow \infty$ , both  $v_{R_n}$  and  $w_{R_n}$

converge to  $\hat{U}$ . For each  $n$ , we can find  $T_n > 0$  such that  $h(t) \geq R_n$  for  $t \geq T_n$ . The problem

$$(2.17) \quad \begin{cases} w_t - d\Delta w = w(\alpha(r) - \beta(r)w), & t \geq T_n, \quad r \in [0, R_n], \\ w_r(t, 0) = 0 \quad w(t, R_n) = 0, & t \geq T_n, \\ w(T_n, r) = u(T_n, r), & r \in [0, R_n], \end{cases}$$

admits a unique positive solution  $w_n(t, r)$  and

$$(2.18) \quad w_n(t, r) \rightarrow v_{R_n}(r) \quad \text{uniformly for } r \in [0, R_n] \text{ as } t \rightarrow +\infty.$$

By the comparison principle, we have

$$w_n(t, r) \leq u(t, r) \quad \text{for } t \geq T_n \text{ and } r \in [0, R_n].$$

Therefore,

$$\underline{\lim}_{t \rightarrow +\infty} u(t, r) \geq v_{R_n}(r) \quad \text{uniformly in } r \in [0, R_n].$$

Sending  $n \rightarrow \infty$ , we obtain

$$(2.19) \quad \underline{\lim}_{t \rightarrow +\infty} u(t, r) \geq \hat{U}(r) \quad \text{locally uniformly for } r \in [0, +\infty).$$

Analogously, by arguments similar to those in the proof of Theorem 4.1 of [10], we see that

$$\overline{\lim}_{t \rightarrow +\infty} u(t, r) \leq w_{R_n}(r) \quad \text{uniformly for } r \in [0, R_n],$$

which implies (by sending  $n \rightarrow \infty$ )

$$(2.20) \quad \overline{\lim}_{t \rightarrow +\infty} u(t, r) \leq \hat{U}(r) \quad \text{locally uniformly for } r \in [0, +\infty).$$

From (2.19) and (2.20) we see that (2.15) holds.  $\square$

Combing Lemmas 2.2 and 2.3, we immediately obtain the following spreading-vanishing dichotomy:

**Theorem 2.4.** *Let  $(u(t, r), h(t))$  be the solution of the free boundary problem (1.4). Then the following alternative holds:*

*Either*

(i) Spreading:  $h_\infty = +\infty$  and

$$\lim_{t \rightarrow +\infty} u(t, r) = \hat{U}(r) \quad \text{locally uniformly for } r \in [0, \infty),$$

*or*

(ii) Vanishing:  $h_\infty \leq R^*$  and  $\lim_{t \rightarrow +\infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0$ .

We next decide exactly when each of the two alternatives occurs. We need to divide our discussion into two cases:

$$(a) \quad h_0 \geq R^*, \quad (b) \quad h_0 < R^*.$$

In case (a), due to  $h'(t) > 0$  for  $t > 0$ , we must have  $h_\infty > R^*$ . Hence Lemma 2.2 implies the following result.

**Theorem 2.5.** *If  $h_0 \geq R^*$ , then  $h_\infty = +\infty$ .*

As in [8], in order to study case (b), and also for later applications, we need a comparison principle which can be used to estimate both  $u(t, r)$  and the free boundary  $r = h(t)$ .

**Lemma 2.6.** *Suppose that  $T \in (0, \infty)$ ,  $\bar{h} \in C^1([0, T])$ ,  $\bar{u} \in C^{1,2}(D_T^*)$  with  $D_T^* = \{(t, r) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq r \leq \bar{h}(t)\}$ , and*

$$\begin{cases} \bar{u}_t - d\Delta\bar{u} \geq \bar{u}[\alpha(r) - \beta(r)\bar{u}], & t > 0, 0 < r < \bar{h}(t), \\ \bar{u} = 0, \quad \bar{h}'(t) \geq -\mu\bar{u}_r, & t > 0, r = \bar{h}(t), \\ \bar{u}_r(t, 0) \leq 0, & t > 0. \end{cases}$$

If

$$h_0 \leq \bar{h}(0) \text{ and } u_0(r) \leq \bar{u}(0, r) \text{ in } [0, h_0],$$

then the solution  $(u, h)$  of the free boundary problem (1.4) satisfies

$$h(t) \leq \bar{h}(t) \text{ in } (0, T], u(r, t) \leq \bar{u}(r, t) \text{ for } t \in (0, T] \text{ and } r \in (0, h(t)).$$

**Proof:** This is almost identical to the proof of Lemma 3.5 in [8]. So we omit the details.  $\square$

**Remark 2.7.** *The pair  $(\bar{u}, \bar{h})$  in Lemma 2.6 is usually called an upper solution of the problem (1.4). We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 2.6 for lower solutions.*

Let us now consider case (b), where  $h_0 < R^*$ . We first examine the case that  $\mu$  is large, then we look at the case  $\mu > 0$  is small, and finally we use Lemma 2.6 and Remark 2.7 to prove the existence of a critical  $\mu^*$  so that spreading occurs when  $\mu > \mu^*$  and vanishing happens if  $\mu \in (0, \mu^*]$ .

**Lemma 2.8.** *Suppose  $h_0 < R^*$ . Then there exists  $\mu^0 > 0$  depending on  $u_0$  such that spreading occurs if  $\mu \geq \mu^0$ .*

**Proof:** We argue indirectly. Suppose that there is an increasing sequence  $\{\mu_n\}$  satisfying  $\mu_n \rightarrow +\infty$  as  $n \rightarrow \infty$  such that the unique solution  $(u^n, h^n)$  of (1.4) with  $\mu = \mu_n$  satisfies  $h_\infty^n := \lim_{t \rightarrow \infty} h^n(t) < +\infty$  for all  $n$ . Then it follows from Lemma 2.2 that  $h_\infty^n \leq R^*$  and hence

$$(2.21) \quad u^n(t, r) \leq w^*(t, r) \text{ for } t > 0 \text{ and } r \in [0, h^n(t)],$$

where  $w^*(t, r)$  is the unique positive solution of the problem

$$\begin{cases} w_t - d\Delta w = w(\alpha(r) - \beta(r)w), & t > 0, r \in [0, R^*], \\ w_r(t, 0) = 0 \quad w(t, R^*) = 0, & t > 0, \\ w(0, r) = \hat{u}_0(r), & r \in [0, R^*], \end{cases}$$

with

$$\hat{u}_0(r) = \begin{cases} u_0(r) & r \in [0, h_0] \\ 0 & r \in (h_0, R^*] \end{cases}$$

The fact that  $1 = \lambda_1(d, \alpha, R^*)$  implies that

$$(2.22) \quad \lim_{t \rightarrow +\infty} \|w^*(t, \cdot)\|_{C([0, R^*])} \rightarrow 0.$$

This and (2.21) imply that there is  $T > 0$  independent of  $n$  such that

$$u^n(t, r) \leq \frac{\kappa_1}{\kappa_2} \text{ for } t \geq T \text{ and } r \in [0, h^n(t)].$$

For convenience, we will omit  $n$  from  $u^n$ ,  $h^n$ ,  $h_\infty^n$  and  $\mu_n$  in the following argument.

Direct calculation gives

$$\begin{aligned} \frac{d}{dt} \int_0^{h(t)} r^{N-1} u(t, r) dr &= \int_0^{h(t)} r^{N-1} u_t(t, r) dr + h^{N-1}(t) h'(t) u(t, h(t)) \\ &= d \int_0^{h(t)} r^{N-1} \Delta u dr + \int_0^{h(t)} u [\alpha(r) - \beta(r) u] r^{N-1} dr \\ &= d \int_0^{h(t)} (r^{N-1} u_r(r))_r dr + \int_0^{h(t)} u [\alpha(r) - \beta(r) u] r^{N-1} dr \\ &= -\frac{d}{\mu} h^{N-1}(t) h'(t) + \int_0^{h(t)} u [\alpha(r) - \beta(r) u] r^{N-1} dr. \end{aligned}$$

Integrating from  $T$  to  $t > T$  yields

$$\begin{aligned} \int_0^{h(t)} r^{N-1} u(t, r) dx &= \int_0^{h(T)} r^{N-1} u(T, r) dr + \frac{d}{N\mu} (h(T)^N - h(t)^N) \\ &\quad + \int_T^t \int_0^{h(s)} u [\alpha(r) - \beta(r) u] r^{N-1} dr ds \\ &\geq \frac{d}{N\mu} (h(T)^N - h(t)^N) + \int_0^{h(T)} r^{N-1} u(T, r) dr, \end{aligned}$$

since the fact that  $0 < u(t, r) \leq \frac{\kappa_1}{\kappa_2}$  for  $t \geq T$  and  $r \in [0, h(t)]$  implies

$$\alpha(r) - \beta(r) u(t, r) \geq \kappa_1 - \kappa_2 u(t, r) \geq 0 \text{ for } t \geq T \text{ and } r \in [0, h(t)].$$

Sending  $t \rightarrow +\infty$  we obtain, in view of (2.21) and (2.22),

$$\frac{d}{N\mu} (h(T)^N - h_\infty^N) + \int_0^{h(T)} r^{N-1} u(T, r) dr \leq 0$$

and hence

$$(2.23) \quad \mu \leq \frac{d [(R^*)^N - h(T)^N]}{N \int_0^{h(T)} r^{N-1} u(T, r) dr}.$$

By Lemma 2.6,  $u^n(t, x)$  and  $h^n(t)$  are increasing in  $n$ . Therefore

$$u^n(t, x) \geq u^1(t, x) \text{ and } h^n(t) \geq h^1(t).$$

Thus from (2.23) we deduce

$$\mu_n \leq \frac{d [(R^*)^N - h^1(T)^N]}{N \int_0^{h^1(T)} r^{N-1} u^1(T, r) dr}.$$

This contradicts our assumption that  $\mu_n \rightarrow +\infty$  as  $n \rightarrow \infty$ . □

**Lemma 2.9.** *Suppose  $h_0 < R^*$ . Then there exists  $\mu_0 > 0$  depending on  $u_0$  such that vanishing happens if  $\mu \leq \mu_0$ .*

**Proof:** We are going to construct a suitable upper solution to (1.4) and then apply Lemma 2.6. For  $t > 0$  and  $r \in [0, \sigma(t)]$ , we define

$$\sigma(t) = h_0(1 + \delta - \frac{\delta}{2}e^{-\gamma t}), \quad w(t, r) = Me^{-\gamma t}V\left(\frac{h_0}{\sigma(t)}r\right),$$

where  $M, \delta, \gamma$  are positive constants to be chosen later and  $V(|x|)$  is the first eigenfunction of the problem

$$\begin{cases} -d\Delta V = \lambda_1(d, \alpha, h_0)\alpha(|x|)V & \text{in } B_{h_0} \\ V = 0 & \text{on } \partial B_{h_0} \end{cases}$$

with  $V \geq 0$  and  $\|V\|_\infty = 1$ . Since  $h_0 < R^*$ , we have

$$1 < \lambda_1(d, \alpha, h_0).$$

We also observe that  $V'(0) = 0$  and

$$-d(r^{N-1}V')' = r^{N-1}\lambda_1(d, \alpha, h_0)\alpha(r)V > 0 \quad \text{for } 0 < r < h_0$$

imply

$$V'(r) < 0 \quad \text{for } 0 < r \leq h_0.$$

Set  $\tau(t) = 1 + \delta - \frac{\delta}{2}e^{-\gamma t}$  so that  $\sigma(t) = h_0\tau(t)$ . Direct calculations yield

$$\begin{aligned} & w_t - d\Delta w - w[\alpha(r) - \beta(r)w] \\ &= Me^{-\gamma t}[-\gamma V - r\tau^{-2}\tau'(t)V' - d\tau^{-2}V'' - d(N-1)r^{-1}\tau^{-1}V' \\ &\quad - V(\alpha(r) - \beta(r)Me^{-\gamma t}V)] \\ &= Me^{-\gamma t}[-\gamma V - r\tau'(t)\tau^{-2}V' + \tau^{-2}\lambda_1(d, \alpha, h_0)\alpha\left(\frac{r}{\tau}\right)V \\ &\quad - V(\alpha(r) - \beta(r)Me^{-\gamma t}V)] \\ &\geq Me^{-\gamma t}V[-\gamma + \tau^{-2}\lambda_1(d, \alpha, h_0)\alpha\left(\frac{r}{\tau}\right) - \alpha(r) + \beta(r)Me^{-\gamma t}V] \\ &\geq Me^{-\gamma t}V[-\gamma + \frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2}\alpha\left(\frac{r}{\tau}\right) - \alpha(r) + \beta(r)Me^{-\gamma t}V] \\ &= Me^{-\gamma t}V\left[-\gamma + \left(\frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2}\frac{\alpha\left(\frac{r}{\tau}\right)}{\alpha(r)} - 1\right)\alpha(r) + \beta(r)Me^{-\gamma t}V\right]. \end{aligned}$$

Clearly

$$1 + \frac{\delta}{2} \leq \tau(t) \leq 1 + \delta, \quad h_0(1 + \frac{\delta}{2}) \leq \sigma(t) \leq h_0(1 + \delta).$$

Hence, due to  $1 < \lambda_1(d, \alpha, h_0)$ , we can choose  $\delta > 0$  sufficiently small such that

$$(2.24) \quad \varrho := \min_{t>0, r \in [0, \sigma(t)]} \frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2} \frac{\alpha\left(\frac{r}{\tau}\right)}{\alpha(r)} - 1 > 0.$$

Setting  $\gamma = \varrho\kappa_1$ , we deduce

$$w_t - d\Delta w - w[\alpha(r) - \beta(r)w] \geq 0 \quad \text{for } t > 0, r \in [0, \sigma(t)].$$

We now choose  $M > 0$  sufficiently large such that

$$u_0(r) \leq MV \left( \frac{r}{(1 + \delta/2)} \right) = w(0, r) \quad \text{for } r \in [0, h_0].$$

We calculate

$$\begin{aligned} \sigma'(t) &= \frac{1}{2} h_0 \gamma \delta e^{-\gamma t}, \\ -\mu w_r(t, \sigma(t)) &= \mu M e^{-\gamma t} \frac{h_0}{\sigma(t)} |V_r(h_0)| \leq \mu M e^{-\gamma t} \frac{|V_r(h_0)|}{1 + \delta/2}. \end{aligned}$$

Hence if we take

$$\mu_0 = \frac{\delta(1 + \delta/2)\gamma h_0}{2M|V_r(h_0)|},$$

then for any  $0 < \mu \leq \mu_0$ ,

$$\sigma'(t) \geq -\mu w_r(t, \sigma(t)),$$

and thus  $(w, \sigma)$  satisfies

$$\begin{cases} w_t - d\Delta w \geq w[\alpha(r) - \beta(r)w], & t > 0, \quad 0 < r < \sigma(t), \\ w = 0, \quad \sigma'(t) \geq -\mu w_r, & t > 0, \quad r = \sigma(t), \\ w_r(t, 0) = 0, & t > 0, \\ \sigma(0) = (1 + \frac{\delta}{2})h_0 > h_0. \end{cases}$$

Hence we can apply Lemma 2.6 to conclude that  $h(t) \leq \sigma(t)$  and  $u(t, r) \leq w(t, r)$  for  $0 \leq r \leq h(t)$  and  $t > 0$ . It follows that  $h_\infty \leq \lim_{t \rightarrow \infty} \sigma(t) = h_0(1 + \delta) < \infty$ .  $\square$

We are now ready to apply Lemma 2.6 to prove the existence of a threshold value  $\mu^*$  of  $\mu$  that governs the alternatives in the spreading-vanishing dichotomy for the case  $h_0 < R^*$ .

**Theorem 2.10.** *If  $h_0 < R^*$ , then there exists  $\mu^* > 0$  depending on  $u_0$  such that vanishing occurs if  $\mu \leq \mu^*$ , and spreading happens if  $\mu > \mu^*$ .*

**Proof:** This is similar to the proof of Theorem 3.9 in [8]. We give the details below for completeness.

Define  $\Sigma := \{\mu > 0 : h_\infty \leq R^*\}$ . By Lemmas 2.9 and 2.2 we have  $\Sigma \supset (0, \mu_0]$ . Using Lemma 2.8 we find on the other hand  $\Sigma \cap [\mu^0, \infty) = \emptyset$ . Therefore  $\mu^* := \sup \Sigma \in [\mu_0, \mu^0]$ . By this definition and Lemma 2.2, we find that  $h_\infty = +\infty$  when  $\mu > \mu^*$ .

We claim that  $\mu^* \in \Sigma$ . Otherwise  $h_\infty = \infty$  for  $\mu = \mu^*$ . Hence we can find  $T > 0$  such that  $h(T) > R^*$ . To stress the dependence of the solution  $(u, h)$  of (1.4) on  $\mu$ , we now write  $(u_\mu, h_\mu)$  instead of  $(u, h)$ . So we have  $h_{\mu^*}(T) > R^*$ . By the continuous dependence of  $(u_\mu, h_\mu)$  on  $\mu$ , we can find  $\epsilon > 0$  small so that  $h_\mu(T) > R^*$  for all  $\mu \in [\mu^* - \epsilon, \mu^* + \epsilon]$ . It follows that for all such  $\mu$ ,

$$\lim_{t \rightarrow +\infty} h_\mu(t) > h_\mu(T) > R^*.$$

This implies that  $[\mu^* - \epsilon, \mu^* + \epsilon] \cap \Sigma = \emptyset$ , and  $\sup \Sigma \leq \mu^* - \epsilon$ , contradicting the definition of  $\mu^*$ . This proves our claim that  $\mu^* \in \Sigma$ .

For  $\mu \in (0, \mu^*)$ ,  $(u_{\mu^*}, h_{\mu^*})$  is an upper solution of (1.4). Hence we can use Lemma 2.6 to deduce that  $h_\mu(t) \leq h_{\mu^*}(t)$  for  $t > 0$ . It follows that

$$\lim_{t \rightarrow \infty} h_\mu(t) \leq \lim_{t \rightarrow \infty} h_{\mu^*}(t) \leq R^*.$$

Hence  $\mu \in \Sigma$ . Thus we have proved that  $\Sigma = (0, \mu^*]$ . The proof is complete.  $\square$

### 3. ESTIMATES OF SPREADING SPEED

In this section we estimate the spreading speed of the expanding front  $r = h(t)$  when spreading occurs. We will find  $0 < k_* \leq k^* < \infty$  such that

$$(3.1) \quad k_* \leq \underline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} \leq \overline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} \leq k^*.$$

Moreover, if  $\alpha(r) \rightarrow \alpha_*$  and  $\beta(r) \rightarrow \beta_*$  as  $r \rightarrow \infty$ , we show that  $k_* = k^*$  and hence

$$\lim_{t \rightarrow +\infty} \frac{h(t)}{t} = k_*.$$

The constants  $k_*$  and  $k^*$  are determined through the following result, which is Proposition 4.1 of [8]:

**Proposition 3.1.** *Let  $d > 0$  be given in (1.4). For any given constants  $a > 0$ ,  $b > 0$  and  $k \geq 0$ , the problem*

$$(3.2) \quad -dU'' + kU' = aU - bU^2 \quad \text{in } (0, \infty), \quad U(0) = 0$$

*admits a unique positive solution  $U = U_k = U_{a,b,k}$ , and it satisfies  $U(r) \rightarrow \frac{a}{b}$  as  $r \rightarrow +\infty$ . Moreover,  $U'_k(r) > 0$  for  $r \geq 0$ ,  $U'_{k_1}(0) > U'_{k_2}(0)$ ,  $U_{k_1}(r) > U_{k_2}(r)$  for  $r > 0$  and  $k_1 < k_2$ , and for each  $\mu > 0$ , there exists a unique  $k_0 = k_0(\mu, a, b) > 0$  such that  $\mu U'_{k_0}(0) = k_0$ .*

It was shown in [8] that  $k_0(\mu, a, b)$  is increasing in  $\mu$  and  $a$ , and is decreasing in  $b$ . More precisely,

$$\mu_1 \geq \mu_2, \quad a_1 \geq a_2 \quad \text{and} \quad b_1 \leq b_2 \quad \text{imply} \quad k_0(\mu_1, a_1, b_1) \geq k_0(\mu_2, a_2, b_2),$$

with strict inequality holding when  $(\mu_1, a_1, b_1) \neq (\mu_2, a_2, b_2)$ . It can also be easily shown that  $k_0(\mu, a, b)$  is a continuous function.

By (1.5), we have

$$\alpha^\infty := \overline{\lim}_{r \rightarrow +\infty} \alpha(r) \leq \kappa_2, \quad \alpha_\infty := \underline{\lim}_{r \rightarrow +\infty} \alpha(r) \geq \kappa_1;$$

$$\beta^\infty := \overline{\lim}_{r \rightarrow +\infty} \beta(r) \leq \kappa_2, \quad \beta_\infty := \underline{\lim}_{r \rightarrow +\infty} \beta(r) \geq \kappa_1.$$

We will show that in (3.1), one can take

$$k_* = k_0(\mu, \alpha_\infty, \beta^\infty), \quad k^* = k_0(\mu, \alpha^\infty, \beta_\infty).$$

To prove this, we need some preparations. Firstly we need some simple variants of Lemma 2.6 and Remark 2.7.

**Lemma 3.2.** *Let  $d_1(s), d_2(s), a(s), b(s)$  and  $l(s)$  be Hölder continuous functions for  $s \geq 0$ , all positive except possibly for  $d_2(s)$ . Suppose that  $T \in (0, \infty)$ ,  $\bar{h} \in C^1([0, T])$ ,  $\bar{u} \in C^{1,2}(D_T^*)$  with  $D_T^* = \{(t, r) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq r \leq \bar{h}(t)\}$ , and*

$$\begin{cases} \bar{u}_t - d_1(r)\bar{u}_{rr} - d_2(r)\bar{u}_r \geq \bar{u}[a(r) - b(r)\bar{u}], & t > 0, 0 < r < \bar{h}(t), \\ \bar{u} = 0, \quad \bar{h}'(t) \geq -\mu\bar{u}_r, & t > 0, r = \bar{h}(t), \\ \bar{u}(t, 0) \geq l(t), & t > 0. \end{cases}$$

*If  $h \in C^1([0, T])$  and  $u \in C^{1,2}(D_T)$  with  $D_T = \{(t, r) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq r \leq h(t)\}$  satisfy*

$$0 < h(0) \leq \bar{h}(0), \quad 0 < u(0, r) \leq \bar{u}(0, r) \text{ for } 0 \leq r \leq h(0),$$

*and*

$$(3.3) \quad \begin{cases} u_t - d_1(r)u_{rr} - d_2(r)u_r = u[a(r) - b(r)u], & t > 0, 0 < r < h(t), \\ u = 0, \quad h'(t) = -\mu u_r, & t > 0, r = h(t), \\ u(t, 0) = l(t), & t > 0, \end{cases}$$

*then*

$$h(t) \leq \bar{h}(t) \text{ in } (0, T], \quad u(r, t) \leq \bar{u}(r, t) \text{ for } t \in (0, T] \text{ and } r \in (0, h(t)).$$

**Proof:** This is similar to the proof of Lemma 3.5 in [8]. For small  $\epsilon > 0$ , let  $(u_\epsilon, h_\epsilon)$  denote the unique solution of (3.3) with  $h_0 := h(0)$  replaced by  $h_0^\epsilon := h_0(1 - \epsilon)$ , with  $\mu$  replaced by  $\mu_\epsilon := \mu(1 - \epsilon)$ , and with  $u_\epsilon(0, r) = u_0^\epsilon(r)$  for some  $u_0^\epsilon \in C^2([0, h_0^\epsilon])$  satisfying

$$0 < u_0^\epsilon(r) \leq u(0, r) \text{ in } [0, h_0^\epsilon], \quad u_0^\epsilon(h_0^\epsilon) = 0,$$

and as  $\epsilon \rightarrow 0$ ,

$$u_0^\epsilon \left( \frac{h_0}{h_0^\epsilon} r \right) \rightarrow u(0, r)$$

in the  $C^2([0, h_0])$  norm. The fact that such a unique solution exists can be proved in the same way as for (1.4).

We claim that  $h_\epsilon(t) < \bar{h}(t)$  for all  $t \in (0, T]$ . Clearly this is true for small  $t > 0$ . If our claim does not hold, then we can find a first  $t^* \leq T$  such that  $h_\epsilon(t) < \bar{h}(t)$  for  $t \in (0, t^*)$  and  $h_\epsilon(t^*) = \bar{h}(t^*)$ . It follows that

$$(3.4) \quad h'_\epsilon(t^*) \geq \bar{h}'(t^*).$$

We now compare  $u_\epsilon$  and  $\bar{u}$  over the region

$$\Omega_{t^*} := \{(t, r) \in \mathbb{R}^2 : 0 < t \leq t^*, 0 \leq r < h_\epsilon(t)\}.$$

The strong maximum principle yields  $u_\epsilon(t, r) < \bar{u}(t, r)$  in  $\Omega_{t^*}$ . Hence  $w(t, r) := \bar{u}(t, r) - u_\epsilon(t, r) > 0$  in  $\Omega_{t^*}$  with  $w(t^*, h_\epsilon(t^*)) = 0$ . It follows that  $w_r(t^*, h_\epsilon(t^*)) \leq 0$ , from which we deduce, in view of  $(u_\epsilon)_r(t^*, h_\epsilon(t^*)) < 0$  and  $\mu_\epsilon < \mu$ , that  $h'_\epsilon(t^*) < \bar{h}'(t^*)$ . But this contradicts (3.4). This proves our claim that  $h_\epsilon(t) < \bar{h}(t)$  for all  $t \in (0, T]$ . We may now apply the usual comparison principle over  $\Omega_T$  to conclude that  $u_\epsilon < \bar{u}$  in  $\Omega_T$ .

Since the unique solution  $(u_\epsilon, h_\epsilon)$  depends continuously on the parameter  $\epsilon$ , as  $\epsilon \rightarrow 0$ ,  $(u_\epsilon, h_\epsilon)$  converges to  $(u, h)$ . The desired result then follows by letting  $\epsilon \rightarrow 0$  in the inequalities  $u_\epsilon < \bar{u}$  and  $h_\epsilon < \bar{h}$ .  $\square$

Similar to Remark 2.7, we have the following analogue of Lemma 3.2:

**Lemma 3.3.** *Let  $d_1(s), d_2(s), a(s), b(s)$  and  $l(s)$  be as in Lemma 3.2. Suppose that  $T \in (0, \infty)$ ,  $\underline{h} \in C^1([0, T])$ ,  $\underline{u} \in C^{1,2}(D_T^\dagger)$  with  $D_T^\dagger = \{(t, r) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq r \leq \underline{h}(t)\}$ , and*

$$\begin{cases} \underline{u}_t - d_1(r)\underline{u}_{rr} - d_2(r)\underline{u}_r \leq \underline{u}[a(r) - b(r)\underline{u}], & t > 0, 0 < r < \underline{h}(t), \\ \underline{u} = 0, \quad \underline{h}'(t) \leq -\mu\underline{u}_r, & t > 0, r = \underline{h}(t), \\ \underline{u}(t, 0) \leq l(t), & t > 0. \end{cases}$$

If  $h \in C^1([0, T])$  and  $u \in C^{1,2}(D_T)$  satisfy (3.3) and

$$h(0) \geq \underline{h}(0), \quad u(0, r) \geq \underline{u}(0, r) \quad \text{for } 0 \leq r \leq \underline{h}(0), \quad u(0, r) \geq 0 \quad \text{for } \underline{h}(0) \leq r \leq h(0),$$

then

$$h(t) \geq \underline{h}(t) \quad \text{in } (0, T], \quad u(r, t) \geq \underline{u}(r, t) \quad \text{for } t \in (0, T] \quad \text{and } r \in (0, \underline{h}(t)).$$

We also need the following result:

**Lemma 3.4.** *Suppose that  $d, a$  and  $b$  are given positive constants, that  $c(s)$  and  $l(s)$  are Hölder continuous functions for  $s \geq 0$  with  $l(s)$  positive, and that  $\sigma(t)$  is a continuous positive function for  $t \geq 0$ . Let  $v \in C^{1,2}(D)$  ( $D = \{(t, r) : 0 \leq r \leq \sigma(t), t \geq 0\}$ ) be a solution of*

$$(3.5) \quad \begin{cases} v_t - dv_{rr} + c(r)v_r = v(a - bv), & t > 0, 0 < r < \sigma(t), \\ v(t, 0) = l(t), \quad v(t, \sigma(t)) = 0, & t > 0, \\ v(0, r) = v_0(r) \geq 0, & 0 < r < \sigma(0). \end{cases}$$

Suppose that

$$\lim_{r \rightarrow \infty} c(r) = 0, \quad \lim_{t \rightarrow \infty} l(t) = l_\infty \in \left[\frac{a}{b}, \infty\right), \quad \lim_{t \rightarrow \infty} \sigma(t) = \infty.$$

Then

$$\underline{\lim}_{t \rightarrow \infty} v(t, r) \geq \frac{a}{b} \quad \text{locally uniformly for } r \in [0, \infty).$$

**Proof:** By the maximum principle,  $v(t, r) > 0$  for  $t > 0$  and  $0 \leq r < \sigma(t)$ . For any given  $R > 0$  and small  $\epsilon > 0$ , we can find  $T_R > 0$  such that  $\sigma(t) > R$  and  $l(t) \geq l_\infty - \epsilon$  for all  $t \geq T_R$ . We now consider the auxiliary problem

$$(3.6) \quad \begin{cases} w_t - dw_{rr} + c(r)w_r = w(a - bw), & t > T_R, 0 < r < R, \\ w(t, 0) = l_\infty - \epsilon, \quad w(t, R) = 0, & t > T_R, \\ w(T_R, r) = v(T_R, r), & 0 < r < R. \end{cases}$$

By the comparison principle we have  $w(t, r) \leq v(t, r)$  for  $t > T_R$  and  $0 \leq r \leq R$ . By [22], problem (3.6) with initial function  $v(T_R, r)$  replaced by 0 has a unique solution  $w_*(t, r)$  which is increasing in  $t$ , and the same problem with initial function  $v(T_R, r)$  replaced by a large constant  $M$  has a unique solution  $w^*(t, r)$  which is decreasing in  $t$ , and moreover,  $w_*(t, r) < w^*(t, r)$  for all  $t > T_R$  and  $0 \leq r \leq R$ , and

$$\underline{w}(r) := \lim_{t \rightarrow \infty} w_*(t, r), \quad \bar{w}(r) := \lim_{t \rightarrow \infty} w^*(t, r)$$

are positive solutions of the elliptic problem

$$(3.7) \quad -dw_{rr} + c(r)w_r = w(a - bw) \quad \text{for } 0 < r < R, \quad w(0) = l_\infty - \epsilon, \quad w(R) = 0.$$

By Lemma 2.1 of [10], we deduce  $\underline{w} = \overline{w} = w_R$ , the unique positive solution of (3.7). Hence

$$\lim_{t \rightarrow \infty} w_*(t, r) = \lim_{t \rightarrow \infty} w^*(t, r) = w_R(r).$$

By the comparison principle, the solution of (3.6) satisfies  $w_*(t, r) \leq w(t, r) \leq w^*(t, r)$  for  $t \geq T_R$  and  $0 \leq r \leq R$ . It follows that

$$\lim_{t \rightarrow \infty} w(t, r) = w_R(r).$$

From [22] we also know that the above convergence is uniform in  $r$ . It follows that

$$\underline{\lim}_{t \rightarrow \infty} v(t, r) \geq w_R(r)$$

uniformly in  $r \in [0, R]$ .

As in [10], one can easily show that as  $R$  increases to infinity,  $w_R(r)$  increases to the minimal positive solution  $W$  of

$$(3.8) \quad -dw_{rr} + c(r)w_r = w(a - bw) \text{ for } r > 0, \quad w(0) = l_\infty - \epsilon.$$

It follows that

$$(3.9) \quad \underline{\lim}_{t \rightarrow \infty} v(t, r) \geq W(r)$$

locally uniformly in  $r \in [0, \infty)$ .

We show next that  $W(r) \geq \min\{l_\infty - \epsilon, a/b\}$  for  $r \geq 0$ . We first prove that  $W(r) \rightarrow a/b$  as  $r \rightarrow \infty$ . Indeed, for any increasing positive sequence  $r_n \rightarrow \infty$ , we define  $c_n(r) = c(r_n + r)$  for  $-r_n/2 \leq r \leq r_n/2$ . Clearly  $\|c_n(r)\|_{L^\infty([-r_n/2, r_n/2])} \rightarrow 0$  as  $n \rightarrow \infty$ .

Since  $a > 0$  and  $r_n \rightarrow \infty$ , for all large  $n$ , the logistic problem

$$-dw'' + c_n(r)w' = w(a - bw) \text{ in } (-r_n/2, r_n/2), \quad w(-r_n/2) = w(r_n/2) = 0$$

has a unique positive solution  $w_n$ . By the comparison principle (see Lemma 2.1 of [10]), we have

$$W(r) \geq w_n(r - r_n) \text{ for } r_n/2 < r < 3r_n/2.$$

Hence  $W(r_n) \geq w_n(0)$ .

On the other hand, by a standard elliptic regularity argument, one finds that, by passing to a subsequence,  $w_n$  converges in  $C_{loc}^1(\mathbb{R}^1)$  to a positive solution  $w_\infty$  of

$$-dw'' = w(a - bw) \text{ in } \mathbb{R}^1.$$

By Theorem 1.1 of [10], we find that  $w_\infty \equiv a/b$ . Hence the entire sequence  $w_n$  converges to  $a/b$  and

$$\underline{\lim}_{n \rightarrow \infty} W(r_n) \geq \lim_{n \rightarrow \infty} w_n(0) = a/b.$$

If we use  $w^n$  to denote the unique positive solution of the boundary blow-up problem

$$-dw'' + c_n(r)w' = w(a - bw) \text{ in } (-r_n/2, r_n/2), \quad w(-r_n/2) = w(r_n/2) = \infty,$$

we similarly deduce  $W(r_n) \leq w^n(0)$  and  $w^n(0) \rightarrow a/b$  as  $n \rightarrow \infty$ . Thus

$$\overline{\lim}_{n \rightarrow \infty} W(r_n) \leq a/b.$$

Hence we must have  $\lim_{r \rightarrow \infty} W(r) = a/b$ .

For any small  $\delta > 0$ , define  $v_\delta(r) = \min\{l_\infty - \epsilon, a/b - \delta\}$ . Then for all large  $R$ ,  $v_\delta(r) \leq W(r)$  for  $r \in \{0, R\}$ , and

$$-dv_\delta'' < v_\delta(a - bv_\delta).$$

Hence we can apply Lemma 2.1 of [10] to conclude that  $W(r) \geq v_\delta$  in  $[0, R]$ . Since  $R$  can be arbitrarily large, this implies that  $W(r) \geq \min\{l_\infty - \epsilon, a/b - \delta\}$  for all  $r \geq 0$ . Since  $\delta > 0$  can be arbitrarily small, we finally deduce  $W(r) \geq \min\{l_\infty - \epsilon, a/b\}$  for  $r \geq 0$ .

We may now use (3.9) to obtain

$$\underline{\lim}_{t \rightarrow \infty} v(t, r) \geq \min\{l_\infty - \epsilon, a/b\}.$$

Since  $\epsilon > 0$  can be arbitrarily small, this implies

$$\underline{\lim}_{t \rightarrow \infty} v(t, r) \geq \min\{l_\infty, a/b\} = a/b.$$

□

**Remark 3.5.** Under the assumptions of Lemma 3.4, it can be shown that

$$\lim_{t \rightarrow \infty} v(t, r) = V(r) \text{ locally uniform for } r \in [0, \infty),$$

where  $V$  is the unique positive solution of

$$-dV'' = V(a - bV) \text{ in } (0, \infty), \quad V(0) = l_\infty.$$

This conclusion is not needed in this paper though.

We are now ready to prove the main result of this section.

**Theorem 3.6.** If  $h_\infty = +\infty$ , then

$$(3.10) \quad \overline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} \leq k_0(\mu, \alpha^\infty, \beta_\infty), \quad \underline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} \geq k_0(\mu, \alpha_\infty, \beta^\infty).$$

**Proof:** By Theorem 1.1 of [7], the unique positive solution  $\hat{U}$  of (2.16) satisfies

$$\overline{\lim}_{r \rightarrow \infty} \hat{U}(r) \leq \frac{\alpha^\infty}{\beta_\infty}, \quad \underline{\lim}_{r \rightarrow \infty} \hat{U}(r) \geq \frac{\alpha_\infty}{\beta^\infty}.$$

For any  $\epsilon > 0$ , there is  $R := R(\epsilon) > 1$  such that for  $r \geq R$ ,

$$\alpha(r) \leq \alpha_\epsilon^\infty := \alpha^\infty + \epsilon, \quad \alpha(r) \geq \alpha_\infty^\epsilon := \alpha_\infty - \epsilon,$$

$$\beta(r) \leq \beta_\epsilon^\infty := \beta^\infty + \epsilon, \quad \beta(r) \geq \beta_\infty^\epsilon := \beta_\infty - \epsilon,$$

and

$$\frac{\alpha_\infty^{\epsilon/2}}{\beta_\epsilon^{\infty/2}} < \hat{U}(r) < \frac{\alpha_\infty^\epsilon}{\beta_\infty^{\epsilon/2}}.$$

Since  $h_\infty = +\infty$  and  $\lim_{t \rightarrow \infty} u(t, r) = \hat{U}(r)$ , there exists  $T := T(R) > 0$  such that

$$h(T) > 3R \text{ and } u(t + T, 2R) < \frac{\alpha_\infty^\epsilon}{\beta_\infty^\epsilon} \text{ for all } t \geq 0.$$

Setting

$$\tilde{u}(t, r) = u(t + T, r + 2R) \text{ and } \tilde{h}(t) = h(t + T) - 2R,$$

and denoting

$$\tilde{\Delta}u = u_{rr} + \frac{N-1}{r+2R}u_r,$$

we obtain

$$(3.11) \quad \begin{cases} \tilde{u}_t - d\tilde{\Delta}\tilde{u} = \tilde{u}[\alpha(r+2R) - \beta(r+2R)\tilde{u}], & t > 0, 0 < r < \tilde{h}(t), \\ \tilde{u}(t, 0) = u(t+T, 2R), \quad \tilde{u}(t, \tilde{h}(t)) = 0, & t > 0, \\ \tilde{h}'(t) = -\mu\tilde{u}_r(\tilde{h}(t)), & t > 0, \\ \tilde{u}(0, r) = u(T, r+2R), & 0 < r < \tilde{h}(0). \end{cases}$$

By our choice of  $R$ , for  $r \geq 0$ ,

$$\alpha(r+2R) \leq \alpha_\varepsilon^\infty, \quad \beta(r+2R) \geq \beta_\infty^\varepsilon.$$

Let  $u^*(t)$  be the unique solution of the problem

$$(3.12) \quad \frac{du^*}{dt} = u^*(\alpha_\varepsilon^\infty - \beta_\infty^\varepsilon u^*) \quad \text{for } t > 0; \quad u^*(0) = \max\{\alpha_\varepsilon^\infty/\beta_\infty^\varepsilon, \|\tilde{u}(0, \cdot)\|_\infty\}.$$

Then

$$u^*(t) \geq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon} \quad \text{for all } t \geq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} u^*(t) = \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}.$$

Now we have

$$u^*(0) \geq \tilde{u}(0, r), \quad \tilde{u}(t, 0) \leq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon} \leq u^*(t), \quad \tilde{u}(t, \tilde{h}(t)) = 0 \leq u^*(t),$$

and

$$u_t^* - d\tilde{\Delta}u^* = u^*(\alpha_\varepsilon^\infty - \beta_\infty^\varepsilon u^*) \geq u^*[\alpha(r+2R) - \beta(r+2R)u^*].$$

Hence we can apply the comparison principle to deduce

$$(3.13) \quad \tilde{u}(t, r) \leq u^*(t) \quad \text{for } 0 < r < \tilde{h}(t), \quad t > 0.$$

As a consequence, there exists  $\tilde{T} = \tilde{T}_\varepsilon > 0$  such that

$$\tilde{u}(t, r) \leq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}(1-\varepsilon)^{-1}, \quad \forall t \geq \tilde{T}, \quad r \in [0, \tilde{h}(t)].$$

Let  $U_\varepsilon := U_{\alpha_\varepsilon^\infty, \beta_\infty^\varepsilon, k^\varepsilon}$  denote the unique positive solution of (3.2) with  $a = \alpha_\varepsilon^\infty$ ,  $b = \beta_\infty^\varepsilon$  and  $k = k^\varepsilon := k_0(\mu, \alpha_\varepsilon^\infty, \beta_\infty^\varepsilon)$ . Since  $U_\varepsilon(r) \rightarrow \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}$  as  $r \rightarrow +\infty$ , there exists  $R_0 := R_0(\varepsilon) > 2R$  such that

$$U_\varepsilon(r) > \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon}(1-\varepsilon) \quad \text{for } r \geq R_0.$$

We now define

$$\begin{aligned} \xi(t) &= (1-\varepsilon)^{-2}k^\varepsilon t + R_0 + \tilde{h}(\tilde{T}) \quad \text{for } t \geq 0, \\ w(t, r) &= (1-\varepsilon)^{-2}U_\varepsilon(\xi(t) - r) \quad \text{for } t \geq 0, \quad 0 \leq r \leq \xi(t). \end{aligned}$$

Then

$$\begin{aligned} \xi'(t) &= (1-\varepsilon)^{-2}k^\varepsilon, \\ -\mu w_r(t, \xi(t)) &= \mu(1-\varepsilon)^{-2}U_\varepsilon'(0) = (1-\varepsilon)^{-2}k^\varepsilon, \end{aligned}$$

and so we have

$$\xi'(t) = -\mu w_r(t, \xi(t)).$$

Clearly

$$w(t, \xi(t)) = 0.$$

Moreover, for  $0 \leq r \leq \tilde{h}(\tilde{T})$ ,

$$w(0, r) = (1 - \varepsilon)^{-2} U_\varepsilon(\xi(0) - r) \geq (1 - \varepsilon)^{-2} U_\varepsilon(R_0) \geq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon} (1 - \varepsilon)^{-1} \geq \tilde{u}(\tilde{T}, r)$$

and  $w(0, r) > 0$  for  $\tilde{h}(\tilde{T}) < r < \xi(0)$ . It is also easily seen that for  $t > 0$ ,

$$w(t, 0) = (1 - \varepsilon)^{-2} U_\varepsilon(\xi(t)) \geq (1 - \varepsilon)^{-2} U_\varepsilon(R_0) \geq \frac{\alpha_\varepsilon^\infty}{\beta_\infty^\varepsilon} (1 - \varepsilon)^{-1} \geq \tilde{u}(t + \tilde{T}, 0).$$

Direct calculations show that, for  $t > 0$  and  $0 < r < \xi(t)$ ,

$$\begin{aligned} w_t - d\tilde{\Delta}w &= (1 - \varepsilon)^{-2} [U'_\varepsilon \xi' - dU''_\varepsilon + d(N - 1)(r + 2R)^{-1} U'_\varepsilon] \\ &= (1 - \varepsilon)^{-2} [(1 - \varepsilon)^{-2} k^\varepsilon U'_\varepsilon - dU''_\varepsilon + d(N - 1)(r + 2R)^{-1} U'_\varepsilon] \\ &\geq (1 - \varepsilon)^{-2} (k^\varepsilon U'_\varepsilon - dU''_\varepsilon) \quad (\text{due to } U'_\varepsilon \geq 0) \\ &= (1 - \varepsilon)^{-2} (\alpha_\varepsilon^\infty U_\varepsilon - \beta_\infty^\varepsilon U_\varepsilon^2) \\ &= \alpha_\varepsilon^\infty w - (1 - \varepsilon)^2 \beta_\infty^\varepsilon w^2 \\ &\geq \alpha_\varepsilon^\infty w - \beta_\infty^\varepsilon w^2. \end{aligned}$$

Hence we can use Lemma 3.2 to conclude that

$$\tilde{u}(t + \tilde{T}, r) \leq w(t, r), \quad \tilde{h}(t + \tilde{T}) \leq \xi(t) \quad \text{for } t \geq 0, 0 \leq r \leq \tilde{h}(t + \tilde{T}).$$

It follows that

$$\overline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} = \overline{\lim}_{t \rightarrow +\infty} \frac{\tilde{h}(t)}{t} \leq \lim_{t \rightarrow +\infty} \frac{\xi(t - \tilde{T})}{t} = k^\varepsilon (1 - \varepsilon)^{-2}.$$

Since  $\varepsilon > 0$  can be arbitrarily small, and  $k^\varepsilon \rightarrow k_0(\mu, \alpha^\infty, \beta_\infty)$  as  $\varepsilon \rightarrow 0$ , we deduce

$$\overline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} \leq k_0(\mu, \alpha^\infty, \beta_\infty).$$

Next we show

$$\underline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} \geq k_0(\mu, \alpha_\infty, \beta^\infty)$$

by constructing a suitable lower solution. To this end, we denote

$$k_\varepsilon = k_0(\mu, \alpha_\infty^\varepsilon, \beta_\infty^\varepsilon) \quad \text{and} \quad V_\varepsilon = U_{\alpha_\infty^\varepsilon, \beta_\infty^\varepsilon, k_\varepsilon}.$$

Then consider the auxiliary problem

$$(3.14) \quad \begin{cases} v_t - d\tilde{\Delta}v = v(\alpha_\infty^\varepsilon - \beta_\infty^\varepsilon v), & t > 0, 0 < r < \tilde{h}(t), \\ v(t, 0) = \tilde{u}(t, 0), \quad v(t, \tilde{h}(t)) = 0, & t > 0, \\ v(0, r) = \tilde{u}(0, r), & r \in [0, \tilde{h}(0)], \end{cases}$$

where  $\tilde{u}$  and  $\tilde{h}$  are defined as before. Since

$$\lim_{t \rightarrow \infty} \tilde{u}(t, 0) \rightarrow \hat{U}(2R) > \frac{\alpha_\infty^{\varepsilon/2}}{\beta_\infty^{\varepsilon/2}},$$

we can apply Lemma 3.4 to (3.14) to conclude that

$$(3.15) \quad \underline{\lim}_{t \rightarrow +\infty} v(t, r) \geq \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty} \text{ locally uniformly for } r \in [0, \infty).$$

Since

$$\alpha(r + 2R) \geq \alpha_\infty^\varepsilon, \quad \beta(r + 2R) \leq \beta_\varepsilon^\infty,$$

from the comparison principle we deduce

$$\tilde{u}(t, r) \geq v(t, r) \text{ for } t > 0, r \in [0, \tilde{h}(t)],$$

and hence, in view of (3.15), we have

$$(3.16) \quad \underline{\lim}_{t \rightarrow +\infty} \tilde{u}(t, r) \geq \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty} \text{ locally uniformly for } r \in [0, \infty).$$

Define

$$\eta(t) = (1 - \varepsilon)^2 k_\varepsilon t + \tilde{h}(0) \text{ for } t \geq 0,$$

and

$$w(t, r) = (1 - \varepsilon)^2 V_\varepsilon(\eta(t) - r) \text{ for } t \geq 0, 0 \leq r \leq \eta(t).$$

Then

$$\begin{aligned} \eta'(t) &= (1 - \varepsilon)^2 k_\varepsilon, \\ -\mu w_r(t, \eta(t)) &= \mu(1 - \varepsilon)^2 V_\varepsilon'(0) = (1 - \varepsilon)^2 k_\varepsilon, \end{aligned}$$

and so we have

$$\eta'(t) = -\mu w_r(t, \eta(t)).$$

Clearly,  $w(t, \eta(t)) = 0$ . Since  $V_\varepsilon'(r) \geq 0$  for  $r > 0$  and

$$\lim_{r \rightarrow +\infty} V_\varepsilon(r) = \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty},$$

we must have

$$V_\varepsilon(r) < \frac{\alpha_\infty^\varepsilon}{\beta_\varepsilon^\infty} \text{ for } r > 0.$$

Therefore, due to (3.16) we can find some  $\hat{T} = \hat{T}(\varepsilon) > 0$  such that

$$(3.17) \quad \tilde{u}(t + \hat{T}, 0) \geq w(t, 0) \text{ for } t \geq 0$$

and

$$(3.18) \quad \tilde{u}(\hat{T}, r) \geq w(0, r) \text{ for } r \in [0, \eta(0)].$$

Direct calculations yield

$$\begin{aligned} w_t - d\tilde{\Delta}w &= (1 - \varepsilon)^2 V_\varepsilon' \eta' - d(1 - \varepsilon)^2 \left[ V_\varepsilon'' - \frac{N-1}{r+2R} V_\varepsilon' \right] \\ &= (1 - \varepsilon)^2 \left[ (1 - \varepsilon)^2 V_\varepsilon' k_\varepsilon - dV_\varepsilon'' + \frac{d(N-1)}{r+2R} V_\varepsilon' \right] \\ &= (1 - \varepsilon)^2 \left[ \left( (1 - \varepsilon)^2 k_\varepsilon + \frac{d(N-1)}{r+2R} \right) V_\varepsilon' - dV_\varepsilon'' \right] \\ &\leq (1 - \varepsilon)^2 [k_\varepsilon V_\varepsilon' - dV_\varepsilon''] \quad (\text{since } V_\varepsilon' \geq 0) \\ &\leq w[\alpha_\infty^\varepsilon - \beta_\varepsilon^\infty w] \text{ for } t \geq 0, 0 \leq r \leq \eta(t), \end{aligned}$$

where we have used the fact that for large  $R$ ,

$$(1 - \varepsilon)^2 k_\varepsilon + \frac{d(N - 1)}{r + 2R} \leq k_\varepsilon.$$

Hence, we can use Lemma 3.3 to conclude that

$$\tilde{u}(t + \hat{T}, r) \geq w(t, r), \quad \tilde{h}(t + \hat{T}) \geq \eta(t) \quad \text{for } t \geq 0, 0 \leq r \leq \eta(t).$$

It follows that

$$\underline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} = \underline{\lim}_{t \rightarrow +\infty} \frac{\tilde{h}(t)}{t} \geq \lim_{t \rightarrow +\infty} \frac{\eta(t - \hat{T})}{t} = (1 - \varepsilon)^2 k_\varepsilon.$$

Since  $\varepsilon > 0$  can be arbitrarily small, this implies

$$\underline{\lim}_{t \rightarrow +\infty} \frac{h(t)}{t} \geq k_0(\mu, \alpha_\infty, \beta^\infty).$$

The proof of the theorem is now complete.  $\square$

The result below follows trivially from Theorem 3.6.

**Corollary 3.7.** *Assume that  $h_\infty = +\infty$  and*

$$(3.19) \quad \alpha(r) \rightarrow \alpha_*, \quad \beta(r) \rightarrow \beta_* \quad \text{as } r \rightarrow +\infty,$$

then

$$\lim_{t \rightarrow +\infty} \frac{h(t)}{t} = k_0(\mu, \alpha_*, \beta_*).$$

Finally we note that, if we fix  $a, b, \mu$  and regard  $k_0(\mu, a, b)$  from Proposition 3.1 as a function of  $d$ , written  $k_0 = k_*(d)$ , then it follows from Proposition 4.3 of [8] that

$$(3.20) \quad \lim_{d \rightarrow +\infty} \sqrt{d} k_*(d) = \sigma_0 \in (0, \infty),$$

$$(3.21) \quad \lim_{d \rightarrow 0} \frac{k_*(d)}{\sqrt{d} |\ln d|} = 0 \quad \text{and} \quad \lim_{d \rightarrow 0} \frac{k_*(d)}{\sqrt{d}} = +\infty.$$

This implies in particular that  $k_*(d) \rightarrow 0$  as  $d \rightarrow 0$  and as  $d \rightarrow \infty$ . Hence  $k_*(d)$  is not a monotone function in  $(0, \infty)$ .

#### 4. PROOF OF THEOREM 2.1

In this section, we prove the existence and uniqueness of a global solution to (1.4). The proof follows that of [8] with suitable modifications. Firstly we prove the local existence and uniqueness result by the contraction mapping theorem, then we use suitable estimates to show that the solution is defined for all  $t > 0$ .

**Theorem 4.1.** *For any given  $u_0$  satisfying (1.6) and any constant  $\nu \in (0, 1)$ , there is a  $T > 0$  such that problem (1.4) admits a unique solution*

$$(u, h) \in C^{(1+\nu)/2, 1+\nu}(D_T) \times C^{1+\nu/2}([0, T]);$$

moreover,

$$(4.1) \quad \|u\|_{C^{(1+\nu)/2, 1+\nu}(D_T)} + \|h\|_{C^{1+\nu/2}([0, T])} \leq C,$$

where  $D_T = \{(t, r) \in \mathbb{R}^2 : t \in [0, T], r \in [0, h(t)]\}$ ,  $C$  and  $T$  only depend on  $h_0, \nu$  and  $\|u_0\|_{C^2([0, h_0])}$ .

**Proof:** As in [8], we first follow [6] and straighten the free boundary. Let  $\zeta(s)$  be a function in  $C^3[0, \infty)$  satisfying

$$\zeta(s) = 1 \text{ if } |s - h_0| < \frac{h_0}{8}, \quad \zeta(s) = 0 \text{ if } |s - h_0| > \frac{h_0}{2}, \quad |\zeta'(s)| < \frac{5}{h_0} \text{ for all } s.$$

Consider the transformation

$$(t, y) \rightarrow (t, x) \text{ where } x = y + \zeta(|y|)(h(t) - h_0)y/|y|, \quad y \in \mathbb{R}^N,$$

which induces the transformation

$$(t, s) \rightarrow (t, r) \text{ with } r = s + \zeta(s)(h(t) - h_0), \quad 0 \leq s < \infty.$$

For fixed  $t \geq 0$ , as long as

$$|h(t) - h_0| \leq \frac{h_0}{8},$$

the transformation  $x \rightarrow y$  determined above is a diffeomorphism from  $\mathbb{R}^N$  onto  $\mathbb{R}^N$ , and the induced transformation  $s \rightarrow r$  is a diffeomorphism from  $[0, \infty)$  onto  $[0, \infty)$ . Moreover, it changes the free boundary  $|x| = h(t)$  to the fixed sphere  $|y| = h_0$ . Now, direct calculations show that

$$\begin{aligned} \frac{\partial s}{\partial r} &= \frac{1}{1 + \zeta'(s)(h(t) - h_0)} := \sqrt{A(h(t), s)}, \\ \frac{\partial^2 s}{\partial r^2} &= -\frac{\zeta''(s)(h(t) - h_0)}{[1 + \zeta'(s)(h(t) - h_0)]^3} := B(h(t), s), \\ -\frac{1}{h'(t)} \frac{\partial s}{\partial t} &= \frac{\zeta(s)}{1 + \zeta'(s)(h(t) - h_0)} := C(h(t), s). \end{aligned}$$

Let us denote

$$\begin{aligned} \frac{(N-1)\sqrt{A}}{s + \zeta(s)(h(t) - h_0)} &:= D(h(t), s), \\ \alpha(s + \zeta(s)(h(t) - h_0)) &:= \tilde{\alpha}(h(t), s), \\ \beta(s + \zeta(s)(h(t) - h_0)) &:= \tilde{\beta}(h(t), s). \end{aligned}$$

If we set

$$u(t, r) = u(t, s + \zeta(s)(h(t) - h_0)) = w(t, s),$$

then

$$\begin{aligned} u_t &= w_t - h'(t)C(h(t), s)w_s, \quad u_r = \sqrt{A(h(t), s)}w_s, \\ u_{rr} &= A(h(t), s)w_{ss} + B(h(t), s)w_s \end{aligned}$$

and the free boundary problem (1.4) becomes

$$(4.2) \quad \begin{cases} w_t - Adw_{ss} - (Bd + h'C + Dd)w_s = w(\tilde{\alpha} - \tilde{\beta}w), & t > 0, \quad 0 < s < h_0, \\ w = 0, \quad h'(t) = -\mu w_s, & t > 0, \quad s = h_0, \\ w_s(t, 0) = 0, & t > 0, \\ h(0) = h_0, \quad w(0, s) = u_0(s), & 0 \leq s \leq h_0, \end{cases}$$

where  $A = A(h(t), s)$ ,  $B = B(h(t), s)$ ,  $C = C(h(t), s)$ ,  $D = D(h(t), s)$ ,  $\tilde{\alpha} = \tilde{\alpha}(h(t), s)$  and  $\tilde{\beta} = \tilde{\beta}(h(t), s)$ .

Denote  $\tilde{h}_0 = -\mu u'_0(h_0)$ , and for  $0 < T \leq \frac{h_0}{8(1+\tilde{h}_0)}$ , define  $\Delta_T = [0, T] \times [0, h_0]$ ,

$$\mathcal{D}_{1T} = \{w \in C(\Delta_T) : w(0, s) = u_0(s), \|w - u_0\|_{C(\Delta_T)} \leq 1\},$$

$$\mathcal{D}_{2T} = \{h \in C^1([0, T]) : h(0) = h_0, h'(0) = \tilde{h}_0, \|h' - \tilde{h}_0\|_{C([0, T])} \leq 1\}.$$

It is easily seen that  $\mathcal{D} := \mathcal{D}_{1T} \times \mathcal{D}_{2T}$  is a complete metric space with the metric

$$d((w_1, h_1), (w_2, h_2)) = \|w_1 - w_2\|_{C(\Delta_T)} + \|h'_1 - h'_2\|_{C([0, T])}.$$

Let us note that for  $h_1, h_2 \in \mathcal{D}_{2T}$ , due to  $h_1(0) = h_2(0) = h_0$ ,

$$(4.3) \quad \|h_1 - h_2\|_{C([0, T])} \leq T \|h'_1 - h'_2\|_{C([0, T])}.$$

Next, we shall prove the existence and uniqueness result by using the contraction mapping theorem. Firstly we observe that due to our choice of  $T$ , for any given  $(w, h) \in \mathcal{D}_{1T} \times \mathcal{D}_{2T}$ , we have

$$|h(t) - h_0| \leq T(1 + \tilde{h}_0) \leq \frac{h_0}{8}.$$

Therefore the transformation  $(t, s) \rightarrow (t, r)$  introduced at the beginning of the proof is well-defined. Moreover, for  $0 \leq s \leq h_0/2$ , we have  $\zeta(s) \equiv 0$  and hence for such  $s$ ,

$$A \equiv 1, \quad B \equiv C \equiv 0, \quad D \equiv (N - 1)/s.$$

Therefore

$$-Adw_{ss} - (Bd + h'C + Dd)w_s = -d\Delta w \text{ in the ball } |y| \leq h_0/2.$$

Thus although  $D = D(h(t), s)$  is singular at  $s = 0$ ,

$$Adw_{ss} + (Bd + h'C + Dd)w_s$$

actually represents an elliptic operator acting on  $w = w(t, y) (= w(t, |y|))$  over the ball  $|y| \leq h_0$ , whose coefficients are continuous in  $(t, y)$  when  $h \in \mathcal{D}_{2T}$ .

Applying standard  $L^p$  theory and then the Sobolev imbedding theorem ([15]), we find that for any  $(w, h) \in \mathcal{D}$ , the following initial boundary value problem

$$(4.4) \quad \begin{cases} \bar{w}_t - Ad\bar{w}_{ss} - (Bd + h'C + Dd)\bar{w}_s = w(\tilde{\alpha} - \tilde{\beta}w), & t > 0, 0 \leq s < h_0, \\ \bar{w}_s(t, 0) = 0, \quad \bar{w}(t, h_0) = 0, & t > 0, \\ \bar{w}(0, s) = u_0(s), & 0 \leq s \leq h_0 \end{cases}$$

admits a unique solution  $\bar{w} \in C^{(1+\nu)/2, 1+\nu}(\Delta_T)$ , and

$$(4.5) \quad \|\bar{w}\|_{C^{(1+\nu)/2, 1+\nu}(\Delta_T)} \leq C_1,$$

where  $C_1$  is a constant dependent on  $h_0, \nu$  and  $\|u_0\|_{C^2[0, h_0]}$ .

Setting

$$(4.6) \quad \bar{h}(t) := h_0 - \int_0^t \mu \bar{w}_s(\tau, h_0) d\tau,$$

we have

$$\bar{h}'(t) = -\mu \bar{w}_s(t, h_0), \quad \bar{h}(0) = h_0, \quad \bar{h}'(0) = -\mu \bar{w}_s(0, h_0) = \tilde{h}_0,$$

and hence  $\bar{h}' \in C^{\nu/2}([0, T])$  with

$$(4.7) \quad \|\bar{h}'\|_{C^{\nu/2}([0, T])} \leq C_2 := \mu C_1.$$

We now define  $\mathcal{F} : \mathcal{D} \rightarrow C(\Delta_T) \times C^1([0, T])$  by

$$\mathcal{F}(w, h) = (\bar{w}, \bar{h}).$$

Clearly  $(w, h) \in \mathcal{D}$  is a fixed point of  $\mathcal{F}$  if and only if it solves (4.2).

By (4.5) and (4.7), we have

$$\|\bar{h}' - \tilde{h}_0\|_{C([0, T])} \leq \|\bar{h}'\|_{C^{\nu/2}([0, T])} T^{\nu/2} \leq \mu C_1 T^{\nu/2},$$

$$\|\bar{w} - u_0\|_{C(\Delta_T)} \leq \|\bar{w} - u_0\|_{C^{(1+\nu)/2, 0}(\Delta_T)} T^{(1+\nu)/2} \leq C_1 T^{(1+\nu)/2}.$$

Therefore if we take  $T \leq \min\{(\mu C_1)^{-2/\nu}, C_1^{-2/(1+\nu)}\}$ , then  $\mathcal{F}$  maps  $\mathcal{D}$  into itself.

Next we prove that  $\mathcal{F}$  is a contraction mapping on  $\mathcal{D}$  for  $T > 0$  sufficiently small. Indeed, let  $(w_i, h_i) \in \mathcal{D}$  ( $i = 1, 2$ ) and denote  $(\bar{w}_i, \bar{h}_i) = \mathcal{F}(w_i, h_i)$ . Then it follows from (4.5) and (4.7) that

$$\|\bar{w}_i\|_{C^{(1+\nu)/2, 1+\nu}(\Delta_T)} \leq C_1, \quad \|\bar{h}'_i(t)\|_{C^{\nu/2}([0, T])} \leq C_2.$$

Setting  $W = \bar{w}_1 - \bar{w}_2$ , we find that  $W(t, s)$  satisfies

$$\begin{aligned} & W_t - A(h_2, s)dW_{ss} - [B(h_2, s)d + h'_2 C(h_2, s) + D(h_2, s)d]W_s \\ &= [A(h_1, s) - A(h_2, s)]d\bar{w}_{1,ss} + [B(h_1, s) - B(h_2, s) + D(h_1, s) - D(h_2, s)]d\bar{w}_{1,s} \\ & \quad + [h'_1 C(h_1, s) - h'_2 C(h_2, s)]\bar{w}_{1,s} + (w_1 - w_2)(\tilde{\alpha}(h_1, s) - \tilde{\beta}(h_1, s)(w_1 + w_2)) \\ & \quad + w_2[(\tilde{\alpha}(h_1, s) - \tilde{\alpha}(h_2, s)) - (\tilde{\beta}(h_1, s) - \tilde{\beta}(h_2, s))w_2], \quad t > 0, \quad 0 \leq s < h_0, \\ & W_s(t, 0) = 0, \quad W(t, h_0) = 0, \quad t > 0, \\ & W(0, s) = 0, \quad 0 \leq s \leq h_0. \end{aligned}$$

Using the  $L^p$  estimates for parabolic equations and Sobolev's imbedding theorem, we obtain

$$(4.8) \quad \|\bar{w}_1 - \bar{w}_2\|_{C^{(1+\nu)/2, 1+\nu}(\Delta_T)} \leq C_3(\|w_1 - w_2\|_{C(\Delta_T)} + \|h_1 - h_2\|_{C^1([0, T])}),$$

where  $C_3$  depends on  $C_1, C_2$  and the functions  $A, B, C$  and  $D$  in the definition of the transformation  $(t, s) \rightarrow (t, r)$ . Taking the difference of the equations for  $\bar{h}_1, \bar{h}_2$  results in

$$(4.9) \quad \|\bar{h}'_1 - \bar{h}'_2\|_{C^{\nu/2}([0, T])} \leq \mu \left( \|\bar{w}_{1,s} - \bar{w}_{2,s}\|_{C^{\nu/2, 0}(\Delta_T)} \right).$$

Combining (4.3), (4.8) and (4.9), and assuming  $T \leq 1$ , we obtain

$$\begin{aligned} & \|\bar{w}_1 - \bar{w}_2\|_{C^{(1+\nu)/2, 1+\nu}(\Delta_T)} + \|\bar{h}'_1 - \bar{h}'_2\|_{C^{\nu/2}([0, T])} \\ & \leq C_4(\|w_1 - w_2\|_{C(\Delta_T)} + \|h'_1 - h'_2\|_{C[0, T]}), \end{aligned}$$

with  $C_4$  depending on  $C_3$  and  $\mu$ . Hence for

$$T := \min \left\{ 1, \left( \frac{1}{2C_4} \right)^{2/\nu}, (\mu C_1)^{-2/\nu}, C_1^{-2/(1+\nu)}, \frac{h_0}{8(1 + \tilde{h}_0)} \right\},$$

we have

$$\begin{aligned}
& \|\bar{w}_1 - \bar{w}_2\|_{C(\Delta_T)} + \|\bar{h}'_1 - \bar{h}'_2\|_{C([0,T])} \\
& \leq T^{(1+\nu)/2} \|\bar{w}_1 - \bar{w}_2\|_{C^{(1+\nu)/2, 1+\nu}(\Delta_T)} + T^{\nu/2} \|\bar{h}'_1 - \bar{h}'_2\|_{C^{\nu/2}([0,T])} \\
& \leq C_4 T^{\nu/2} (\|w_1 - w_2\|_{C(\Delta_T)} + \|h'_1 - h'_2\|_{C([0,T])}) \\
& \leq \frac{1}{2} (\|w_1 - w_2\|_{C(\Delta_T)} + \|h'_1 - h'_2\|_{C([0,T])}).
\end{aligned}$$

This shows that for this  $T$ ,  $\mathcal{F}$  is a contraction mapping on  $\mathcal{D}$ . It now follows from the contraction mapping theorem that  $\mathcal{F}$  has a unique fixed point  $(w, h)$  in  $\mathcal{D}$ . Moreover, by the Schauder estimates, we have additional regularity for  $(w, h)$  as a solution of (4.2), namely,  $h \in C^{1+\nu/2}(0, T]$  and  $w \in C^{1+\nu/2, 2+\nu}((0, T] \times [0, h_0])$ , and (4.5), (4.7) hold. In other words,  $(w(t, s), h(t))$  is a unique local classical solution of the problem (4.2).  $\square$

To show that the local solution obtained in Theorem 4.1 can be extended to all  $t > 0$ , we need the following estimate.

**Lemma 4.2.** *Let  $(u, h)$  be a solution to problem (1.4) defined for  $t \in (0, T_0)$  for some  $T_0 \in (0, +\infty]$ . Then there exist constants  $C_1$  and  $C_2$  independent of  $T_0$  such that*

$$0 < u(t, r) \leq C_1, \quad 0 < h'(t) \leq C_2 \quad \text{for } 0 \leq r < h(t), \quad t \in (0, T_0).$$

**Proof:** Using the strong maximum principle to the equation of  $u$  we immediately obtain

$$u(t, r) > 0, \quad u_r(t, h(t)) < 0 \quad \text{for } 0 < t < T_0, 0 \leq r < h(t).$$

Hence  $h'(t) > 0$  for  $t \in (0, T_0)$ .

Since (1.5) holds, it follows from the comparison principle that  $u(t, r) \leq \bar{u}(t)$  for  $t \in (0, T_0)$  and  $r \in [0, h(t)]$ , where

$$\bar{u}(t) := \frac{\kappa_2}{\kappa_1} e^{\frac{\kappa_2}{\kappa_1} t} \left( e^{\frac{\kappa_2}{\kappa_1} t} - 1 + \frac{\kappa_2}{\kappa_1 \|u_0\|_\infty} \right)^{-1},$$

which is the solution of the problem

$$(4.10) \quad \frac{d\bar{u}}{dt} = \bar{u}(\kappa_2 - \kappa_1 \bar{u}) \quad t > 0; \quad \bar{u}(0) = \|u_0\|_\infty.$$

Thus we have

$$u(t, r) \leq C_1 := \sup_{t \geq 0} \bar{u}(t).$$

It remains to show that  $h'(t) \leq C_2$  for all  $t \in (0, T_0)$  and some  $C_2$  independent of  $T_0$ . To this end, we define

$$\Omega = \Omega_M := \{(t, r) : 0 < t < T_0, h(t) - M^{-1} < r < h(t)\}$$

and construct an auxiliary function

$$w(t, r) := C_1 [2M(h(t) - r) - M^2(h(t) - r)^2].$$

We will show that  $M$  can be chosen so that  $w(t, r) \geq u(t, r)$  holds over  $\Omega$ .

Direct calculations show that, for  $(t, r) \in \Omega$ ,

$$\begin{aligned} w_t &= 2C_1 M h'(t) [1 - M(h(t) - r)] \geq 0, \\ -w_r &= 2MC_1 [1 - M(h(t) - r)] \geq 0, \\ -w_{rr} &= 2C_1 M^2, \quad w[\alpha(r) - \beta(r)w] \leq \kappa_2 C_1. \end{aligned}$$

It follows that

$$w_t - d(w_{rr} + \frac{N-1}{r}w_r) \geq 2dC_1 M^2 \geq \kappa_2 u \text{ in } \Omega$$

if  $M^2 \geq \frac{\kappa_2}{2d}$ . On the other hand,

$$w(t, h(t) - M^{-1}) = C_1 \geq u(t, h(t) - M^{-1}), \quad w(t, h(t)) = 0 = u(t, h(t)).$$

Thus, if we can choose  $M$  such that  $u_0(r) \leq w(0, r)$  for  $r \in [h_0 - M^{-1}, h_0]$ , then we can apply the maximum principle to  $w - u$  over  $\Omega$  to deduce that  $u(t, r) \leq w(t, r)$  for  $(t, r) \in \Omega$ . It would then follow that

$$u_r(t, h(t)) \geq w_r(t, h(t)) = -2MC_1, \quad h'(t) = -\mu u_r(t, h(t)) \leq C_2 := 2MC_1\mu.$$

To complete the proof, we only have to find some  $M$  independent of  $T_0$  such that  $u_0(r) \leq w(0, r)$  for  $r \in [h_0 - M^{-1}, h_0]$ . We calculate

$$w_r(0, r) = -2C_1 M [1 - M(h_0 - r)] \leq -C_1 M \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Therefore upon choosing

$$M := \max \left\{ \sqrt{\frac{\kappa_2}{2d}}, \frac{4\|u_0\|_{C^1([0, h_0])}}{3C_1} \right\},$$

we will have

$$w_r(0, r) \leq u'_0(r) \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Since  $w(0, h_0) = u_0(h_0) = 0$ , the above inequality implies

$$w(0, r) \geq u_0(r) \text{ for } r \in [h_0 - (2M)^{-1}, h_0].$$

Moreover, for  $r \in [h_0 - M^{-1}, h_0 - (2M)^{-1}]$ , we have

$$w(0, r) \geq \frac{3}{4}C_1, \quad u_0(r) \leq \|u_0\|_{C^1([0, h_0])} M^{-1} \leq \frac{3}{4}C_1.$$

Therefore  $u_0(r) \leq w(0, r)$  for  $r \in [h_0 - M^{-1}, h_0]$ . This completes the proof.  $\square$

**Theorem 4.3.** *The solution of problem (1.4) exists and is unique for all  $t \in (0, \infty)$ .*

**Proof:** Let  $[0, T_{max})$  be the maximal time interval in which the solution exists. By Theorem 4.1,  $T_{max} > 0$ . It remains to show that  $T_{max} = \infty$ . Arguing indirectly, we assume that  $T_{max} < \infty$ . By Lemma 4.2, there exist  $C_1$  and  $C_2$  independent of  $T_{max}$  such that for  $t \in [0, T_{max})$  and  $r \in [0, h(t)]$ ,

$$0 \leq u(t, r) \leq C_1, \quad h_0 \leq h(t) \leq h_0 + C_2 t, \quad 0 \leq h'(t) \leq C_2.$$

We now fix  $\delta_0 \in (0, T_{max})$  and  $\tilde{T} > T_{max}$ . By standard parabolic regularity, we can find  $C_3 > 0$  depending only on  $\delta_0, \tilde{T}, C_1$  and  $C_2$  such that  $\|u(t, \cdot)\|_{C^2([0, h(t)])} \leq C_3$  for  $t \in [\delta_0, T_{max})$ . It then follows from the proof of Theorem 4.1 that there exists a  $\tau > 0$  depending only on  $C_3, C_2$  and  $C_1$  such that the solution of problem (1.4)

with initial time  $T_{max} - \tau/2$  can be extended uniquely to the time  $T_{max} - \tau/2 + \tau$ . But this contradicts the assumption. The proof is complete.  $\square$

**Remark 4.4.** *It follows from the uniqueness of the solution to (1.4) and some standard compactness argument that the unique solution  $(u, h)$  depends continuously on  $u_0$  and the parameters appearing in (1.4).*

## REFERENCES

- [1] D.G. Aronson and H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in *Partial Differential Equations and Related Topics*, Lecture Notes in Math., Vol. **446**, Springer, Berlin, 1975, pp. 5–49.
- [2] D.G. Aronson and H.F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, *Adv. Math.* **30** (1978), 33–76.
- [3] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems. I. Periodic framework, *J. Eur. Math. Soc.* **7** (2005), no. 2, 173–213.
- [4] H. Berestycki, F. Hamel and L. Roques, Analysis of the periodically fragmented environment model. II. Biological invasions and pulsating travelling fronts, *J. Math. Pures Appl.* **84** (2005), no. 8, 1101–1146.
- [5] R.S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-diffusion Equations*, John Wiley & Sons Ltd, 2003.
- [6] X. F. Chen and A. Friedman, A free boundary problem arising in a model of wound healing, *SIAM J. Math. Anal.* **32** (2000), 778–800.
- [7] W. Dong and Y. Du, Unbounded principal eigenfunctions and the logistic equation on  $R^N$ , *Bull. Australian Math. Soc.* **67** (2003), 413–427.
- [8] Y. Du and Z.G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, to appear.
- [9] Y. Du and L.S. Liu, Remarks on the uniqueness problem for the logistic equation on the entire space, *Bull. Austral. Math. Soc.* **73** (2006), 129–137.
- [10] Y. Du and L. Ma, Logistic type equations on  $\mathbb{R}^N$  by a squeezing method involving boundary blow-up solutions, *J. London Math. Soc.* **64** (2001), 107–124.
- [11] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* **7** (1937), 335–369.
- [12] F. Hamel and N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in  $\mathbb{R}^N$ , *Arch. Ration. Mech. Anal.* **157** (2001), no. 2, 91–163.
- [13] D. Hilhorst, M. Mimura, R. Schatzle, Vanishing latent heat limit in a Stefan-like problem arising in biology, *Nonlinear Anal. RWA* **4** (2003), 261–285.
- [14] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. Moscou Sér. Internat.* **A1** (1937), 1–26; English transl. in: *Dynamics of Curved Fronts*, P. Pelcé (ed.), Academic Press, 1988, 105–130.
- [15] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc., Providence, RI, 1968.
- [16] X. Liang and X-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.* **60** (2007), no. 1, 1–40.
- [17] Z. G. Lin, A free boundary problem for a predator-prey model, *Nonlinearity* **20** (2007), 1883–1892.
- [18] J. L. Lockwood, M.F. Hoopes and M.P. Marchetti, *Invasion Ecology*, Blackwell Publishing, 2007.
- [19] M. Mimura, Y. Yamada, S. Yotsutani, A free boundary problem in ecology, *Japan J. Appl. Math.* **2**(1985), 151–186.
- [20] M. Mimura, Y. Yamada, S. Yotsutani, Stability analysis for free boundary problems in ecology, *Hiroshima Math. J.* **16** (1986), 477–498.

- [21] M. Mimura, Y. Yamada, S. Yotsutani, Free boundary problems for some reaction-diffusion equations, *Hiroshima Math. J.* **17** (1987), 241-280.
- [22] D.H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.* **21** (1972), 979-1000.
- [23] N. Shigesada and K. Kawasaki, *Biological Invasions: Theory and Practice*, Oxford Series in Ecology and Evolution, Oxford Univ. Press., Oxford, 1997.
- [24] J.G. Skellam, Random dispersal in theoretical populations, *Biometrika* **38** (1951), 196-218.
- [25] H.F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, *J. Math. Biol.* **45** (2002), no. 6, 511-548.
- [26] H.F. Weinberger, M.A. Lewis and B. Li, Anomalous spreading speeds of cooperative recursion systems, *J. Math. Biol.* **55** (2007), no. 2, 207-222.
- [27] J.X. Xin, Front propagation in heterogeneous media, *SIAM Rev.* **42** (2000), no. 2, 161-230.