SPREADING SPEED AND PROFILE FOR NONLINEAR STEFAN PROBLEMS IN HIGH SPACE DIMENSIONS

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ABSTRACT. We consider nonlinear diffusion problems of the form $u_t = \Delta u + f(u)$ with Stefan type free boundary conditions, where the nonlinear term $f(u)$ is of monostable, bistable or combustion type. Such problems arise as an alternative model (to the corresponding Cauchy problem) to describe the spreading of a biological or chemical species, where the free boundary represents the expanding front. We are interested in its long-time spreading behavior in the radially symmetric case, where the equation is satisfied in $|x| < h(t)$, with $|x| = h(t)$ the free boundary, and $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} u(t, |x|) = 1$. For the case of one space dimension ($N = 1$), Du and Lou [8] proved that $\lim_{t\to\infty} \frac{h(t)}{t} = c^*$ for some $c^* > 0$. Subsequently, sharper estimate of the spreading speed was obtained by the authors of the current paper in [11], in the form that $\lim_{t\to\infty}[h(t) - c^*t] = \hat{H} \in \mathbb{R}^1$. In this paper, we consider the case $N \geq 2$ and show that a logarithmic shifting occurs, namely there exists $c_* > 0$ independent of N such that $\lim_{t\to\infty}[h(t) - c^*t + (N-1)c_* \log t] = \hat{h} \in \mathbb{R}^1$. At the same time, we also obtain a rather clear description of the spreading profile of $u(t, r)$. These results contrast sharply with those for the corresponding Cauchy problem, where the logarithmic shifting for the monostable case is significantly different from that for the bistable and combustion cases.

1. INTRODUCTION

We are interested in obtaining exact long-time limit of the spreading speed and profile determined by the following free boundary problem:

$$
\begin{cases}
u_t = \Delta u + f(u), & 0 < r < h(t),\ t > 0,\\
u_r(t,0) = 0,\ u(t,h(t)) = 0, & t > 0,\\
h'(t) = -\mu u_r(t,h(t)), & t > 0,\\
h(0) = h_0, \quad u(0,r) = u_0(r), & 0 \leq r \leq h_0,
\end{cases}
\tag{1.1}
$$

where $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, $r = h(t)$ is the moving boundary to be determined, μ and h_0 are given positive constants. The initial function u_0 is chosen from

$$
\mathscr{K}(h_0) := \Big\{\psi \in C^2([0,h_0]) : \psi'(0) = \psi(h_0) = 0,\ \psi(r) > 0 \text{ in } [0,h_0)\Big\}.
\tag{1.2}
$$

For any given $h_0 > 0$ and $u_0 \in \mathscr{K}(h_0)$, by a classical solution of (1.1) on the time-interval $[0,T]$ we mean a pair $(u(t,r), h(t))$ belonging to $C^{1,2}(D_T) \times C^1([0,T])$, such that all the identities in (1.1) are satisfied pointwisely, where

$$
D_T := \big\{(t,r) : t \in (0,T],\ r \in [0,h(t)]\big\}.
$$

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The nonlinearity $f(u)$ is assumed to be of monostable, bistable or combustion type, whose meanings will be made precise below.

When $f(u) \equiv 0$, (1.1) reduces to the classical one-phase Stefan problem, which arises in the study of melting of ice in contact with water. Our motivation to study the nonlinear Stefan problem (1.1) mainly comes from the wish to better understand the spreading of a new species, where u is viewed as the density of such a species, and the free boundary represents the spreading front, beyond which the species cannot be observed (i.e., the species has density 0).

The spreading process is usually modeled by the Cauchy problem

$$\begin{cases} U_t - \Delta U = f(U) & \text{for } x \in \mathbb{R}^N,\ t > 0, \\ U(0, x) = U_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $U_0(x)$ is nonnegative and has nonempty compact support. In such a case, $U(t, x) > 0$ for all $x \in \mathbb{R}^N$ once $t > 0$, but one may specify a certain level set $\Gamma_\delta(t) := \{x : U(t, x) = \delta\}$ as the spreading front, where $\delta > 0$ is small, and $\Omega_\delta(t) := \{x : U(t, x) > \delta\}$ is regarded as the range where the species can be observed. A striking feature of the long time behavior of the front $\Gamma_\delta(t)$ is that, when spreading happens (i.e., $U(t, x) \to 1$ as $t \to \infty$), $\Gamma_\delta(t)$ goes to infinity at a constant asymptotic speed in all directions, namely, for any small $\epsilon > 0$, there exists $T > 0$ so that

$$\Gamma_\delta(t) \subset A_\epsilon(t) := \{x \in \mathbb{R}^N : (c_0 - \epsilon)t \le |x| \le (c_0 + \epsilon)t\} \ \text{ for } t \ge T. \tag{1.4}$$

The number c_0 is usually called the spreading speed of (1.3), and is determined by the well-known traveling wave problem

$$Q'' - cQ' + f(Q) = 0,\ Q > 0 \text{ in } \mathbb{R}^1,\ Q(-\infty) = 0,\ Q(+\infty) = 1,\ Q(0) = 1/2. \tag{1.5}$$

More precisely, in the monostable case, $c_0 > 0$ is the minimal value of c such that (1.5) has a solution Q_c (more accurately Q_c exists if and only if $c \ge c_0$); in the bistable and combustion cases, c_0 is the unique value of c such that (1.5) has a solution Q_c. Moreover, Q_c is unique when it exists for a given c. When $U_0(x)$ is radially symmetric, then $U(t, x)$ is radially symmetric in x for any $t > 0$, and better estimates of the spreading speed and the profile of U near the front are available, which will be recalled briefly below.

Problem (1.1) is the spherically symmetric version of the general nonlinear Stefan problem studied in [6] and [10], which has the form

$$\begin{cases} u_t - \Delta u = f(u) & \text{for } x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu |\nabla_x u|^2 & \text{for } x \in \Gamma(t), t > 0, \\ u(0, x) = u_0(x) & \text{for } x \in \Omega_0, \end{cases} \tag{1.6}$$

where $\Omega(t) \subset \mathbb{R}^N$ ($N \ge 1$) is bounded by the free boundary $\Gamma(t)$ (i.e., $\Gamma(t) = \partial\Omega(t)$), with $\Omega(0) = \Omega_0$, which is a bounded domain that agrees with the interior of its closure $\overline{\Omega}_0$, $\partial\Omega_0$ satisfies the interior ball condition, and $u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0)$ is positive in Ω_0 and vanishes on $\partial\Omega_0$. If $u_0(x)$ in (1.6) is radially symmetric, then (1.6) reduces to (1.1).

It follows from [6] that (1.6) has a unique weak solution which is defined for all $t > 0$. One of the main results in [10] for the general problem (1.6) implies the following:

Theorem A. *$\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$. Moreover, $\Omega_\infty := \cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^N$, or it is a bounded set. Furthermore, when $\Omega_\infty = \mathbb{R}^N$, for all large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^N$, and there exists a continuous function $M(t)$ such that*

$$\Gamma(t) \subset \{x : M(t) - \frac{d_0}{2}\pi \le |x| \le M(t)\}; \tag{1.7}$$

and when Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t, \cdot)\|_{L^\infty(\Omega(t))} = 0$. Here d_0 is the diameter of Ω_0.

It can be shown (see [9]) that when spreading happens (i.e., $u(t,x) \to 1$ as $t \to \infty$), there exists $c^* > 0$ such that

$$\lim_{t\to\infty} \frac{M(t)}{t} = c^*. \tag{1.8}$$

The number c^* is therefore called the asymptotic spreading speed of (1.6), which is determined by the following problem,

$$q'' - cq' + f(q) = 0,\ q > 0 \ \text{ in } (0,\infty),\ \ q(0) = 0,\ q(\infty) = 1. \tag{1.9}$$

The above discussion shows that when spreading happens, (1.3) and (1.6) exhibit similar asymptotic behavior: Their fronts can be approximated by spheres, which go to infinity at some constant asymptotic speed. Moreover, by [6], if u and $\Omega(t)$ in (1.6) are denoted by u_μ and $\Omega_\mu(t)$, respectively, then as $\mu \to \infty$,

$$\Omega_\mu(t) \to \mathbb{R}^N (\forall t > 0),\ \ u_\mu \to U \text{ in } C^{(1+\nu)/2,1+\nu}_{loc}((0,\infty)\times\mathbb{R}^N)\ (\forall \nu \in (0,1)),$$

where U is the unique solution of (1.3) with $U_0 = u_0$. Thus the Cauchy problem (1.3) may be regarded as the limiting problem of (1.6) as $\mu \to \infty$.

It turns out that underneath these similarities, there exist fundamental differences between (1.6) and (1.3). This paper is devoted to revealing these differences. It is our hope that this may provide further insights to the understanding of the mechanisms underlying so many different spreading processes.

For such a purpose, we will restrict to the simpler spherically symmetric case (1.1), for which we are able to gain deeper understanding of the spreading profile of the free boundary model. If we take

$$U_0(x) = \begin{cases} u_0(|x|), & |x| < h_0, \\ 0, & |x| \geq h_0, \end{cases} \tag{1.10}$$

with u_0 given in (1.1), then the unique solution of (1.3) is radially symmetric: $U = U(t,|x|)$. Thus for such U_0, (1.1) provides an alternative to (1.3) for the description of the spreading of a certain species with initial density u_0. We will closely examine the spreading behavior determined by (1.1) and compare it with that of (1.3).

While the Cauchy problem (1.3) has been extensively studied in the past several decades and relatively well understood (some relevant results for (1.3) will be recalled below), the study of the nonlinear free boundary problem (1.1) is rather recent. Problem (1.1) with $f(u) = au - bu^2$ was investigated in [5], continuing a study initiated in [7] for the one space dimension case. A deduction of the free boundary condition based on ecological assumptions can be found in [4], but generally speaking, the role of this free boundary condition in the mechanism of spreading is still poorly understood.

In [8], problem (1.1) with a rather general $f(u)$ but in one space dimension was considered. In particular, if $f(u)$ is of monostable, or bistable, or combustion type, it was shown in [8] that (1.1) has a unique solution which is defined for all $t > 0$, and as $t \to \infty$, $h(t)$ either increases to a finite number h_∞, or it increases to $+\infty$. Moreover, in the former case, $u(t,r) \to 0$ uniformly in r, while in the latter case, $u(t,r) \to 1$ locally uniformly in $r \in [0,+\infty)$ (except for a transition case when f is of bistable or combustion type). The situation that

$$u \to 0 \text{ and } h \to h_\infty < +\infty$$

is called the **vanishing** case, and

$$u \to 1 \text{ and } h \to +\infty$$

is called the **spreading** case.

When spreading happens, it was shown in [8] that there exists $c^* > 0$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = c^*.$$

The number c^* is the same as in (1.8). These conclusions remain valid in higher space dimensions ([9]).

Next we will describe the results more accurately. Firstly, let us recall in detail the three types of nonlinearities of f mentioned above:

$$(\mathrm{f_M})\ \text{ monostable case, }\ (\mathrm{f_B})\ \text{ bistable case, }\ (\mathrm{f_C})\ \text{ combustion case.}$$

In the monostable case $(\mathrm{f_M})$, we assume that f is C^1 and it satisfies

$$f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0,\ (1-u)f(u) > 0 \text{ for } u > 0, u \neq 1.$$

A typical example is $f(u) = u(1-u)$.

In the bistable case $(\mathrm{f_B})$, we assume that f is C^1 and it satisfies

$$\begin{cases} f(0) = f(\theta) = f(1) = 0, \\ f(u) < 0 \text{ in } (0,\theta),\ f(u) > 0 \text{ in } (\theta, 1),\ f(u) < 0 \text{ in } (1,\infty), \end{cases}$$

for some $\theta \in (0,1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(s)ds > 0.$$

A typical example is $f(u) = u(u-\theta)(1-u)$ with $\theta \in \left(0, \frac{1}{2}\right)$.

In the combustion case $(\mathrm{f_C})$, we assume that f is C^1 and it satisfies

$$f(u) = 0 \text{ in } [0,\theta],\quad f(u) > 0 \text{ in } (\theta,1),\ f'(1) < 0,\ f(u) < 0 \text{ in } [1,\infty)$$

for some $\theta \in (0,1)$, and there exists a small $\delta_0 > 0$ such that

$$f(u) \text{ is nondecreasing in } (\theta, \theta + \delta_0).$$

The asymptotic spreading speed c^* is determined in the following way.

Proposition 1.1 (Proposition 1.8 and Theorem 6.2 of [8])**.** *Suppose that f is of* $(\mathrm{f_M})$*, or* $(\mathrm{f_B})$*, or* $(\mathrm{f_C})$ *type. Then for any $\mu > 0$ there exists a unique $c^* = c^*(\mu) > 0$ and a unique solution q_{c^*} to* (1.9) *with $c = c^*$ such that $q'_{c^*}(0) = \frac{c^*}{\mu}$.*

We remark that this function q_{c^*} is shown in [8] to satisfy $q'_{c^*}(z) > 0$ for $z \geq 0$. We call q_{c^*} a *semi-wave with speed* c^*, since the function $v(t,x) := q_{c^*}(c^*t - x)$ satisfies

$$\begin{cases} v_t = v_{xx} + f(v) \ \text{ for }\ t \in \mathbb{R}^1, x < c^*t, \\ v(t, c^*t) = 0,\ -\mu v_x(t, c^*t) = c^*,\ v(t, -\infty) = 1. \end{cases}$$

In [11], sharper estimate of the spreading speed in one space dimension was obtained. More precisely it was shown in [11] that when spreading happens for (1.1), there exists $\hat{H} \in \mathbb{R}$ such that

$$\lim_{t\to\infty} (h(t) - c^*t - \hat{H}) = 0,\ \lim_{t\to\infty} h'(t) = c^*, \tag{1.11}$$

$$\lim_{t\to\infty} \sup_{r\in[0,\,h(t)]} \Big| u(t,r) - q_{c^*}(h(t) - r) \Big| = 0. \tag{1.12}$$

In this paper, we consider the case that the space dimension $N \geq 2$, and spreading happens for (1.1), namely

$$\lim_{t\to\infty} h(t) = \infty \text{ and } \lim_{t\to\infty} u(t,r) = 1 \text{ locally uniformly for } r \in [0,\infty).$$

We will show that in such a case, we still have (1.12) and $\lim_{t\to\infty} h'(t) = c^*$, but there exists $c_* > 0$ independent of N such that

$$\lim_{t\to\infty} \big[h(t) - c^*t + (N-1)c_* \log t\big] = \hat{h} \in \mathbb{R}^1. \tag{1.13}$$

Moreover, the constant c_* is given by

$$c_* = \frac{1}{\zeta\, c^*},\ \ \zeta = 1 + \frac{c^*}{\mu^2 \int_0^\infty q'_{c^*}(z)^2 e^{-c^* z} dz}.$$

The term $(N-1)c_* \log t$ in (1.13) will be called a logarithmic shifting term. For simplicity of notation, we will write $c_N = (N-1)c_*$. Thus from (1.13) and (1.12) we obtain

$$\lim_{t\to\infty} \sup_{r\in[0,\,h(t)]} \Big| u(t,r) - q_{c^*}(c^*t - c_N \log t + \hat{h} - r)\Big| = 0.$$

For convenience of comparison, we now recall some relevant results for the corresponding Cauchy problem (1.3). The classical paper of Aronson and Weinberger [2] contains a systematic investigation of this problem (see [1] for the case of one space dimension). Various sufficient conditions for $\lim_{t\to\infty} U(t,x) = 1$ ("spreading" or "propagation") and for $\lim_{t\to\infty} U(t,x) = 0$ ("vanishing" or "extinction") are known, and the way $U(t,x)$ approaches 1 as $t\to\infty$ has been used to describe the spreading of a (biological or chemical) species. In particular, when spreading happens, it was shown in [2] that, in any space dimension $N \geq 1$, there exists $c_0 > 0$ independent of N, such that, for any small $\epsilon > 0$,

$$\begin{cases} \lim_{t\to\infty} \max_{|x|\geq (c_0+\epsilon)t} U(t,x) = 0, \\ \lim_{t\to\infty} \max_{|x|\leq (c_0-\epsilon)t} |U(t,x) - 1| = 0. \end{cases} \tag{1.14}$$

Clearly (1.4) is a consequence of (1.14) (with the same c_0). The relationship between the spreading speed determined by (1.1) and that determined by (1.3) is given by (see Theorem 6.2 of [8])

$$c_0 = \lim_{\mu\to\infty} c^*(\mu).$$

More details on the spreading behavior of the Cauchy problem can be found, for example, in [1, 2, 13, 14, 18, 19, 20, 24].

As we will explain below, fundamental differences arise between the free boundary problem and the Cauchy problem when we compare their spreading profiles closely. While the spreading profiles of all three basic cases (f_M), (f_B) and (f_C) can be described in a unified fashion for the free boundary model (see (1.11), (1.12) and (1.13)), where no logarithmic shifting occurs in space dimension $N = 1$, and a synchronized logarithmic shifting happens in dimensions $N \geq 2$, this is not the case for the Cauchy problem, where the monostable case behaves very differently from the other two cases: The monostable case gives rise to logarithmic shifting in all dimensions $N \geq 1$, and the shifting is significantly different from the other two cases.

More precisely, in one space dimension, a classical result of Fife and McLeod [13] states that for f of type (f_B), and for appropriate initial function U_0 that guarantees $U(t,x) \to 1$ as $t\to\infty$, where U is the unique solution to (1.3), the spreading profile of U is described by

$$|U(t,x) - Q_{c_0}(c_0 t + x + C_-)| < Ke^{-\omega t} \text{ for } x < 0,$$

$$|U(t,x) - Q_{c_0}(c_0 t - x + C_+)| < Ke^{-\omega t} \text{ for } x > 0.$$

Here (c_0, Q_{c_0}) is the unique solution of (1.5), $C_\pm \in \mathbb{R}$, and K, ω are suitable positive constants. So no logarithmic shifting occurs in this case.

The monostable case of (1.3) has very different behavior. Firstly we recall that (1.5) already behaves differently in the monostable case. Secondly, a logarithmic shifting occurs: When (f_M) holds and furthermore $f(u) \leq f'(0)u$ for $u \in (0,1)$ (so f falls to the so called "pulled" case), there exist constants $C_\pm$ such that

$$\lim_{t\to\infty} \max_{x\geq 0} \left| U(t,x) - Q_{c_0}\Big(c_0 t - \frac{3}{c_0}\log t - x + C_+\Big) \right| = 0,$$

and

$$\lim_{t\to\infty} \max_{x\leq 0} \left| U(t,x) - Q_{c_0}\Big(c_0 t - \frac{3}{c_0}\log t + x + C_-\Big) \right| = 0.$$

Here the logarithmic shifting term $\frac{3}{c_0}\log t$ is known as the logarithmic Bramson correction term; see [3, 17, 22, 24] for more details.

For space dimension $N \geq 2$, if $U_0(x)$ is given by (1.10) and hence the unique solution U of (1.3) is spherically symmetric ($U = U(t,|x|)$), results in [16, 25] indicate that the Bramson correction term for the monostable case (with some extra conditions on f) becomes

$$\frac{N+2}{c_0}\log t \text{ (for the pulled case of } f\text{)}, \quad \text{or} \quad \frac{N-1}{c_0}\log t \text{ (for the pushed case of } f\text{)},$$

that is, there exists some constant C such that for the pulled case of f,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| U(t,|x|) - Q_{c_0}\Big(c_0 t - \frac{N+2}{c_0}\log t + C - |x|\Big) \right| = 0,$$

and for the pushed case of f,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| U(t,|x|) - Q_{c_0}\Big(c_0 t - \frac{N-1}{c_0}\log t + C - |x|\Big) \right| = 0.$$

In the bistable case (as well as the combustion case), the Fife-McLeod result should be changed to (see [25])

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| U(t,|x|) - Q_{c_0}\Big(c_0 t - \frac{N-1}{c_0}\log t + L - |x|\Big) \right| = 0,$$

where L is some constant.

The above comparisons indicate that the singular behavior of the monostable case observed in the Cauchy problem does not exist anymore in the free boundary model, where all three cases behave in a rather synchronized manner.

The rest of the paper is organized as follows. In section 2, we describe how the constant c_N in the logarithmic shifting term is defined. In section 3, we estimate $h(t)$ in several steps until the sharp term $c_N \log t$ appears in the upper and lower bounds of $h(t)$. The main convergence results of this paper are proved in section 4, where our arguments are based on the estimates obtained in section 3, and on a new device very different from the energy methods used in [11] and [13].

A key step in this research is to find the exact form of the logarithmic shifting term $c_N \log t$. This relies on the discovery that sharp upper and lower solutions to (1.1) can be obtained by suitable perturbations of

$$h(t) = c^* t - c_N \log t, \; u(t,r) = \phi(\mu(c^* - c_N t^{-1}), r - h(t)),$$

with the functions $\phi(\mu, z)$ and $\mu(\xi)$ defined in (2.1) and (2.6), respectively. This approach is completely different from that used for treating the corresponding Cauchy problem, and from that used to handle the one space dimension case in [11].

Our method to prove the convergence result in section 4 also relies on innovative ideas. The method is very powerful and should have applications elsewhere. The spirit of the method is close to those in [26] and [12].

2. Formula for c_N

In this section, we describe how c_N in the logarithmic shifting term is defined, and also give a key identity (see (2.7) below) to be used in the next section.

Let q_{c^*} be given by Proposition 1.1 and we define $\phi(z)$ to be the unique solution of the following initial value problem

$$\phi'' + c^*\phi' + f(\phi) = 0,\ \phi(0) = 0,\ \phi'(0) = -c^*/\mu. \tag{2.1}$$

Clearly

$$\phi(z) = q_{c^*}(-z) \text{ for } z \le 0.$$

To stress its dependence on μ, we write $\phi(z) = \phi(\mu, z)$. Similarly we write $c^* = c^*(\mu)$. It is easily seen that for each given $\mu_0 > 0$, we can find $\epsilon_0 > 0$ such that $\phi(\mu, z)$ is defined for $z \in (-\infty, \epsilon_0]$ with

$$\phi_z(\mu, z) < 0,\ \phi(\mu, \epsilon_0) < -\eta_0 < 0 \text{ for } \mu \in [\mu_0/2, 2\mu_0] \text{ and } z \le \epsilon_0.$$

From [8] we see that $\mu \to c^*(\mu)$ is strictly increasing. We will show below that it is a C^2 function. To this end, we need to recall some details contained in [8]. Under the assumptions of Proposition 1.1, it was shown in [8] that there exists a unique $c_0 > 0$ such that for each $c \in [0, c_0]$, the problem

$$P' = c - \frac{f(q)}{P} \text{ in } [0,1),\ P(1) = 0,\ P'(1) < 0 \tag{2.2}$$

has a unique solution $P_c(q)$, which necessarily satisfies

$$P_c'(1) = \frac{c - \sqrt{c^2 - 4f'(1)}}{2},\ P_c(q) > 0 \text{ in } (0,1).$$

Furthermore, the following monotonicity and continuity result holds.

Lemma 2.1 (Lemma 6.1 of [8])**.** *For any $0 \le c_1 < c_2 \le c_0$ and $\bar{c} \in [0, c_0]$,*

$$P_{c_1}(q) > P_{c_2}(q) \text{ in } [0,1),\ \lim_{c \to \bar{c}} P_c(q) = P_{\bar{c}}(q) \text{ uniformly in } [0,1].$$

Moreover, $P_{c_0}(0) = 0$ and $P_{c_0}(q) > 0$ in $(0,1)$.

From the proof of Theorem 6.2 in [8], we see that, for $\mu > 0$, $c^*(\mu)$ is the unique solution of

$$P_c(0) - \frac{c}{\mu} = 0,\ c \in [0, c_0].$$

We show below that $c \to P_c(0)$ is a C^2 function for $c \in (0, c_0)$.

Fix $c \in (0, c_0)$ and let $h \ne 0$ be sufficiently small so that $c + h \in (0, c_0)$. We then consider

$$\hat{P}_h(q) := \frac{P_{c+h}(q) - P_c(q)}{h},\ q \in [0,1].$$

Clearly

$$\hat{P}_h'(q) = 1 + \frac{f(q)}{P_c(q)P_{c+h}(q)}\hat{P}_h(q) \text{ in } [0,1),\ \hat{P}_h(1) = 0. \tag{2.3}$$

The unique solution of (2.3) is given by

$$\hat{P}_h(q) = -\int_q^1 e^{\int_q^\xi \frac{-f(s)}{P_c(s)P_{c+h}(s)}ds} d\xi,\ q \in [0,1).$$

Let us note that for q close to 1, $f(q)$ is close to $f'(1)(q-1)$ and $P_c(q)$ is close to $P_c'(1)(q-1)$. Hence, for fixed $q \in [0,1)$,

$$e^{\int_q^\xi \frac{-f(s)}{P_c(s)P_{c+h}(s)}ds} \to 0 \text{ as } \xi \to 1 \text{ uniformly in } c, h.$$

It follows that the integrand function

$$e^{\int_q^\xi \frac{-f(s)}{P_c(s)P_{c+h}(s)}ds}$$

is uniformly bounded in the set $\{(q,\xi): 0\le q\le \xi\le 1\}$. Letting $h\to 0$ in the expression for $\hat{P}_h(q)$ we obtain

$$\lim_{h\to 0}\hat{P}_h(q) = -\int_q^1 e^{\int_q^\xi \frac{-f(s)}{P_c(s)^2}ds} d\xi,\ q\in[0,1).$$

Therefore

$$\frac{d}{dc}P_c(q) = -\int_q^1 e^{\int_q^\xi \frac{-f(s)}{P_c(s)^2}ds} d\xi < 0 \text{ for } q\in[0,1). \tag{2.4}$$

By Lemma 2.1, we easily see from the above identity that $\frac{d}{dc}P_c(q)$ is continuous in c for $c\in(0,c_0)$. Moreover, $\frac{d}{dc}P_c(1)=0$ and the continuity of $\frac{d}{dc}P_c(q)$ in c is uniform in $q\in[0,1]$.

From (2.4) we further obtain

$$\frac{d^2}{dc^2}P_c(0) = -2\int_0^1 \left[e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds}\int_0^\xi \frac{f(s)}{P_c(s)^3}\frac{d}{dc}P_c(s)ds\right]d\xi, \tag{2.5}$$

provided that we can show the integral above is convergent. By (2.4) we can find $C_1>0$ such that

$$\left|\frac{d}{dc}P_c(s)\right| \le C_1 \text{ for } s\in[0,1].$$

For $\epsilon\in(0,1)$ sufficiently small, there exist $C_2, C_3>0$ such that

$$\frac{-f(s)}{P_c(s)^2} \le -C_2(1-s)^{-1},\ \left|\frac{f(s)}{P_c(s)^3}\right| \le C_3(1-s)^{-2} \text{ for } s\in[1-\epsilon,1].$$

Hence, for $\xi\in[1-\epsilon,1]$,

$$\begin{aligned}
&\left|e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds}\int_0^\xi \frac{f(s)}{P_c(s)^3}\frac{d}{dc}P_c(s)ds\right| \\
&\le C_1 e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds}\left[\int_0^{1-\epsilon} + \int_{1-\epsilon}^\xi\right]\left|\frac{f(s)}{P_c(s)^3}\right|ds \\
&\le C_1C_\epsilon e^{-C_2\int_{1-\epsilon}^\xi (1-s)^{-1}ds}\left[C_\epsilon + C_3\int_{1-\epsilon}^\xi (1-s)^{-2}ds\right] \\
&\le C_\epsilon[(1-\xi)^{C_2} + (1-\xi)^{C_2-1}],
\end{aligned}$$

where we have used C_ϵ to denote various positive constants that depend on ϵ. Clearly this implies the convergence of the integral in the formula for $\frac{d^2}{dc^2}P_c(0)$ in (2.5). Moreover, by the continuous dependence of $P_c(q)$ and $\frac{d}{dc}P_c(q)$ on c, we find from (2.5) that $\frac{d^2}{dc^2}P_c(0)$ is continuous in c for $c\in(0,c_0)$. We have thus proved the following result.

Lemma 2.2. *The function $c\to P_c(0)$ is C^2 for $c\in(0,c_0)$.*

Define $\zeta(c,\mu) := P_c(0) - \frac{c}{\mu}$. Then

$$\partial_c\zeta(c,\mu) = \frac{d}{dc}P_c(0) - \frac{1}{\mu} < -\frac{1}{\mu} < 0.$$

Hence by the implicit function theorem we find that the unique solution $c=c^*(\mu)$ of $\zeta(c,\mu)=0$, as a function of μ, is as smooth as ζ, and hence is C^2. Moreover

$$c^{*\prime}(\mu) = -\frac{\partial_\mu\zeta(c^*(\mu),\mu)}{\partial_c\zeta(c^*(\mu),\mu)} = -\frac{\mu^{-2}c^*(\mu)}{\partial_c\zeta(c^*(\mu),\mu)} > 0,$$

and

$$\left(\frac{c^*(\mu)}{\mu}\right)' = \frac{c^{*\prime}(\mu)}{\mu} - \frac{c^*(\mu)}{\mu^2} = \mu^{-2}c^*(\mu)\left[\frac{1}{-\mu\partial_c\zeta} - 1\right] < 0$$

since $-\partial_c\zeta > \mu^{-1}$.

From [8] we further have

$$\lim_{\mu\to\infty}\frac{c^*(\mu)}{\mu} = 0,\ \lim_{\mu\to 0}\frac{c^*(\mu)}{\mu} = P_0(0) > 0.$$

We now fix $\mu_0 > 0$ and denote $c_0^* = c^*(\mu_0)$. Therefore for each $\xi \in (0, \mu_0 P_0(0))$ there exists a unique $\mu = \mu(\xi)$ such that

$$\frac{c^*(\mu(\xi))}{\mu(\xi)} = \frac{\xi}{\mu_0},\ \xi \to \mu(\xi) \text{ is } C^2,\ \ \mu'(\xi) < 0,\ \ \mu(c_0^*) = \mu_0. \tag{2.6}$$

Here we have used the implicit function theorem and $\mu \to \frac{c^*(\mu)}{\mu}$ is C^2 to conclude that $\xi \to \mu(\xi)$ is C^2.

Let $g(\xi) := c^*(\mu(\xi))$. Then g is C^2 and $g'(\xi) = c^{*\prime}(\mu(\xi))\mu'(\xi) < 0$. The following identity will play a crucial role in the estimates of the next section.

$$g(c_0^* - c_N t^{-1}) - g(c_0^*) = -g'(c_0^*)(c_N t^{-1}) + \frac{1}{2}g''(\theta_t)(c_N^2 t^{-2}) \tag{2.7}$$

with $\theta_t \in (c_0^* - c_N t^{-1}, c_0^*)$, where c_N is given by

$$c_N = \Big[1 - g'(c_0^*)\Big]^{-1}\frac{N-1}{c_0^*}, \tag{2.8}$$

and $g'(c_0^*)$ can be calculated by the following formula:

Lemma 2.3.

$$g'(c_0^*) = -\frac{c_0^*}{\mu_0^2\int_0^\infty q'_{c_0^*}(z)^2 e^{-c_0^* z}dz}. \tag{2.9}$$

Proof. By definition, $g'(c_0^*) = c^{*\prime}(\mu_0)\mu'(c_0^*)$. Using $c^*(\mu(\xi)) = \mu_0^{-1}\xi\mu(\xi)$, we obtain

$$c^{*\prime}(\mu(\xi))\mu'(\xi) = \mu_0^{-1}[\mu(\xi) + \xi\mu'(\xi)],\ \mu'(\xi) = \frac{\mu_0^{-1}\mu(\xi)}{c^{*\prime}(\mu(\xi)) - \mu_0^{-1}\xi}.$$

Hence

$$\mu'(c_0^*) = \frac{1}{c^{*\prime}(\mu_0) - \mu_0^{-1}c_0^*}.$$

By our earlier calculation, we have

$$c^{*\prime}(\mu_0) = -\left.\frac{\mu_0^{-2}c_0^*}{\frac{d}{dc}P_c(0) - \mu_0^{-1}}\right|_{c=c_0^*}.$$

Hence

$$g'(c_0^*) = \frac{c^{*\prime}(\mu_0)}{c^{*\prime}(\mu_0) - \mu_0^{-1}c_0^*} = \frac{1}{1 - \mu_0^{-1}c_0^* c^{*\prime}(\mu_0)^{-1}} = \left.\frac{1}{\mu_0\frac{d}{dc}P_c(0)}\right|_{c=c_0^*}.$$

From (2.4) we obtain

$$\frac{d}{dc}P_c(0) = -\int_0^1 e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds}d\xi.$$

From [8] we know that

$$P_c(s) = P_c(q_c(z)) = q_c'(z) \text{ with } s = q_c(z),\ \text{or equivalently } z = q_c^{-1}(s).$$

Therefore, making use of the change of variable $s = q_c(z)$, and the identity $f(q_c(z)) = -q_c''(z)+cq_c'(z)$, we obtain

$$\begin{aligned}\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds &= \int_0^{q_c^{-1}(\xi)} \frac{-f(q_c(z))}{q_c'(z)^2}q_c'(z)dz \\ &= \int_0^{q_c^{-1}(\xi)} \frac{q_c''(z) - cq_c'(z)}{q_c'(z)}dz \\ &= \log\left[\frac{q_c'(q_c^{-1}(\xi))}{q_c'(0)}\right] - cq_c^{-1}(\xi).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dc}P_c(0) &= -\int_0^1 e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds}d\xi \\ &= -\int_0^1 \frac{q_c'(q_c^{-1}(\xi))}{q_c'(0)}e^{-cq_c^{-1}(\xi)}d\xi \\ &= -\int_0^\infty \frac{q_c'(z)}{q_c'(0)}e^{-cz}q_c'(z)dz \\ &= -\frac{\mu}{c}\int_0^\infty q_c'(z)^2e^{-cz}dz.\end{aligned}$$

Hence

$$g'(c_0^*) = \frac{-c_0^*}{\mu_0^2\int_0^\infty q_{c_0^*}'(z)^2e^{-c_0^*z}dz}.$$

□

3. Sharp bounds

In this section we give some sharp estimates for $h(t)$. We always assume that f satisfies the conditions of Proposition 1.1. We fix $\mu_0 > 0$ and suppose that $(u(t,r),h(t))$ is the unique solution of (1.1) with $\mu = \mu_0$. Let c_0^* and c_N be defined as in the previous section (see (2.8)), and suppose that spreading happens:

(3.1) $$\lim_{t\to\infty} h(t) = \infty,\ \lim_{t\to\infty} u(t,r) = 1 \text{ uniformly for } r \text{ in compact subsets of } [0,\infty).$$

We make these assumptions throughout this section. Our aim is to show the following result.

Theorem 3.1. *There exist positive constants C and T such that*

(3.2) $$|h(t) - (c_0^*t - c_N\log t)| \le C \text{ for } t \ge T.$$

Moreover, for any $c \in (0,c_0^*)$, there exist positive constants M and σ such that

(3.3) $$|u(t,r) - 1| \le Me^{-\sigma t} \text{ for } t > 0,\ r \in [0,ct].$$

These conclusions will be proved by a sequence of lemmas.

3.1. **Rough bounds.** We start with some rough bounds for u and h.

Lemma 3.2. *The following conclusions hold:*

(i) *For any $c \in (0,c_0^*)$ and $\delta \in (0,-f'(1))$, there exist a positive constants $T_* > 0$ and $M > 0$ such that*

$$u(t,r) \le 1 + Me^{-\delta t},\ h(t) \ge ct \text{ for } t \ge T_* \text{ and } r \in [0,h(t)].$$

(ii) *There exists* $\tilde{c} \in (0, c_0^*)$, $\tilde{\delta} \in (0, -f'(1))$, *and* $\tilde{T}_* > 0$, $\tilde{M} > 0$ *such that*

$$u(t,r) \geq 1 - \tilde{M}e^{-\tilde{\delta}t} \quad \textit{for } r \in [0, \tilde{c}t] \quad \text{and} \quad t \geq \tilde{T}_*.$$

Proof. (i) Consider the equation $\eta'(t) = f(\eta)$ with initial value $\eta(0) = \|u_0\|_{L^\infty} + 1$. Then η is an upper solution of (1.1). So $u(t,x) \leq \eta(t)$ for all $t \geq 0$. Since $f(u) < 0$ for $u > 1$, $\eta(t)$ is a decreasing function converging to 1 as $t \to \infty$. Hence there exists $T_* > 0$ such that $\eta(t) < 1 + \rho$ and $\eta'(t) = f(\eta) \leq \delta(1-\eta)$ for $t \geq T_*$. It follows that

$$u(t,x) \leq \eta(t) \leq 1 + \rho e^{-\delta(t-T_*)} \quad \text{for} \quad 0 \leq |x| \leq h(t), t \geq T_*.$$

Next we take any $c \in (0, c_0^*)$ and show that for all large t, $h(t) \geq ct$. We construct a lower solution similar to the proof of Lemma 6.5 in [8]. Let us recall that for each $c \in (0, c_0^*)$, there exists a function $q^c(z)$ defined for $z \in [0, z^c]$ such that

$$q'' - cq' + f(q) = 0 \text{ in } [0, z^c]; \; q(0) = q'(z^c) = 0; \; q'(z) > 0 \text{ in } [0, z^c).$$

Moreover, $Q^c := q^c(z^c) < 1$ and as $c \nearrow c_0^*$,

$$Q^c \nearrow 1, \; z^c \nearrow +\infty, \; \|q^c - q_{c_0^*}\|_{L^\infty([0,z^c])} \to 0.$$

See page 38 of [8] for details.

We now choose c_1, $c_2 \in (c, c_0^*)$ satisfying $c_1 < c_2$, $f(Q^{c_2}) > 0$, and define

$$k(t) := z^{c_2} + c_2 t - \frac{N-1}{c_1} \log t.$$

We can find $T_1 > 0$ such that

$$c_1 t \leq c_2 t - \frac{N-1}{c_1} \log t$$

for $t \geq T_1$. Set

$$w(t,r) := \begin{cases} q^{c_2}(k(t) - r), & c_2 t - \frac{N-1}{c_1} \log t \leq r \leq k(t), \\ q^{c_2}(z^{c_2}), & 0 \leq r \leq c_2 t - \frac{N-1}{c_1} \log t. \end{cases}$$

Since spreading happens we can find $T_2 > T_1$ such that

$$\begin{aligned} &k(T_1) \leq h(T_2) \\ &w(T_1, r) \leq u(T_2, r) \quad \text{for } r \in [0, k(T_1)] \end{aligned}$$

We note that

$$w_r(t,r) = 0 \quad \text{when} \quad 0 \leq r \leq c_2 t - \frac{N-1}{c_1} \log t.$$

Moreover, by (6.7) in [8],

$$k'(t) = c_2 - \frac{N-1}{c_1 t} \leq c_2 < \mu (q^{c_2})'(0) = -\mu w_r(t, k(t))$$

and

$$\begin{aligned} &w_t - \Delta w \\ &= k'(t)(q^{c_2})'(k(t) - r) - (q^{c_2})''(k(t) - r) + \frac{N-1}{r}(q^{c_2})'(k(t) - r) \\ &= f(q^{c_2}(k(t) - r)) + \left(\frac{N-1}{r} - \frac{N-1}{c_1 t}\right)(q^{c_2})'(k(t) - r) \\ &\leq f(w) \end{aligned}$$

for $r \in \left[c_2 t - \frac{N-1}{c_1}\log t, k(t)\right] \subset [c_1 t, k(t)]$ and

$$w_t - \Delta w = 0 < f(Q^{c_2}) = f(w)$$

for $r \in \left[0, c_2 t - \frac{N-1}{c_1}\log t\right]$.

Since w is C^1 in r, the above discussions show that $(w(t-T_2+T_1, r), k(t-T_2+T_1))$ is a lower solution of (1.1) for $t \geq T_2$. Hence there exists some $T_3 \geq T_2$ such that for $t \geq T_3$,

$$\begin{aligned} h(t) &\geq k(t-T_2+T_1) = z^{c_2} + c_2(t-T_2+T_1) - \frac{N-1}{c_1}\log(t-T_2+T_1) \\ &\geq z^{c_2} + c_1(t-T_2+T_1) \geq ct \end{aligned}$$

and

$$u(t,r) \geq w(t-T_2+T_1, r) \quad \text{for} \quad r \in [0, k(t-T_1+T_2)] \supset [0, ct].$$

(ii) Since $w(t-T_2+T_1, r) \equiv q^{c_2}(z^{c_2}) = Q^{c_2} > Q^c$ for $r \leq ct$ for all $t \geq T_3$, we find from the above estimates for u and h that

$$h(t) \geq ct, \;\; u(t,r) \geq Q^c \text{ for } 0 \leq r \leq ct, \;\; t \geq T_3 \tag{3.4}$$

Since $f'(1) < 0$, for any $\delta \in (0, -f'(1))$ we can find $\rho = \rho(\delta) \in (0,1)$ such that

$$f(u) \geq \delta(1-u) \;\, (u \in [1-\rho, 1]), \quad f(u) \leq \delta(1-u) \;\, (u \in [1, 1+\rho]).$$

Since $Q^c \to 1$ as $c \nearrow c_0^*$, we may assume that $Q^c > 1-\rho$.

Now for a given domain D we consider a solution $\psi = \psi_D$ to the following auxiliary problem:

$$\begin{cases} \psi_t - \Delta\psi = -\delta(\psi - 1), & t > 0, x \in D, \\ \psi \equiv Q^c, & t > 0, x \in \partial D, \\ \psi \equiv Q^c, & t = 0, x \in D. \end{cases} \tag{3.5}$$

The function $\Psi = \Psi_D = e^{\delta t}(\psi_D - Q^c)$ satisfies

$$\begin{cases} \Psi_t - \Delta\Psi = \delta e^{\delta t}(1 - Q^c), & t > 0, x \in D, \\ \Psi \equiv 0, & t > 0, x \in \partial D, \\ \Psi \equiv 0, & t = 0, x \in D. \end{cases} \tag{3.6}$$

Take

$$D = Q_{\tilde{c}T} := \{x \in \mathbb{R}^N \,|\, -\tilde{c}T \leq x_i \leq \tilde{c}T, \; i = 1, 2, \cdots, N\}$$

with $\tilde{c} = c/\sqrt{N}$. Let $G(x,t;\xi,\tau)$ be the Green function for the problem (3.6). From page 84 of [15] one sees that this Green function can be expressed as follows:

$$G(x,t;\xi,\tau) = \prod_{i=1}^{N} \tilde{G}(x_i, t; \xi_i, \tau)$$

where $\tilde{G}$ is the Green function of the one space dimension problem:

$$\begin{cases} \Psi_t - \Psi_{xx} = g(t,x), & t > 0, -\tilde{c}T \leq x \leq \tilde{c}T, \\ \Psi \equiv 0, & t > 0, x = \pm\tilde{c}T, \\ \Psi \equiv 0, & t = 0, -\tilde{c}T \leq x \leq \tilde{c}T. \end{cases}$$

Thus

$$\Psi_{Q_{\tilde{c}T}}(t,x) = \int_0^t \delta e^{\delta\tau}(1 - Q^c) \int_{Q_{\tilde{c}T}} G(x,t;\xi,\tau) d\xi d\tau$$

For $\varepsilon \in (0,1)$, consider $(t,x) \in \mathbb{R}^{N+1}$ satisfying

$$|x_i| \le (1-\varepsilon)\tilde{c}T,\ i = 1,2,\cdots,N,\ 0 \le t \le \frac{\varepsilon^2\tilde{c}^2 T}{4}.$$

From the proof of Lemma 6.5 in [8] we find that for such (t,x), there exists $T_4 \ge T_3$ such that for $T \ge T_4$,

$$\int_{-\tilde{c}T}^{\tilde{c}T} \tilde{G}(x_i,t;\xi_i,\tau)d\xi_i \ge 1 - \frac{4}{\sqrt{\pi}}e^{-T/2} \ge 0.$$

Hence, for sufficiently large $T > 0$ there exists $M_0 > 0$ such that

$$\int_{Q_{\tilde{c}T}} G(x,t;\xi,\tau)d\xi \ge 1 - M_0 e^{-T/2}.$$

From the above estimate we obtain

$$\begin{aligned}\Psi_{Q_{\tilde{c}T}}(t,x) &\ge \delta(1-Q^c)\int_0^t e^{\delta\tau}(1-M_0e^{-T/2})d\tau \\ &= (1-Q^c)(1-M_0e^{-T/2})(e^{\delta t}-1)\end{aligned}$$

for $T \ge T_4$, $|x_i| \le (1-\varepsilon)\tilde{c}T$, $i = 1,2,\cdots,N$, $0 \le t \le \frac{\varepsilon^2\tilde{c}^2}{4}T$.

Since $B_{\tilde{c}T} \subset Q_{\tilde{c}T} \subset B_{\sqrt{N}\tilde{c}T} \subset B_{cT}$, using (3.4) and a simple comparison argument we obtain

$$\psi_{Q_{\tilde{c}T}}(t,x) \le \psi_{B_{cT}}(t,x) \le u(t+T,|x|) \text{ for } t \ge 0,\ x \in Q_{\tilde{c}T}.$$

Hence

$$u(t+T,|x|) \ge \psi_{Q_{\tilde{c}T}}(t,x) \text{ for } t > 0, x \in Q_{\tilde{c}T}. \tag{3.7}$$

Fix $T \ge T_4$. We have

$$\psi_{Q_{\tilde{c}T}}(t,x) = \Psi_{Q_{\tilde{c}T}}(t,x)e^{-\delta t} + Q^c \ge 1 - M_0e^{-T/2} - e^{-\delta t}$$

for $|x_i| \le \tilde{c}T(1-\varepsilon)$, $i = 1,2,\cdots,N$, $0 \le t \le \frac{\varepsilon^2\tilde{c}^2}{4}T$. Taking $t = \frac{\varepsilon^2\tilde{c}^2}{4}T$ we obtain

$$\psi_{Q_{\tilde{c}T}}\left(\frac{\varepsilon^2\tilde{c}^2}{4}T, x\right) \ge 1 - M_0e^{-T/2} - e^{-\varepsilon^2\tilde{c}^2\delta T/4}.$$

We only focus on small $\varepsilon > 0$ such that $\varepsilon^2\tilde{c}^2\delta < 2$ so

$$\begin{aligned}\psi_{Q_{\tilde{c}T}}\left(\frac{\varepsilon^2\tilde{c}^2}{4}T, x\right) &\ge 1 - M_0e^{-\varepsilon^2\tilde{c}^2\delta T/4} - e^{-\varepsilon^2\tilde{c}^2\delta T/4} \\ &= 1-(M_0+1)e^{-\varepsilon^2\tilde{c}^2\delta T/4}.\end{aligned}$$

This holds for $|x_i| \le (1-\varepsilon)\tilde{c}T$, $i = 1,2,\cdots,N$, $T \ge T_4$. Thus, by (3.7), for such T and x, we have

$$u\left(\frac{\varepsilon^2\tilde{c}^2}{4}T + T, |x|\right) \ge 1-(M_0+1)e^{-\varepsilon^2\tilde{c}^2\delta T/4}.$$

Finally, if we rewrite

$$t = \frac{\varepsilon^2\tilde{c}^2}{4}T + T$$

then

$$T = \left(1 + \frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1} t,$$

and

$$u(t,|x|) \ge 1-(M_0+1)e^{-\tilde{\delta}t}$$

for $|x_i| \leq (1-\varepsilon)\left(1+\frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1}\tilde{c}t$, $i=1,2,\cdots,N$ and $t \geq T_5$ where $\tilde{\delta} := \frac{\varepsilon^2\tilde{c}^2}{4}\left(1+\frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1}\delta$ and $T_5 = \frac{\varepsilon^2\tilde{c}^2}{4}T_4 + T_4$. This is also true for $|x| \leq (1-\varepsilon)\left(1+\frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1}\tilde{c}t$. Since this is true for any $\tilde{c} \in (0, c_0^*/\sqrt{N})$ and for any small $\varepsilon > 0$, the above estimate implies the conclusion in (ii). This completes the proof. □

Lemma 3.3. *For any $c \in (0, c_0^*)$ there exist $M' > 0$, $T' > 0$ and $\delta' \in (0, -f'(1))$ such that*

$$u(t,r) \geq 1 - M'e^{-\delta' t},\ h(t) \geq c_0^* t - M' \log t \text{ for } r \in [0, ct] \text{ and } t \geq T'.$$

Proof. We first construct a lower solution. Define

$$\begin{aligned}
&\underline{u}(t,r) = (1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\underline{h}(t)-r),\\
&\underline{h}(t) = c_0^*(t-T_{**}) + \tilde{c}T_{**} - \frac{N-1}{\tilde{c}}\log\frac{t}{T_{**}} - \sigma\tilde{M}(e^{-\tilde{\delta}T_{**}} - e^{-\tilde{\delta}t}),\\
&\underline{g}(t) = \tilde{c}t,
\end{aligned}$$

where $\tilde{M}, \tilde{\delta}$ and $\tilde{c}$ are given in Lemma 3.2, $\sigma > 0$ and $T_{**} > T_*$ (T_* as in Lemma 3.2) will be chosen later. We will check that $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution, that is,

$$\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) \leq f(\underline{u}) \quad \text{for} \quad t > T_{**}, \underline{g}(t) < r < \underline{h}(t), \tag{3.8}$$

$$\underline{u} \leq u \text{ for } \ t \geq T_{**},\ r = \underline{g}(t), \tag{3.9}$$

$$\underline{u} = 0,\ \underline{h}'(t) \leq -\mu\underline{u}_r \ \text{ for } \ t \geq T_{**},\ r = \underline{h}(t), \tag{3.10}$$

$$\underline{h}(T_{**}) \leq h(T_{**}),\ \underline{u}(T_{**}, r) \leq u(T_{**}, r) \ \text{ for } \ r \in [\underline{g}(T_{**}), \underline{h}(T_{**})]. \tag{3.11}$$

We first see that $\underline{h}(T_{**}) = \tilde{c}T_{**} \leq h(T_{**})$ from Lemma 3.2. Thus we have

$$\underline{u}(T_{**}, r) \leq 1 - \tilde{M}e^{-\tilde{\delta}T_{**}} \leq u(T_{**}, r) \ \text{ for } r \in [\underline{g}(T_{**}), \underline{h}(T_{**})]$$

from Lemma 3.2. Similarly we have

$$\underline{u}(t, \underline{g}(t)) = \underline{u}(t, \tilde{c}t) \leq 1 - \tilde{M}e^{\tilde{\delta}t} \leq u(t, \tilde{c}t) = u(t, \underline{g}(t))$$

for $t \geq T_{**}$ by Lemma 3.2.

Clearly $\underline{u}(t, \underline{h}(t)) = 0$. Next we calculate

$$\begin{aligned}
&\underline{h}'(t) = c_0^* - \frac{N-1}{\tilde{c}t} - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t} \leq c_0^* - \sigma\tilde{\delta}\tilde{M}e^{-\tilde{\delta}t},\\
&\underline{u}_r(t, \underline{h}(t)) = -(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(0) = -\frac{c_0^* - c_0^*\tilde{M}e^{-\tilde{\delta}t}}{\mu},\\
&-\mu\underline{u}_r(t, \underline{h}(t)) = c_0^* - c_0^*\tilde{M}e^{-\tilde{\delta}t}.
\end{aligned}$$

Hence if we take $\sigma > 0$ so that $c_0^* \leq \sigma\tilde{\delta}$, then

$$\underline{h}'(t) \leq -\mu\underline{u}_r(t, \underline{h}(t)) \text{ for } t \geq T_{**}.$$

It remains to prove $\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u}) \leq 0$. Put $\zeta = \underline{h}(t) - r$. Since

$$\begin{aligned}
&\underline{u}_t = \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})\underline{h}'(t)q'_{c_0^*}(\zeta),\\
&\underline{u}_r = -(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta),\\
&\underline{u}_{rr} = (1-\tilde{M}'e^{-\tilde{\delta}t})q''_{c_0^*}(\zeta),
\end{aligned}$$

we have for $t \geq T_{**}$ and $r \in (\tilde{c}t, \underline{h}(t))$,

$$
\begin{aligned}
&\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u}) \\
&= \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})\underline{h}'(t)q'_{c_0^*}(\zeta) \\
&\quad - (1-\tilde{M}e^{-\tilde{\delta}t})q''_{c_0^*}(\zeta) + \frac{N-1}{r}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)) \\
&= \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})\left(c_0^* - \frac{N-1}{\tilde{c}t} - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}\right)q'_{c_0^*}(\zeta) \\
&\quad - (1-\tilde{M}e^{-\tilde{\delta}t})q''_{c_0^*}(\zeta) + \frac{N-1}{r}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)) \\
&= \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})(c_0^*q'_{c_0^*}(\zeta) - q''_{c_0^*}(\zeta)) \\
&\quad - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})\left(\frac{N-1}{r} - \frac{N-1}{\tilde{c}t}\right)q'_{c_0^*}(\zeta) \\
&\qquad\quad - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)) \\
&\leq \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad + (1-\tilde{M}e^{-\tilde{\delta}t})f(q_{c_0^*}(\zeta)) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)).
\end{aligned}
$$

Let us consider the term $(1-\tilde{M}e^{-\tilde{\delta}t})f(q_{c_0^*}(\zeta)) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta))$, which is of the form

$$(1+\xi)f(u) - f((1+\xi)u).$$

The mean value theorem implies that

$$\xi f(u) + f(u) - f((1+\xi)u) = \xi f(u) - \xi f'(u + \theta_{\xi,u}\xi u)u$$

for some $\theta_{\xi,u} \in (0,1)$. Since $0 < \tilde{\delta} < -f'(1)$, we can find an $\eta > 0$ such that

$$
\left\{
\begin{array}{l}
\tilde{\delta} \leq -f'(u) \ \text{ for } \ 1-\eta \leq u \leq 1+\eta, \\
f(u) \geq 0 \ \text{ for } \ 1-\eta \leq u \leq 1.
\end{array}
\right.
\tag{3.12}
$$

Since $q_{c_0^*}(\zeta) \to 1$ as $\zeta \to \infty$, there exists $\zeta_\eta > 0$ such that $q_{c_0^*}(\zeta) \geq 1-\eta/2$ for $\zeta \geq \zeta_\eta$. We may assume that $\tilde{M}e^{-\tilde{\delta}t} \leq \eta/2$ for $t \geq T_{**}$.

For $\zeta \geq \zeta_\eta$, we have

$$
\begin{aligned}
&\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u}) \\
&\leq \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad - \tilde{M}e^{-\tilde{\delta}t}\Big\{f(q_{c_0^*}(\zeta)) - f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big)q_{c_0^*}(\zeta)\Big\} \\
&= \ -\tilde{M}e^{-\tilde{\delta}t}f(q_{c_0^*}(\zeta)) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad + \tilde{M}e^{-\tilde{\delta}t}\Big\{f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big) + \tilde{\delta}\Big\}q_{c_0^*}(\zeta) \leq 0,
\end{aligned}
$$

for some $\theta'_{\zeta,t} \in (0,1)$. Here we have use the fact that

$$q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) \geq q_{c_0^*}(\zeta) - \tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) \geq 1-\eta$$

and hence $f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big) + \tilde{\delta} \leq 0$.

For $0 \le \zeta \le \zeta_\eta$, we have

$$\begin{aligned}
&\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u}) \\
\le& \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t} q_{c_0^*}(\zeta) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad - \tilde{M}e^{-\tilde{\delta}t}\Big\{ f(q_{c_0^*}(\zeta)) - f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big)q_{c_0^*}(\zeta)\Big\} \\
=& -\tilde{M}e^{-\tilde{\delta}t} f(q_{c_0^*}(\zeta)) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad + \tilde{M}e^{-\tilde{\delta}t}\Big\{ f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big) + \tilde{\delta}\Big\} q_{c_0^*}(\zeta) \\
\le& -\tilde{M}e^{-\tilde{\delta}t}\min_{0\le s\le 1} f(s) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad + \tilde{M}e^{-\tilde{\delta}t}\Big\{ \max_{0\le s\le 1} f'(s) + \tilde{\delta}\Big\} \\
=& \tilde{M}e^{-\tilde{\delta}t}\Big\{ -\min_{0\le s\le 1} f(s) + \max_{0\le s\le 1} f'(s) + \tilde{\delta} - \sigma\tilde{\delta}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta)\Big\} \\
\le& 0,
\end{aligned}$$

for sufficiently large $\sigma > 0$ and all large t. Finally we note that we can take $T_{**} > T_*$ so large that the above holds and $\tilde{c}t \le \underline{h}(t)$ for $t \ge T_{**}$.

Thus we have shown that (3.8)-(3.11) hold and $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution of (1.1). It follows that

$$u(t,r) \ge \underline{u}(t,r),\ h(t) \ge \underline{h}(t) \text{ for } t \ge T_{**} \text{ and } r \in [\underline{g}(t), \underline{h}(t)].$$

Hence

$$\begin{aligned}
u(t,r) &\ge (1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\underline{h}(t) - r) \\
&\ge q_{c_0^*}(\underline{h}(t) - r) - \tilde{M}e^{-\tilde{\delta}t}
\end{aligned}$$

for $t \ge T_{**}$ and $\tilde{c}t \le r \le \underline{h}(t)$.

For any $c \in (0, c_0^*)$ and any $\kappa \in (0, c_0^* - c)$, there exists $T_{***} > 0$ such that for $t \ge T_{***}$ and $r \in [0, ct]$, we have

$$\underline{h}(t) - r \ge (c_0^* - c)t - \frac{N-1}{\tilde{c}}\log\frac{t}{T_{**}} + \tilde{c}T_{**} - \sigma\tilde{M} \ge \kappa t.$$

Since there exist $C > 0$ and $\beta > 0$ such that $q_{c_0^*}$ satisfies $q_{c_0^*}(z) \ge 1 - Ce^{-\beta z}$ for $z \ge 0$, we thus obtain

$$u(t,r) \ge 1 - Ce^{-\beta\kappa t} - \tilde{M}e^{-\tilde{\delta}t} = 1 - \tilde{M}'e^{-\delta' t} \tag{3.13}$$

for $t \ge T_{***}$ and $r \in [\tilde{c}t, ct]$, where $\delta' = \min\{\beta\kappa, \tilde{\delta}\}$.

Moreover, if $M_0 > (N-1)/\tilde{c}$, then

$$h(t) \ge \underline{h}(t) = c_0^* t - \frac{N-1}{\tilde{c}}\log t - \tilde{C} \ge c_0^* t - M_0 \log t \text{ for all large } t.$$

Thus combined with (3.14) and Lemma 3.2, we find that

$$u(t,r) \ge 1 - M'e^{-\delta' t},\ h(t) \ge c_0^* t - M' \log t$$

for $t \ge T'$ and $r \in [0, ct]$ provided that M' and T' are chosen large enough. This completes the proof of Lemma 3.3. □

Clearly (3.3) follows directly from Lemmas 3.2 and 3.3. Let us note that from the proof of Lemma 3.3, we have, for $t \ge T'$ and $r \in [\tilde{c}t, c_0^* t - M' \log t]$,

$$u(t,r) \ge (1 - \tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(c_0^* t - M' \log t - r).$$

Since

$$q_{c_0^*}(z) \geq 1 - M_1 e^{-\delta_1 z} \text{ for } z \in [0,\infty) \text{ and some } M_1, \delta_1 > 0,$$

we immediately obtain

$$u(t,r) \geq (1-\tilde{M}e^{-\tilde{\delta}t})(1-M_1 e^{-\delta_1(c_0^* t - M'\log t - r)}) \tag{3.14}$$

for $t \geq T'$ and $r \in [\tilde{c}t, c_0^* t - M'\log t]$.

3.2. **Sharp bounds.** We now make use of the rough bounds for u and h to obtain sharp bounds for h. We first improve the estimate for $h(t)$ in Lemma 3.3.

Lemma 3.4. *There exist $C > 0$ and $T > 0$ such that*

$$h(t) \geq c_0^* t - c_N \log t - C \text{ for } t \geq T,$$

where c_N is given by (2.8).

Proof. With $B > 0$ a constant to be determined, and $\phi(z) = \phi(\mu, z)$ given in (2.1), we set

$$\tilde{k}(t) = c_0^* t - c_N \log t + Bt^{-1}\log t,$$
$$\underline{v}(t,r) = \phi\left(\mu(c_0^* - c_N t^{-1}), r - \tilde{k}(t)\right) - t^{-2}\log t.$$

We have $\underline{v}(t,\tilde{k}(t)) = -t^{-2}\log t < 0$ for $t > 1$, and

$$\underline{v}(t,\tilde{k}(t) - t^{-1}) = \phi\left(\mu(c_0^* - c_N t^{-1}), -t^{-1}\right) - t^{-2}\log t = -\phi_r(\mu_0, 0)t^{-1} + o(t^{-1}) > 0$$

for all large t. Moreover,

$$\underline{v}_r(t,r) = \phi_r\left(\mu(c_0^* - c_N t^{-1}), r - \tilde{k}(t)\right) < 0 \text{ for all } t > 0 \text{ and } r \in (0, \tilde{k}(t)].$$

Therefore, there exists a unique $\underline{k}(t) \in (\tilde{k}(t) - t^{-1}, \tilde{k}(t))$ such that

$$\underline{v}(t, \underline{k}(t)) = 0 \text{ for all large } t.$$

By the implicit function theorem we know that $t \to \underline{k}(t)$ is smooth, and by the mean value theorem we obtain

$$\left[\phi_r(\mu_0, 0) + o(1)\right]\left[\underline{k}(t) - \tilde{k}(t)\right] = t^{-2}\log t.$$

Since $\phi_r(\mu_0, 0) = -c_0^*/\mu_0$, we thus obtain

$$\underline{k}(t) - \tilde{k}(t) = \left[-\frac{\mu_0}{c_0^*} + o(1)\right] t^{-2}\log t \quad \text{for all large } t. \tag{3.15}$$

Using $\underline{v}_t(t,\underline{k}(t)) + \underline{v}_r(t,\underline{k}(t))\underline{k}'(t) = 0$ we obtain

$$\phi_\mu \cdot \mu' \cdot c_N t^{-2} + \phi_r \cdot \left[\underline{k}'(t) - \tilde{k}'(t)\right] + \left[1 + o(1)\right]2t^{-3}\log t = 0.$$

It follows that

$$\underline{k}'(t) = \tilde{k}'(t) + O(t^{-2}) = c_0^* - c_N t^{-1} - Bt^{-2}\log t + O(t^{-2})$$

for all large t.

We want to show that there exist positive constants M and T such that $(\underline{v}(t,r), \underline{k}(t))$ satisfies, for $t \geq T$ and $\underline{k}(t) - M\log t \leq r \leq \underline{k}(t)$,

$$\underline{v}(t,\underline{k}(t)) = 0, \ \ \underline{k}'(t) \leq -\mu_0 \underline{v}_r(t,\underline{k}(t)), \tag{3.16}$$

$$\underline{v}(t,\underline{k}(t) - M\log t) \leq u(t+s, k(t+s) - M\log(t+s)), \ \forall s > 0, \tag{3.17}$$

$$\underline{v}_t - \underline{v}_{rr} - \frac{N-1}{r}\underline{v}_r - f(\underline{v}) \leq 0. \tag{3.18}$$

Moreover, we will show that the above inequalities imply

$$\underline{k}(t) \le h(t+T_1),\ \underline{v}(t,r) \le u(t+T_1,r) \text{ for } r \in (\underline{k}(t) - M\log t, \underline{k}(t)) \text{ and } t \ge T. \tag{3.19}$$

Clearly the required estimate for $h(t)$ follows directly from (3.19) and (3.15).

By the definition of $\underline{k}(t)$, we have $\underline{v}(t,\underline{k}(t)) = 0$. We now calculate

$$\begin{aligned}\underline{v}_r(t,\underline{k}(t)) &= \phi_r(\mu(c_0^* - c_N t^{-1}), \underline{k}(t) - \tilde{k}(t)) \\ &= \phi_r(\mu(c_0^* - c_N t^{-1}), 0) + \left[\phi_{rr}(\mu_0, 0) + o(1)\right]\left[\underline{k}(t) - \tilde{k}(t)\right] \\ &= -\frac{1}{\mu_0}(c_0^* - c_N t^{-1}) + \left[\phi_{rr}(\mu_0,0) + o(1)\right]\left[-\frac{\mu_0}{c_0^*} + o(1)\right] t^{-2}\log t.\end{aligned}$$

Using

$$\phi_{rr}(\mu_0, r) + c_0^*\phi_r(\mu_0, r) + f(\phi(\mu_0, r)) = 0$$

and $f(\phi(\mu_0,0)) = f(0) = 0$, we obtain

$$\phi_{rr}(\mu_0, 0) = -c_0^*\phi_r(\mu_0, 0) = \frac{c_0^{*2}}{\mu_0}.$$

It follows that

$$\begin{aligned}-\mu_0\underline{v}_r(t,\underline{k}(t)) &= c_0^* - c_N t^{-1} + \mu_0 c_0^* t^{-2}\log t + o(t^{-2}\log t) \\ &> c_0^* - c_N t^{-1} - Bt^{-2}\log t + O(t^{-2}) \\ &= \underline{k}'(t) \text{ for all large } t.\end{aligned}$$

Hence (3.16) holds.

Since

$$c_0^* t - M'\log t - \left[\underline{k}(t) - M\log t\right] = (c_N + M - M')\log t + o(1) > (M/2)\log t$$

for all large t, provided that $M > 2M'$, we obtain from (3.14) that

$$u(t, \underline{k}(t) - M\log t) \ge (1 - \tilde{M}e^{-\tilde{\delta}t})\left(1 - M_1 t^{-\delta_1 M/2}\right) > 1 - t^{-2}$$

for all large t, provided that $M > 4/\delta_1$. We now fix M such that $M > \max\{2M', 4/\delta_1\}$. Thus

$$u(t+s, \underline{k}(t+s) - M\log(t+s)) > 1 - (t+s)^{-2} > 1 - t^{-2}\log t > \underline{v}(t, \underline{k}(t) - M\log t)$$

for all large t and every $s > 0$. This proves (3.17).

Next we show (3.18). We have, with $\xi = c_0^* - c_N t^{-1}$,

$$\begin{aligned}\underline{v}_t =& \phi_\mu(\mu(\xi), r - \tilde{k}(t))\mu'(\xi)c_N t^{-2} - \phi_r(\mu(\xi), r - \tilde{k}(t))\tilde{k}'(t) + 2t^{-3}\log t - t^{-3} \\ =& O(t^{-2}) + \phi_r \cdot \big(- c_0^* + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2}\big),\end{aligned}$$

and

$$\underline{v}_r(t,r) = \phi_r(\mu(\xi), r - \tilde{k}(t)),\ \underline{v}_{rr}(t,r) = \phi_{rr}(\mu(\xi), r - \tilde{k}(t)).$$

Hence,

$$\begin{aligned}&\underline{v}_t - \underline{v}_{rr} - \frac{N-1}{r}\underline{v}_r - f(\underline{v}) \\ &= O(t^{-2}) + \phi_r\left[-c_0^* + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2} - \frac{N-1}{r}\right] - \phi_{rr} - f\left(\phi - t^{-2}\log t\right) \\ &= O(t^{-2}) + \phi_r\left[g(\xi) - g(c_0^*) + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2} - \frac{N-1}{r}\right] \\ &\quad - g(\xi)\phi_r - \phi_{rr} - f\left(\phi - t^{-2}\log t\right) \\ &= O(t^{-2}) + \phi_r J + f(\phi) - f\left(\phi - t^{-2}\log t\right),\end{aligned}$$

where

$$J := g(\xi) - g(c_0^*) + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2} - \frac{N-1}{r}.$$

For $r \in [\underline{k}(t) - M\log t, \underline{k}(t)]$, we have

$$\begin{aligned} r &\geq \underline{k}(t) - M\log t \\ &= \tilde{k}(t) - M\log t + O(t^{-2}\log t) \\ &= c_0^* t - (c_N + M)\log t + Bt^{-1}\log t + O(t^{-2}\log t) \\ &\geq c_0^* t - M_2 \log t \quad \text{for all large } t, \end{aligned}$$

where $M_2 = c_N + M$. It follows that, for such r,

$$\begin{aligned} \frac{N-1}{r} &\leq \frac{N-1}{c_0^* t - M_2\log t} \\ &= \frac{N-1}{c_0^* t} + \frac{(N-1)M_2 \log t}{c_0^{*2} t^2}\big[1 + o(1)\big]. \end{aligned}$$

Therefore

$$\begin{aligned} J &\geq -g'(c_0^*)c_N t^{-1} + c_N t^{-1} - \frac{N-1}{c_0^*}t^{-1} + \left[B - \frac{(N-1)M_2}{c_0^{*2}}\right] t^{-2}\log t + o(t^{-2}\log t) \\ &= \left[B - \frac{(N-1)M_2}{c_0^{*2}} + o(1)\right] t^{-2}\log t > 0 \end{aligned}$$

for all large t, provided that B is large enough.

We now fix $\epsilon_0 > 0$ small so that $f'(u) \leq -\sigma_0 < 0$ for $u \in [1-2\epsilon_0, 1+2\epsilon_0]$. Then when $\phi(\mu(\xi), r - \tilde{k}(t)) \in [1-\epsilon_0, 1]$ we have

$$f(\phi) - f(\phi - t^{-2}\log t) \leq -\sigma_0 t^{-2}\log t$$

for all large t. Hence in such a case,

$$O(t^{-2}) + \phi_r J + f(\phi) - f(\phi - t^{-2}\log t) \leq O(t^{-2}) - \sigma_0 t^{-2}\log t < 0$$

for all large t.

If $\phi(\mu(\xi), r - \tilde{k}(t)) \in [0, 1-\epsilon_0]$, then we can find $\sigma_1 > 0$ such that $\phi_r \leq -\sigma_1$, and hence

$$\phi_r J \leq -\sigma_1 \left[B - \frac{(N-1)M_2}{c_0^{*2}} + o(1)\right] t^{-2}\log t.$$

On the other hand, there exists $\sigma_2 > 0$ such that

$$f(\phi) - f(\phi - t^{-2}\log t) \leq \sigma_2 t^{-2}\log t.$$

Thus in this case we have

$$\begin{aligned} &O(t^{-2}) + \phi_r J + f(\phi) - f(\phi - t^{-2}\log t) \\ &\leq -\sigma_1 \left[B - \frac{(N-1)M_2}{c_0^{*2}} + o(1)\right] t^{-2}\log t + \sigma_2 t^{-2}\log t + O(t^{-2}) \\ &< 0 \end{aligned}$$

for all large t, provided that B is large enough. This proves (3.18).

We are now ready to show (3.19). Since as $t \to \infty$, $h(t) \to \infty$ and $u(t,r) \to 1$ locally uniformly in $r \in [0,\infty)$, we can find $T' > T$ such that

$$h(T') > \underline{k}(T),\ u(T', r) > \underline{v}(T, r) \text{ for } r \in [\underline{k}(T) - M\log T, \underline{k}(T)],$$

where $T>0$ is a constant such that (3.16), (3.17) and (3.18) hold for $t\geq T$. We may now use a comparison argument to conclude that

$$h(T'+t)\geq \underline{k}(T+t),\ u(T'+t,r)\geq \underline{v}(T+t,r)$$

for $t>0,\ r\in[\underline{k}(T+t)-M\log(T+t),\underline{k}(T+t)]$, which is equivalent to (3.19) with $T_1=T'-T$. □

Lemma 3.5. *There exist $C>0$ and $T>0$ such that*

$$h(t)\leq c_0^*t-c_N\log t+C \text{ for } t\geq T,$$

where c_N is given by (2.8).

Proof. With $B>0$ and $C>0$ constants to be determined, and $\phi(z)=\phi(\mu,z)$ given in (2.1), we set

$$\hat{k}(t)=c_0^*t-c_N\log t-Bt^{-1}\log t+C,$$
$$\overline{v}(t,r)=\phi\left(\mu(c_0^*-c_Nt^{-1}),r-\hat{k}(t)\right)+t^{-2}\log t.$$

We have $\overline{v}(t,\hat{k}(t))=t^{-2}\log t>0$ for $t>1$, and

$$\overline{v}(t,\hat{k}(t)+t^{-1})=\phi\left(\mu(c_0^*-c_Nt^{-1}),t^{-1}\right)+t^{-2}\log t=\left[\phi_r(\mu_0,0)+o(1)\right]t^{-1}<0$$

for all large t. Moreover,

$$\overline{v}_r(t,r)=\phi_r\left(\mu(c_0^*-c_Nt^{-1}),r-\hat{k}(t)\right)<0 \text{ for all } t>0 \text{ and } r\in(0,\hat{k}(t)].$$

Therefore, there exists a unique $\overline{k}(t)\in(\hat{k}(t),\hat{k}(t)+t^{-1})$ such that

$$\overline{v}(t,\overline{k}(t))=0 \text{ for all large } t.$$

By the implicit function theorem we know that $t\to\overline{k}(t)$ is smooth, and by the mean value theorem we obtain

$$\left[\phi_r(\mu_0,0)+o(1)\right]\left[\overline{k}(t)-\hat{k}(t)\right]=-t^{-2}\log t.$$

Since $\phi_r(\mu_0,0)=-c_0^*/\mu_0$, we thus obtain

$$\overline{k}(t)-\hat{k}(t)=\left[\frac{\mu_0}{c_0^*}+o(1)\right]t^{-2}\log t \quad \text{for all large } t. \tag{3.20}$$

Using $\overline{v}_t(t,\overline{k}(t))+\overline{v}_r(t,\overline{k}(t))\overline{k}\,'(t)=0$ we obtain

$$\phi_\mu\cdot\mu'\cdot c_Nt^{-2}+\phi_r\cdot\left[\overline{k}\,'(t)-\hat{k}\,'(t)\right]-\left[1+o(1)\right]2t^{-3}\log t=0.$$

It follows that

$$\overline{k}\,'(t)=\hat{k}\,'(t)+O(t^{-2})=c_0^*-c_Nt^{-1}+Bt^{-2}\log t+O(t^{-2})$$

for all large t.

We want to show that, by choosing B and C properly, there exists a positive constant T such that $(\overline{v}(t,r),\overline{k}(t))$ satisfies, for $t\geq T$ and $1\leq r\leq\overline{k}(t)$,

$$\overline{v}(t,\overline{k}(t))=0,\ \ \overline{k}\,'(t)\geq-\mu_0\overline{v}_r(t,k(t)), \tag{3.21}$$

$$\overline{v}(t,1)\geq u(t,1), \tag{3.22}$$

$$\overline{v}_t-\overline{v}_{rr}-\frac{N-1}{r}\overline{v}_r-f(\overline{v})\geq 0, \tag{3.23}$$

and

$$\overline{k}(T)\geq h(T),\ \overline{v}(T,r)\geq u(T,r) \text{ for } r\in[1,h(T)]. \tag{3.24}$$

If these inequalities are proved, then we can apply a comparison argument to conclude that

$$\overline{k}(t)\geq h(t),\ \overline{v}(t,r)\geq u(t,r) \text{ for } r\in[1,h(t)] \text{ and } t\geq T. \tag{3.25}$$

Clearly the required estimate for $h(t)$ follows directly from (3.25) and (3.20).

By the definition of $\overline{k}(t)$, we have $\overline{v}(t,\overline{k}(t))=0$. We now calculate

$$\begin{aligned}\overline{v}_r(t,\overline{k}(t)) &= \phi_r(\mu(c_0^*-c_Nt^{-1}),\overline{k}(t)-\hat{k}(t))\\ &= \phi_r(\mu(c_0^*-c_Nt^{-1}),0)+\big[\phi_{rr}(\mu_0,0)+o(1)\big]\big[\overline{k}(t)-\hat{k}(t)\big]\\ &= -\frac{1}{\mu_0}(c_0^*-c_Nt^{-1})+\big[\phi_{rr}(\mu_0,0)+o(1)\big]\left[\frac{\mu_0}{c_0^*}+o(1)\right]t^{-2}\log t\\ &= -\frac{1}{\mu_0}(c_0^*-c_Nt^{-1})+c_0^*t^{-2}\log t+o(t^{-2}\log t).\end{aligned}$$

It follows that

$$\begin{aligned}-\mu_0\overline{v}_r(t,\overline{k}(t)) &= c_0^*-c_Nt^{-1}-\mu_0c_0^*t^{-2}\log t+o(t^{-2}\log t)\\ &< c_0^*-c_Nt^{-1}+Bt^{-2}\log t+O(t^{-2})\\ &= \overline{k}\,'(t) \text{ for all large } t.\end{aligned}$$

Hence (3.21) holds.

Since

$$\overline{v}(t,1)=\phi\big(\mu(c_0^*-c_Nt^{-1}),1-\hat{k}(t)\big)+t^{-2}\log t\geq 1-M_1e^{\delta_1[1-\hat{k}(t)]}+t^{-2}\log t\geq 1+t^{-2}$$

for all large t, and by Lemma 3.2, $u(t,1)\leq 1+Me^{-\delta t}$ for all $t>0$, we find that

$$u(t,1)<\overline{v}(t,1) \text{ for all large } t.$$

This proves (3.22).

Next we show (3.23). We have, with $\xi=c_0^*-c_Nt^{-1}$,

$$\begin{aligned}\overline{v}_t =&\phi_\mu(\mu(\xi),r-\hat{k}(t))\mu'(\xi)c_Nt^{-2}-\phi_r(\mu(\xi),r-\hat{k}(t))\hat{k}'(t)-2t^{-3}\log t+t^{-3}\\ =&O(t^{-2})+\phi_r\cdot\big(-c_0^*+c_Nt^{-1}-Bt^{-2}\log t+Bt^{-2}\big),\end{aligned}$$

and

$$\overline{v}_r(t,r)=\phi_r(\mu(\xi),r-\hat{k}(t)),\ \overline{v}_{rr}(t,r)=\phi_{rr}(\mu(\xi),r-\hat{k}(t)).$$

Hence,

$$\begin{aligned}&\overline{v}_t-\overline{v}_{rr}-\frac{N-1}{r}\overline{v}_r-f(\overline{v})\\ &= O(t^{-2})+\phi_r\left[-c_0^*+c_Nt^{-1}-Bt^{-2}\log t+Bt^{-2}-\frac{N-1}{r}\right]-\phi_{rr}-f\left(\phi+t^{-2}\log t\right)\\ &= O(t^{-2})+\phi_r\left[g(\xi)-g(c_0^*)+c_Nt^{-1}-Bt^{-2}\log t+Bt^{-2}-\frac{N-1}{r}\right]\\ &\quad -g(\xi)\phi_r-\phi_{rr}-f\left(\phi+t^{-2}\log t\right)\\ &= O(t^{-2})+\phi_r\hat{J}+f(\phi)-f\left(\phi+t^{-2}\log t\right),\end{aligned}$$

where

$$\hat{J}:=g(\xi)-g(c_0^*)+c_Nt^{-1}-Bt^{-2}\log t+Bt^{-2}-\frac{N-1}{r}.$$

For $r\in[1,\overline{k}(t)]$, we have

$$\begin{aligned}\frac{N-1}{r}\geq\frac{N-1}{\overline{k}(t)}&=\frac{N-1}{\hat{k}(t)+o(t^{-1})}\\ &=\frac{N-1}{c_0^*t}+\frac{(N-1)c_N\log t}{c_0^{*2}t^2}\big[1+o(1)\big].\end{aligned}$$

Therefore, for such r,

$$\begin{aligned}\hat{J} &\le -g'(c_0^*)c_N t^{-1} + c_N t^{-1} - \frac{N-1}{c_0^*}t^{-1} - \left[B + \frac{(N-1)c_N}{c_0^{*2}}\right] t^{-2}\log t + o(t^{-2}\log t)\\ &= -\left[B + \frac{(N-1)c_N}{c_0^{*2}} + o(1)\right] t^{-2}\log t < 0\end{aligned}$$

for all large t.

We now fix $\epsilon_0 > 0$ small so that $f'(u) \le -\sigma_0 < 0$ for $u \in [1-2\epsilon_0, 1+2\epsilon_0]$. Then for $\phi(\mu(\xi), r-\hat{k}(t)) \in [1-\epsilon_0, 1]$ we have

$$f(\phi) - f(\phi + t^{-2}\log t) \ge \sigma_0 t^{-2}\log t$$

for all large t. Hence in such a case,

$$O(t^{-2}) + \phi_r \hat{J} + f(\phi) - f(\phi + t^{-2}\log t) \ge O(t^{-2}) + \sigma_0 t^{-2}\log t > 0$$

for all large t.

If $\phi(\mu(\xi), r - \hat{k}(t)) \in [0, 1-\epsilon_0]$, then we can find $\sigma_1 > 0$ such that $\phi_r \le -\sigma_1$, and hence

$$\phi_r \hat{J} \ge \sigma_1 \left[B + \frac{(N-1)c_N}{c_0^{*2}} + o(1)\right] t^{-2}\log t.$$

On the other hand, there exists $\sigma_2 > 0$ such that

$$f(\phi) - f(\phi + t^{-2}\log t) \ge -\sigma_2 t^{-2}\log t.$$

Thus in this case we have

$$\begin{aligned}&O(t^{-2}) + \phi_r \hat{J} + f(\phi) - f(\phi + t^{-2}\log t)\\ &\ge \sigma_1 \left[B + \frac{(N-1)c_N}{c_0^{*2}} + o(1)\right] t^{-2}\log t - \sigma_2 t^{-2}\log t + O(t^{-2})\\ &> 0\end{aligned}$$

for all large t, provided that B is large enough. This proves (3.23).

Finally we show that (3.24) holds if C is chosen suitably. Indeed, we set

$$C = h(T) - c_0^* T + c_N \log T + 2T.$$

Then

$$\overline{k}(T) = \hat{k}(T) + o(T^{-1}) = h(T) - BT^{-1}\log T + 2T + o(T^{-1}) > h(T) + T$$

for T large enough.

By enlarging T if necessary we have, for $r \in [1, h(T)]$,

$$\begin{aligned}\overline{v}(T, r) &\ge \overline{v}(T, h(T)) = \phi(\mu(c_0^* - c_N T^{-1}), h(T) - \hat{k}(T)) + T^{-2}\log T\\ &\ge \phi(\mu(c_0^* - c_N T^{-1}), -T) + T^{-2}\log T\\ &\ge 1 - M_1 e^{-\delta_1 T} + T^{-2}\log T\\ &> 1 + T^{-2},\end{aligned}$$

while

$$u(T, r) \le 1 + Me^{-\delta T}.$$

Therefore

$$\overline{v}(T, r) \ge u(T, r) \text{ for } r \in [1, h(T)]$$

provided that T is large enough. This proves (3.24). The proof of the lemma is now complete. □

4. Convergence

Throughout this section we assume that (u,h) is the unique solution of (1.1) with $\mu=\mu_0>0$, and spreading happens: As $t\to\infty$, $h(t)\to\infty$ and $u(t,r)\to 1$ for r in compact subsets of $[0,\infty)$. We will prove the following convergence result.

Theorem 4.1. *There exists a constant $\hat h\in\mathbb{R}^1$ such that*

$$\lim_{t\to\infty}\left\{h(t)-\left[c_0^*t-c_N\log t\right]\right\}=\hat h,\ \lim_{t\to\infty}h'(t)=c_0^*$$

and

$$\lim_{t\to\infty}\|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])}=0.$$

Again we will prove this theorem by a series of lemmas. By Lemmas 3.4 and 3.5 we know that there exist $C,T>0$ such that

$$-C\le h(t)-[c_0^*t-c_N\log t]\le C \text{ for } t\ge T.$$

We now denote

$$k(t)=c_0^*t-c_N\log t-2C$$

and define

$$v(t,r)=u(t,r+k(t)),\ g(t)=h(t)-k(t),\ t\ge T.$$

Clearly

$$C\le g(t)\le 3C \text{ for } t\ge T.$$

Moreover,

$$u_r=v_r,\ u_{rr}=v_{rr},\ u_t=v_t-(c_0^*-c_Nt^{-1})v_r,$$

and (v,g) satisfies

$$\begin{cases} v_t-v_{rr}-\left[c_0^*-c_Nt^{-1}+\frac{N-1}{r+k(t)}\right]v_r=f(v), & -k(t)\le r<g(t), t>T,\\ v(t,g(t))=0,\ g'(t)=-\mu_0v_r(t,g(t))-c_0^*+c_Nt^{-1}, & t>T.\end{cases}$$

4.1. **Limit along a subsequence of $t_n\to\infty$.** Let $t_n\to\infty$ be an arbitrary sequence satisfying $t_n>T$ for every $n\ge 1$. Define

$$k_n(t)=k(t+t_n),\ v_n(t,r)=v(t+t_n,r),\ g_n(t)=g(t+t_n).$$

Lemma 4.2. *Subject to a subsequence,*

$$g_n\to G \text{ in } C^{1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^1) \text{ and } \|v_n-V\|_{C^{\frac{1+\alpha}{2},1+\alpha}_{loc}(D_n)}\to 0,$$

where $\alpha\in(0,1)$, $D_n=\{(t,r)\in D: r\le g_n(t)\}$, $D=\{(t,r): -\infty<r\le G(t),\ t\in\mathbb{R}^1\}$, and $(V(t,r),G(t))$ satisfies

$$\begin{cases} V_t-V_{rr}-c_0^*V_r=f(V), & (t,r)\in D,\\ V(t,G(t))=0,\ G'(t)=-\mu_0V_r(t,G(t))-c_0^*, & t\in\mathbb{R}^1.\end{cases} \tag{4.1}$$

Proof. By [9] there exists $C_0>0$ such that $0<h'(t)\le C_0$ for all $t>0$. It follows that

$$-c_0^*<g_n'(t)\le C_0 \text{ for } t+t_n \text{ large and every } n\ge 1.$$

Define

$$s=\frac{r}{g_n(t)},\ w_n(t,s)=v_n(t,r).$$

Then $(w_n(t,s), g_n(t))$ satisfies

$$(w_n)_t - \frac{(w_n)_{ss}}{g_n(t)^2} - \left[s g_n'(t) + c_0^* - c_N(t+t_n)^{-1} + \frac{N-1}{g_n(t)s + k_n(t)} \right] \frac{(w_n)_s}{g_n(t)} = f(w_n) \tag{4.2}$$

for $-\frac{k_n(t)}{g_n(t)} \le s < 1, t > T - t_n$, and

$$w_n(t,1) = 0 \text{ for } t > T - t_n, \tag{4.3}$$

$$g_n'(t) = -\mu_0 \frac{(w_n)_s(t,1)}{g_n(t)} - c_0^* + c_N(t+t_n)^{-1} \text{ for } t > T - t_n. \tag{4.4}$$

For any given $R > 0$ and $T_0 \in \mathbb{R}^1$, using the partial interior-boundary L^p estimates (see Theorem 7.15 in [23]) to (4.2) and (4.3) over $[T_0-1, T_0+1] \times [-R-1, 1]$, we obtain, for any $p > 1$,

$$\|w_n\|_{W_p^{1,2}([T_0,T_0+1]\times[-R,1])} \le C_R \text{ for all large } n,$$

where C_R is a constant depending on R and p but independent of n and T_0. Therefore, for any $\alpha' \in (0,1)$, we can choose $p > 1$ large enough and use the Sobolev embedding theorem (see [21]) to obtain

$$\|w_n\|_{C^{\frac{1+\alpha'}{2},1+\alpha'}([T_0,\infty)\times[-R,1])} \le \tilde{C}_R \text{ for all large } n, \tag{4.5}$$

where $\tilde{C}_R$ is a constant depending on R and α' but independent of n and T_0.

From (4.4) and (4.5) we deduce

$$\|g_n\|_{C^{1+\frac{\alpha'}{2}}([T_0,\infty))} \le C_1 \text{ for all large } n,$$

with C_1 a constant independent of T_0 and n. Hence by passing to a subsequence we may assume that, as $n \to \infty$,

$$w_n \to W \text{ in } C_{loc}^{\frac{\alpha+1}{2},1+\alpha}(\mathbb{R}^1 \times (-\infty,1]),\ g_n \to G \text{ in } C_{loc}^{1+\frac{\alpha}{2}}(\mathbb{R}^1),$$

where $\alpha \in (0,\alpha')$. Moreover, using (4.2),(4.3) and (4.4), we find that (W,G) satisfies in the $W_p^{1,2}$ sense (and hence classical sense by standard regularity theory),

$$\begin{cases} W_t - \frac{W_{ss}}{G(t)^2} - (sG'(t) + c_0^*)\frac{W_s}{G(t)} = f(W), & s \in (-\infty,1],\ t \in \mathbb{R}^1, \\ W(t,1) = 0,\ G'(t) = -\mu_0 \frac{W_s(t,1)}{G(t)} - c_0^*, & t \in \mathbb{R}^1. \end{cases}$$

Define $V(t,r) = W(t, \frac{r}{G(t)})$. We easily see that (V,G) satisfies (4.1) and

$$\lim_{n\to\infty} \|v_n - V\|_{C_{loc}^{\frac{1+\alpha}{2},1+\alpha}(D_n)} = 0.$$

□

4.2. **Determine the limit pair** (V,G)**.** We show by a sequence of lemmas that $G(t) \equiv G_0$ is a constant, and hence $V(t,r) = \phi(r - G_0)$.

Since $C \le g(t) \le 3C$ for $t \ge T$, we have

$$C \le G(t) \le 3C \text{ for } t \in \mathbb{R}^1.$$

By the proof of Lemma 3.5, we have, for $r \in [1 - k(t+t_n), g(t+t_n)]$ and $t + t_n$ large,

$$v_n(t,r) \le \phi\big(\mu(c_0^* - c_N(t+t_n)^{-1}), r - 3C\big) + (t+t_n)^{-2}\log(t+t_n).$$

Letting $n \to \infty$ we obtain

$$V(t,r) \le \phi(\mu_0, r - 3C) \text{ for all } t \in \mathbb{R}^1,\ r < G(t).$$

Define

$$R^* = \inf \big\{ R : V(t,r) \le \phi(\mu_0, r-R) \text{ for all } (t,r) \in D \big\}.$$

Then

$$V(t,r) \le \phi(\mu_0, r-R^*) \text{ for all } (t,r) \in D$$

and

$$C \le \inf_{t\in\mathbb{R}^1} G(t) \le \sup_{t\in\mathbb{R}^1} G(t) \le R^* \le 3C.$$

Lemma 4.3. $R^* = \sup_{t\in\mathbb{R}^1} G(t)$.

Proof. Otherwise we have $R^* > \sup_{t\in\mathbb{R}^1} G(t)$. We are going to derive a contradiction.

Choose $\delta > 0$ such that

$$G(t) \le R^* - \delta \text{ for all } t \in \mathbb{R}^1.$$

We derive a contradiction in three steps. To simplify notations we will write $\phi(r)$ instead of $\phi(\mu_0, r)$.

Step 1. $V(t,r) < \phi(r-R^*)$ for all $t \in \mathbb{R}^1$ and $r \le G(t)$.

Otherwise there exists $(t_0, r_0) \in D$ such that

$$V(t_0, r_0) = \phi(r_0 - R^*) \ge \phi(-\delta) > 0.$$

Hence necessarily $r_0 < G(t_0)$. Since $V(t,r) \le \phi(r-R^*)$ in D, and $\phi(r-R^*)$ satisfies the first equation in (4.1), we can apply the strong maximum principle to conclude that $V(t,r) \equiv \phi(r-R^*)$ in $D_0 := \{(t,r) : r < G(t), t \le t_0\}$, which clearly contradicts with the assumption that $G(t) \le R^* - \delta$.

Step 2. $M_r := \inf_{t\in\mathbb{R}^1} \big[\phi(r-R^*) - V(t,r)\big] > 0$ for $r \in (-\infty, R^*-\delta]$. Here we assume that $V(t,r) = 0$ for $r > G(t)$.

Otherwise there exists $r_0 \in (-\infty, R^*-\delta]$ such that $M_{r_0} = 0$, since the definition of R^* implies $M_r \ge 0$ for all $r \le R^* - \delta$. By Step 1 we know that M_{r_0} is not achieved at any finite t. Therefore there exists $s_n \in \mathbb{R}^1$ with $|s_n| \to \infty$ such that

$$\phi(r_0 - R^*) = \lim_{n\to\infty} V(s_n, r_0).$$

Define

$$(V_n(t,r), G_n(t)) = (V(t+s_n, r), G(t+s_n)).$$

Then the same argument used in the proof of Lemma 4.2 shows that, by passing to a subsequence, $(V_n, G_n) \to (\tilde V, \tilde G)$ with $(\tilde V, \tilde G)$ satisfying

$$\begin{cases} \tilde V_t - \tilde V_{rr} - c_0^* \tilde V_r = f(\tilde V), & -\infty < r < \tilde G(t), t \in \mathbb{R}^1, \\ \tilde V(t, \tilde G(t)) = 0, & t \in \mathbb{R}^1. \end{cases} \tag{4.6}$$

Moreover,

$$\tilde V(t,r) \le \phi(r-R^*),\ \tilde G(t) \le R^* - \delta,\ \tilde V(0, r_0) = \phi(r_0 - R^*) > 0. \tag{4.7}$$

Since $\phi(r-R^*)$ satisfies (4.6) with $\tilde G(t)$ replaced by R^*, we can apply the strong maximum principle to conclude, from (4.7), that $\tilde V(t,r) \equiv \phi(r-R^*)$ for $t \le 0, r \le \tilde G(t)$, which is clearly impossible.

Step 3. Reaching a contradiction.

Choose $\epsilon_0 > 0$ small and $R_0 < 0$ large negative such that

$$\phi(r-R^*) \ge 1 - \epsilon_0 \text{ for } r \le R_0,\ f'(u) < 0 \text{ for } u \in [1-2\epsilon_0, 1+2\epsilon_0].$$

Then choose $\epsilon \in (0, \epsilon_0)$ such that

$$\phi(R_0 - R^* + \epsilon) \ge \phi(R_0 - R^*) - M_{R_0},\ \phi(r - R^* + \epsilon) \ge 1 - 2\epsilon_0 \text{ for } r \le R_0.$$

We consider the auxiliary problem

$$\begin{cases} \overline{V}_t - \overline{V}_{rr} - c_0^* \overline{V}_r = f(\overline{V}), & t > 0,\ r < R_0, \\ \overline{V}(t, R_0) = \phi(R_0 - R^* + \epsilon), & t > 0, \\ \overline{V}(0, r) = 1, & r < R_0. \end{cases} \tag{4.8}$$

Since the initial function is an upper solution of the corresponding stationary problem of (4.8), its unique solution $\overline{V}(t,r)$ is decreasing in t. Clearly $\underline{V}(t,r) := \phi(r - R^* + \epsilon)$ is a lower solution of (4.8). It follows from the comparison principle that

$$1 \geq \overline{V}(t,r) \geq \phi(r - R^* + \epsilon) \text{ for all } t > 0,\ r < R_0.$$

Hence

$$V^*(r) := \lim_{t\to\infty} \overline{V}(t,r) \geq \phi(r - R^* + \epsilon),\ \forall r < R_0.$$

Moreover, V^* satisfies

$$-V^*_{rr} - c_0^* V^*_r = f(V^*) \text{ in } (-\infty, R_0),\ V^*(-\infty) = 1,\ V^*(R_0) = \phi(R_0 - R^* + \epsilon). \tag{4.9}$$

Write $\psi(r) = \phi(r - R^* + \epsilon)$. We notice that $\psi(r)$ also satisfies (4.9). Moreover

$$1 - 2\epsilon_0 \leq \psi(r) \leq V^*(r) \leq 1 \text{ for } r \in (-\infty, R_0].$$

Hence $W(r) := V^*(r) - \psi(r) \geq 0$ and there exists $c(r) < 0$ such that

$$f(V^*(r)) - f(\psi(r)) = c(r)W(r) \text{ in } (-\infty, R_0].$$

Therefore

$$-W'' - c_0^* W' = c(r)W \text{ in } (-\infty, R_0),\ W(R_0) = 0,$$

and by the maximum principle we deduce, for any $R < R_0$,

$$W(r) \leq W(R) \text{ for } r \in [R, R_0].$$

Letting $R \to -\infty$ we deduce $W(r) \leq 0$ in $(-\infty, R_0]$. It follows that $W \equiv 0$. Hence

$$V^*(r) \equiv \psi(r) = \phi(r - R^* + \epsilon).$$

We now look at $V(t,r)$, which satisfies the first equation in (4.8), and for any $t \in \mathbb{R}^1$,

$$V(t,r) \leq 1,\ V(t,R_0) \leq \phi(R_0 - R^*) - M_{R_0} \leq \phi(R_0 - R^* + \epsilon).$$

Therefore we can use the comparison principle to deduce that

$$V(s+t, r) \leq \overline{V}(t,r) \text{ for all } t > 0, r < R_0, s \in \mathbb{R}^1.$$

Or equivalently

$$V(t,r) \leq \overline{V}(t-s, r) \text{ for all } t > s, r < R_0, s \in \mathbb{R}^1.$$

Letting $s \to -\infty$ we obtain

$$V(t,r) \leq V^*(r) = \phi(r - R^* + \epsilon) \text{ for all } r < R_0, t \in \mathbb{R}^1. \tag{4.10}$$

By Step 2 and the continuity of M_r in r, we have

$$M_r \geq \sigma > 0 \text{ for } r \in [R_0, R^* - \delta].$$

If $\epsilon_1 \in (0, \epsilon]$ is small enough we have

$$\phi(r - R^* + \epsilon_1) \geq \phi(r - R^*) - \sigma \text{ for } r \in [R_0, R^* - \delta],$$

and hence

$$V(t,r) - \phi(r - R^* + \epsilon_1) \leq \sigma - M_r \leq 0 \text{ for } r \in [R_0, R^* - \delta],\ t \in \mathbb{R}^1.$$

Therefore we can combine with (4.10) to obtain

$$V(t,r) - \phi(r - R^* + \epsilon_1) \leq 0 \text{ for } r \in (-\infty, R^* - \delta],\ t \in \mathbb{R}^1,$$

for all small $\epsilon_1 \in (0, \epsilon)$, which contradicts the definition of R^*. The proof is now complete. □

Lemma 4.4. *There exists a sequence* $\{s_n\} \subset \mathbb{R}^1$ *such that*

$$G(t+s_n) \to R^*,\ V(t+s_n, r) \to \phi(r-R^*) \text{ as } n \to \infty$$

uniformly for (t,r) *in compact subsets of* $\mathbb{R}^1 \times (-\infty, R^*]$.

Proof. There are two possibilities:

(i) $R^* = \sup_{t\in\mathbb{R}^1} G(t)$ is achieved at some finite $t = s_0$,

(ii) $R^* > G(t)$ for all $t \in \mathbb{R}^1$ and $G(s_n) \to R^*$ along some unbounded sequence s_n.

In case (i), necessarily $G'(s_0) = 0$. Since $V(t,r) \le \phi(r-R^*)$ for $r \le G(t)$ and $t \in \mathbb{R}^1$, with $V(s_0, G(s_0)) = \phi(G(s_0) - R^*) = \phi(0) = 0$, we can apply the strong maximum principle and the Hopf boundary lemma to conclude that

$$V_r(s_0, G(s_0)) > \phi'(0) \text{ unless } V(t,r) \equiv \phi(r-R^*) \text{ in } D_0 = \{(t,r) : r \le G(t), t \le s_0\}.$$

On the other hand, we have

$$V_r(s_0, G(s_0)) = -\mu_0^{-1}[G'(s_0) + c_0^*] = -\mu_0^{-1} c_0^* = \phi'(0).$$

Hence we must have $V(t,r) \equiv \phi(r-R^*)$ and $G(t) \equiv R^*$ in D_0. Using the uniqueness of (4.1) with a given initial value, we conclude that $V(t,r) \equiv \phi(r-R^*)$ for all $r \le G(t)$ and $t \in \mathbb{R}^1$. Thus the conclusion of the lemma holds by taking $s_n \equiv s_0$.

In case (ii), we consider the sequence

$$V_n(t,r) = V(t+s_n, r),\ G_n(t) = G(t+s_n).$$

By the same reasoning as in the proof of Lemma 4.2, we can show that, by passing to a subsequence,

$$V_n \to \tilde V \text{ in } C_{loc}^{\frac{1+\alpha}{2}, 1+\alpha}(D),\ G_n \to \tilde G \text{ in } C^1_{loc}(\mathbb{R}^1) \text{ and } (\tilde V, \tilde G) \text{ satisfies (4.1)},$$

where $D := \{(t,r) : -\infty < r \le \tilde G(t),\ t \in \mathbb{R}^1\}$. Moreover,

$$\tilde G(t) \le R^*,\ \tilde G(0) = R^*.$$

Hence we are back to case (i) and thus $\tilde V(t,r) \equiv \phi(r-R^*)$ in D, and $\tilde G \equiv R^*$. The conclusion of the lemma now follows easily. ☐

By the proof of Lemma 3.4, we have

$$v_n(t,r) \ge \phi\big(\mu(c_0^* - c_N(t+t_n)^{-1}), r - C\big) - (t+t_n)^{-2}\log(t+t_n)$$

for $r \in [\underline{k}(t+t_n) - k(t+t_n) - M\log(t+t_n), \underline{k}(t+t_n) - k(t+t_n)]$ and $t+t_n$ large. Letting $n \to \infty$ we obtain

$$V(t,r) \ge \phi(\mu_0, r - C) \text{ for all } t \in \mathbb{R}^1,\ r < G(t).$$

Define

$$R_* = \sup\big\{R : V(t,r) \ge \phi(\mu_0, r-R) \text{ for all } (t,r) \in D\big\}.$$

Then

$$V(t,r) \ge \phi(\mu_0, r - R_*) \text{ for all } (t,r) \in D$$

and

$$C \le R_* \le \inf_{t\in\mathbb{R}^1} G(t) \le \sup_{t\in\mathbb{R}^1} G(t) \le R^* \le 3C.$$

Lemma 4.5. $R_* = \inf_{t\in\mathbb{R}^1} G(t)$, *and there exists a sequence* $\{\tilde s_n\} \subset \mathbb{R}^1$ *such that*

$$G(t+\tilde s_n) \to R_*,\ V(t+\tilde s_n, r) \to \phi(r-R_*) \text{ as } n \to \infty$$

uniformly for (t,r) *in compact subsets of* $\mathbb{R}^1 \times (-\infty, R_*]$.

Proof. The proof uses similar arguments to those used to prove Lemmas 4.3 and 4.4, and we omit the details. ☐

Lemma 4.6. *$R_* = R^*$ and hence $G(t) \equiv G_0$ is a constant, which implies $V(t,r) = \phi(r - G_0)$.*

Proof. Argue indirectly we assume that $R_* < R^*$. Set $\epsilon = (R^* - R_*)/4$. We show next that there exists $T_\epsilon > 0$ such that

$$G(t) - R_* \le \epsilon \text{ and } G(t) - R^* \ge -\epsilon \text{ for } t \ge T_\epsilon, \tag{4.11}$$

which implies $R^* - R_* \le 2\epsilon$. This contradiction would complete the proof.

To prove (4.11), we use Lemmas 4.4 and 4.5, and a modification of the argument in section 3.3 of [11]. Indeed, by using Lemma 4.4 and constructing a suitable lower solution we can show that there exists $n_1 = n_1(\epsilon)$ large such that $G(t) - R^* \ge -\epsilon$ for all $t \ge s_{n_1}$. Similarly we can use Lemma 4.5 and construct a suitable upper solution to show that $G(t) - R_* \le \epsilon$ for all $t \ge \tilde{s}_{n_2}$ with $n_2 = n_2(\epsilon)$ large enough. Hence (4.11) holds for $t \ge T := \max\{s_{n_1}, \tilde{s}_{n_2}\}$. For completeness, the detailed constructions of the above mentioned upper and lower solutions are given in the Appendix at the end of the paper. □

4.3. **Convergence of h and u.**

Lemma 4.7. *There exist a constant $C > 0$ and a function $\xi \in C^1(\mathbb{R}^1_+)$ such that $|\xi(t)| \le C$ for all $t > 0$,*

$$\lim_{t\to\infty} \left\{h(t) - \left[c_0^* t - c_N \log t + \xi(t)\right]\right\} = 0, \ \lim_{t\to\infty} \xi'(t) = 0,$$

and

$$\lim_{t\to\infty} \|u(t,\cdot) - q_{c_0^*}(h(t) - \cdot)\|_{L^\infty([0,h(t)])} = 0.$$

Proof. By Lemmas 4.2 and 4.6, we find that for any sequence $t_n \to \infty$, by passing to a subsequence, $h(t + t_n) - k(t + t_n) \to G_0$ in $C^{1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^1)$. Hence $h'(t + t_n) \to c_0^*$ in $C^{\alpha/2}_{loc}(\mathbb{R}^1)$.

We now define

$$U(t,r) = u(t, r + h(t)) \ \text{ for } t > 0, \ r \in [-h(t), 0],$$

and

$$U_n(t,r) = U(t + t_n, r), \ h_n(t) = h(t + t_n).$$

It is easily checked that

$$\begin{cases} (U_n)_t - \left[h_n'(t) + \frac{N-1}{r+h_n(t)}\right](U_n)_r - (U_n)_{rr} = f(U_n), & t > -t_n, \ r \in (-h_n(t), 0], \\ U_n(t,0) = 0, \ (U_n)_r(t,0) = -h_n'(t)/\mu_0, & t > -t_n. \end{cases} \tag{4.12}$$

By the same reasoning as in the proof of Lemma 4.2, we can use the parabolic regularity to (4.12) plus Sobolev embedding to conclude that, by passing to a further subsequence, as $n \to \infty$,

$$U_n \to U \text{ in } C^{\frac{1+\alpha}{2}, 1+\alpha}_{loc}(\mathbb{R}^1 \times (-\infty, 0]),$$

and U satisfies, in view of $h_n'(t) \to c_0^*$,

$$\begin{cases} U_t - c_0^* U_r - U_{rr} = f(U), & t \in \mathbb{R}^1, \ r \in (-\infty, 0], \\ U(t,0) = 0, \ U_r(t,0) = -c_0^*/\mu_0, & t \in \mathbb{R}^1. \end{cases}$$

This is equivalent to (4.1) with $V = U$ and $G = 0$. Hence we may repeat the argument in Lemmas 4.2-4.5 to conclude that

$$U(t,r) \equiv \phi(\mu_0, r) \text{ for } (t,r) \in \mathbb{R}^1 \times (-\infty, 0].$$

Thus we have proved that, as $n \to \infty$,

$$u(t + t_n, r + h(t + t_n)) - q_{c_0^*}(-r) \to 0 \text{ in } C^{\frac{1+\alpha}{2}, 1+\alpha}_{loc}(\mathbb{R}^1 \times (-\infty, 0]).$$

Since $\{t_n\}$ is an arbitrary sequence converging to ∞, this implies that

$$\lim_{t\to\infty}\left[u(t,r+h(t))-q_{c_0^*}(-r)\right]=0 \text{ uniformly for } r \text{ in compact subsets of } (-\infty,0].$$

Therefore, for every $L>0$,

$$\lim_{t\to\infty}\|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([h(t)-L,h(t)])}=0. \tag{4.13}$$

Similarly, the arbitrariness of $\{t_n\}$ implies that $h'(t)\to c_0^*$ as $t\to\infty$. Hence

$$\xi(t):=h(t)-[c_0^*t-c_N\log t]$$

satisfies

$$\xi'(t)\to 0 \text{ as } t\to\infty.$$

The boundedness of $\xi(t)$ is a direct consequence of (3.2).

It remains to strengthen (4.13) to

$$\lim_{t\to\infty}\|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])}=0.$$

Let $(\underline{v}(t,r),\underline{k}(t))$ be as in the proof of Lemma 3.4, so that (3.16), (3.17) and (3.18) hold. Since as $t\to\infty$, $h(t)\to\infty$ and $u(t,r)\to 1$ locally uniformly in $r\in[0,\infty)$, we can find $T_2>0$ such that

$$h(T_2)>\underline{k}(T),\ u(T_2,r)>\underline{v}(T,r) \text{ for } r\in[0,\underline{k}(T)].$$

We note that $\underline{v}(T,r)$ is a strictly decreasing function of r. We now choose a smooth function $\tilde u_0(r)$ such that

$$\tilde u_0'(0)=\tilde u_0(\tilde h_0)=0,\ \tilde u_0'(r)<0,\ u(T_2,r)>\tilde u_0(r) \text{ in } (0,\tilde h_0], \text{ and } \tilde u_0(r)>\underline{v}(T,r) \text{ in } (0,\underline{k}(T)),$$

where $\tilde h_0\in(\underline{k}(T),h(T_2))$. We next consider the auxiliary problem

$$\begin{cases} u_t=u_{rr}+\frac{N-1}{r}u_r+f(u), & 0<r<h(t),\ t>0,\\ u(t,h(t))=0,\ h'(t)=-\mu_0u_r(t,h(t)), & t>0,\\ h(0)=\tilde h_0,\ \ u(0,r)=\tilde u_0(r), & 0\le r\le\tilde h_0.\end{cases} \tag{4.14}$$

Let $(\tilde u,\tilde h)$ denote the unique solution of (4.14). By the comparison principle we have

$$h(t+T_2)\ge\tilde h(t),\ u(t+T_2,r)\ge\tilde u(t,r) \text{ for } t>0,\ r\in[0,\tilde h(t)].$$

Moreover, since $\tilde u_0'(r)<0$ we can use a reflection argument to show that $\tilde u_r(t,r)<0$ for $t>0$ and $r\in(0,\tilde h(t)]$. This reflection argument is similar in spirit to the well known moving plane argument used for elliptic problems. The idea is to treat (4.14) as an initial boundary value problem for $\tilde u=\tilde u(t,x)$ over the region $\Omega:=\{(t,x):t>0,|x|<\tilde h(t)\}$ in $\mathbb{R}^1\times\mathbb{R}^N$. For each point x_0 in the ball $\{|x|<\tilde h(t)\}$ but away from the origin, we consider a hyperplane H passing through x_0, which divides $\mathbb{R}^N$ into two half spaces H^- and H^+, where H^- denotes the half space that contains the origin. Denote $\Omega^+=\{(t,x)\in\Omega: x\in H^+\}$, and for each point $x\in H^+$, we denote by $x^*\in H^-$ its reflection in H, and define $\tilde u^*(t,x)=\tilde u(t,x^*)$ for $(t,x)\in\Omega^+$. Then on the parabolic boundary of Ω^+, $\tilde u-\tilde u^*\le 0$ but is not identically 0. We thus obtain by the maximum principle that $\tilde u-\tilde u^*\le 0$ in Ω^+ and strict inequality holds in the interior of Ω^+. Since $\tilde u(t,x_0)-\tilde u^*(t,x_0)=0$, we can apply the Hopf boundary lemma to conclude that

$$\partial_\nu\tilde u(t,x_0)=\frac{1}{2}\partial_\nu[\tilde u(t,x_0)-\tilde u^*(t,x_0)]<0,$$

where ν is a normal vector of H pointing away from the origin. The conclusion $\tilde u_r(t,r)<0$ is a simple consequence of this fact.

On the other hand, if T is large enough, our assumptions on $\tilde u(0,r)$ and $\tilde h(0)$ imply that spreading happens for $(\tilde u,\tilde h)$ (see [9]). Hence we can apply Lemma 3.4 to $(\tilde u,\tilde h)$ to conclude that there exist

$\tilde{T} > 0, \tilde{T}_1 > 0$ such that (3.19) holds when (u, h, T, T_1) there is replaced by $(\tilde{u}, \tilde{h}, \tilde{T}, \tilde{T}_1)$. We thus obtain

$$u(t+T_1+T_2, r) \geq \tilde{u}(t+T_1, r) \geq \underline{v}(t,r) \text{ for } r \in [\underline{k}(t) - M\log t, \underline{k}(t)] \text{ and } t \geq \tilde{T}.$$

It follows that

$$\liminf_{t\to\infty} \min_{r\in[0,h(t)-L]} u(t,r) \geq \liminf_{t\to\infty} \tilde{u}(t, h(t)-L) \geq \liminf_{t\to\infty} \underline{v}(t, h(t)-L) \geq \phi(\mu_0, -L+C).$$

Therefore, for any $\epsilon > 0$ there exists $L_\epsilon > 0$ large such that

$$u(t,r) \geq q_{c_0^*}(L_\epsilon - C) \geq 1-\epsilon \text{ for all } r \in [0, h(t)-L_\epsilon] \text{ and all large } t.$$

Since $q_{c_0^*}(r) < 1$ is increasing in r, and by Lemma 3.2, $u(t,r) \leq 1+\epsilon$ for all large t, we deduce

$$|u(t,r) - q_{c_0^*}(h(t)-r)| \leq 2\epsilon \text{ for } r \in [0, h(t)-L_\epsilon] \text{ and all large } t.$$

We may now make use of (4.13) to obtain

$$\limsup_{t\to\infty} \|u(t,\cdot) - q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])} \leq \limsup_{t\to\infty} \|u(t,\cdot) - q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)-L_\epsilon])} \leq 2\epsilon.$$

Since $\epsilon > 0$ can be arbitrarily small, we obtain

$$\lim_{t\to\infty} \|u(t,\cdot) - q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])} = 0,$$

as we wanted. The proof is complete. □

4.4. **Improved convergence result for h.**

Lemma 4.8. *There exists $\hat{h} \in \mathbb{R}^1$ such that*

$$\lim_{t\to\infty} \left[h(t) - c_0^* t + c_N \log t\right] = \hat{h}.$$

Proof. By Lemma 4.7,

$$\xi(t) = h(t) - c_0^* t + c_N \log t \in [-C, C] \text{ for } t > 0.$$

Set

$$\hat{h} = \liminf_{t\to\infty} \xi(t).$$

We will show that for any given small $\epsilon > 0$,

$$\limsup_{t\to\infty} \xi(t) \leq \hat{h} + \epsilon. \tag{4.15}$$

The required conclusion clearly follows from (4.15).

We use a comparison argument to prove (4.15). Let $t_k \to \infty$ be chosen such that $\xi(t_k) \to \hat{h}$ as $k \to \infty$. For given small $\epsilon > 0$, we define

$$\tilde{h}_k(t) = c_0^*(t+t_k) - c_N \log(t+t_k) + B\epsilon(1-e^{-\alpha t}) + \hat{h} + \epsilon,\ t \geq 0,$$

$$\overline{u}_k(t,r) = \phi\big(\mu(c_0^* - c_N(t+t_k)^{-1}), r - \tilde{h}_k(t)\big) + \epsilon e^{-\alpha t},\ r \in [0, \tilde{h}_k(t) + \epsilon_0],$$

where α and B are positive constants to be determined later, and ϕ is given by (2.1), which is defined over $(-\infty, \epsilon_0]$. To simplify notations, we will write

$$\tilde{h}_k(t) = \tilde{h}(t),\ \overline{u}_k(t,r) = \overline{u}(t,r) \text{ unless their dependence on } k \text{ need to be stressed.}$$

We will choose α and B such that for all large k and small ϵ,

$$\limsup_{t\to\infty} \xi(t+t_k) \leq \hat{h} + C_0\epsilon,$$

where $C_0 > 0$ is a constant independent of ϵ and k. This clearly implies (4.15).

By definition, with the notation $\zeta = c_0^* - c_N(t+t_k)^{-1}$,

$$\overline{u}_r(t,r) = \phi_r(\mu(\zeta), r - \tilde{h}(t)) < 0 \text{ for } r \in [0, \tilde{h}(t) + \epsilon_0].$$

Moreover,

$$\overline{u}(t, \tilde{h}(t)) = \phi(\mu(\zeta), 0) + \epsilon e^{-\alpha t} > 0 \quad (\forall t > 0)$$

and

$$\overline{u}(t, \tilde{h}(t) + \epsilon_0) = \phi(\mu(\zeta), \epsilon_0) + \epsilon e^{-\alpha t} < 0 \quad (\forall t > 0)$$

provided that $\epsilon > 0$ is small enough. Hence for such ϵ, there exists a unique $\overline{h}(t) = \overline{h}_k(t) \in (\tilde{h}(t), \tilde{h}(t) + \epsilon_0)$ such that

$$\overline{u}(t, \overline{h}(t)) = 0 \quad (\forall t > 0).$$

Moreover, we could replace ϵ_0 by $C\epsilon$ with $C > 0$ sufficiently large to conclude that $\overline{h}(t) < \tilde{h}(t) + C\epsilon$, and we can apply the implicit function theorem to conclude that $t \to \overline{h}(t)$ is a smooth function.

By the mean value theorem we have

$$\overline{u}(t, \overline{h}(t)) - \overline{u}(t, \tilde{h}(t)) = \big[\phi_r(\mu_0, 0) + o_{\epsilon,k}(1)\big]\big[\overline{h}(t) - \tilde{h}(t)\big] = -\epsilon e^{-\alpha t} \quad (\forall t > 0),$$

where $o_{\epsilon,k}(1) \to 0$ as $\epsilon \to 0$ and $k \to \infty$, uniformly in $t > 0$. It follows that

$$\overline{h}(t) - \tilde{h}(t) = \left[\frac{\mu_0}{c_0^*} + o_{\epsilon,k}(1)\right] \epsilon e^{-\alpha t} \quad (\forall t > 0). \tag{4.16}$$

Using $\frac{d}{dt}\overline{u}(t, \overline{h}(t)) = 0$ we deduce

$$\phi_\mu \cdot \mu' \cdot c_N(t+t_k)^{-2} + \phi_r \cdot \big[\overline{h}'(t) - \tilde{h}'(t)\big] - \alpha\epsilon e^{-\alpha t} = 0.$$

Since $\phi_\mu \cdot \mu' > 0$, it follows that

$$\begin{aligned} \overline{h}'(t) &> \tilde{h}'(t) + [\phi_r]^{-1}\alpha\epsilon e^{-\alpha t} \\ &= c_0^* - c_N(t+t_k)^{-1} + \alpha B\epsilon e^{-\alpha t} - \left[\frac{\mu_0}{c_0^*} + o_{\epsilon,k}(1)\right]\alpha\epsilon e^{-\alpha t} \\ &= c_0^* - c_N(t+t_k)^{-1} + \left[B - \frac{\mu_0}{c_0^*} + o_{\epsilon,k}(1)\right]\alpha\epsilon e^{-\alpha t} \quad (\forall t > 0). \end{aligned}$$

On the other hand, for all large k and small ϵ, we have

$$\begin{aligned} \overline{u}_r(t, \overline{h}(t)) &= \phi_r(\mu(\zeta), \overline{h}(t) - \tilde{h}(t)) \\ &= \phi_r(\mu(\zeta), 0) + \big[\phi_{rr}(\mu_0, 0) + o_{\epsilon,k}(1)\big]\big[\overline{h}(t) - \tilde{h}(t)\big] \\ &> -\frac{1}{\mu_0}\big[c_0^* - c_N(t+t_k)^{-1}\big] \quad (\forall t > 0) \end{aligned}$$

since $\phi_{rr}(\mu_0, 0) = -c_0^*\phi_r(\mu_0, 0) = (c_0^*)^2/\mu_0 > 0$. Therefore if we choose $B > \frac{\mu_0}{c_0^*}$, then for all large k and small ϵ,

$$\overline{h}'(t) > -\mu_0 \overline{u}_r(t, \overline{h}(t)) \quad (\forall t > 0). \tag{4.17}$$

Next we prove that by choosing α suitably small and enlarging B accordingly, we have

$$\overline{u}_t - \overline{u}_{rr} - \frac{N-1}{r}\overline{u}_r - f(\overline{u}) > 0 \text{ for } t > 0,\ r \in (0, \overline{h}(t)] \tag{4.18}$$

and all large k and small ϵ.

We calculate

$$\begin{aligned} \overline{u}_t &= \phi_\mu \cdot \mu' \cdot c_N(t+t_k)^{-2} - \phi_r \cdot \tilde{h}'(t) - \epsilon\alpha e^{-\alpha t} \\ &> -\phi_r\big[c_0^* - c_N(t+t_k)^{-1} + B\epsilon\alpha e^{-\alpha t}\big] - \epsilon\alpha e^{-\alpha t}. \end{aligned}$$

Hence

$$\begin{aligned}
&\overline{u}_t-\overline{u}_{rr}-\frac{N-1}{r}\overline{u}_r-f(\overline{u})\\
&>-\phi_r\big[c_0^*-c_N(t+t_k)^{-1}+B\epsilon\alpha e^{-\alpha t}+\frac{N-1}{r}\big]-\phi_{rr}-f(\phi+\epsilon e^{-\alpha t})-\epsilon\alpha e^{-\alpha t}\\
&=-\phi_r\tilde{J}+f(\phi)-f(\phi+\epsilon e^{-\alpha t})-\epsilon\alpha e^{-\alpha t},
\end{aligned}$$

where

$$\tilde{J}:=c_0^*-g(c_0^*-c_N(t+t_k)^{-1})-c_N(t+t_k)^{-1}+B\epsilon\alpha e^{-\alpha t}+\frac{N-1}{r}.$$

For $r\in(0,\overline{h}(t)]$, we have

$$\begin{aligned}
\frac{N-1}{r}\geq\frac{N-1}{\overline{h}(t)}&=\frac{N-1}{\tilde{h}(t)+o_{\epsilon,k}(1)}\\
&=\frac{N-1}{c_0^*(t+t_k)-c_N\log(t+t_k)+\hat{h}+o_{\epsilon,k}(1)}\\
&=\frac{N-1}{c_0^*(t+t_k)}+\frac{(N-1)c_N\log(t+t_k)}{c_0^{*2}(t+t_k)^2}\big[1+o_{\epsilon,k}(1)\big].
\end{aligned}$$

Moreover,

$$c_0^*-g(c_0^*-c_N(t+t_k)^{-1})=g'(c_0^*)c_N(t+t_k)^{-1}+O_k\big[(t+t_k)^{-2}\big].$$

Therefore,

$$\begin{aligned}
\tilde{J}&\geq\left\{c_N[g'(c_0^*)-1]+\frac{N-1}{c_0^*}\right\}(t+t_k)^{-1}+\frac{(N-1)c_N\log(t+t_k)}{c_0^{*2}(t+t_k)^2}\big[1+o_{\epsilon,k}(1)\big]+B\epsilon\alpha e^{-\alpha t}\\
&=\frac{(N-1)c_N\log(t+t_k)}{c_0^{*2}(t+t_k)^2}\big[1+o_{\epsilon,k}(1)\big]+B\epsilon\alpha e^{-\alpha t}\\
&>B\epsilon\alpha e^{-\alpha t}\quad(\forall t>0)
\end{aligned}$$

for all large k and small ϵ.

Choose $\delta_0>0$ small so that $f'(u)\leq-\sigma_0<0$ for $u\in[1-\delta_0,1+\delta_0]$. Then for $\phi\in[1-\delta_0,1)$ we have

$$f(\phi)-f(\phi+\epsilon e^{-\alpha t})\geq\sigma_0\epsilon e^{-\alpha t}.$$

Thus for all large k and small ϵ and

$$(t,r)\in\Omega^1_{\epsilon,k}:=\big\{(t,r):\phi(\mu(c_0^*-c_N(t+t_k)^{-1}),r-\tilde{h}(t))\in[1-\delta_0,1)\big\},$$

we have

$$-\phi_r\tilde{J}+f(\phi)-f(\phi+\epsilon e^{-\alpha t})-\epsilon\alpha e^{-\alpha t}\geq(\sigma_0-\alpha)\epsilon e^{-\alpha t}>0$$

provided that we take $\alpha=\sigma_0/2$.

For $\phi\in(0,1-\delta_0)$, there exists $\sigma_1>0$ such that $\phi_r\leq-\sigma_1$; moreover, for all small ϵ,

$$f(\phi)-f(\phi+\epsilon e^{-\alpha t})\geq-\sigma_2\epsilon e^{-\alpha t},$$

where $\sigma_2=\max_{u\in[0,1]}|f'(u)|$. Therefore for all large k, small ϵ, and

$$(t,r)\in\Omega^2_{\epsilon,k}:=\big\{(t,r):\phi(\mu(c_0^*-c_N(t+t_k)^{-1}),r-\tilde{h}(t))\in(0,1-\delta_0)\big\},$$

we have

$$\begin{aligned}
&-\phi_r\tilde{J}+f(\phi)-f(\phi+\epsilon e^{-\alpha t})-\epsilon\alpha e^{-\alpha t}\\
&\geq\sigma_1B\epsilon\alpha e^{-\alpha t}-(\sigma_2+\alpha)\epsilon e^{-\alpha t}\\
&=(\sigma_1B\alpha-\sigma_2-\alpha)\epsilon e^{-\alpha t}>0
\end{aligned}$$

provided that $\sigma_1 B\alpha > \sigma_2 + \alpha$. With $\alpha = \sigma_0/2$, this is achieved by taking $B \geq \frac{4\sigma_2+2\sigma_0}{\sigma_1\sigma_0}$. This proves that (4.18) holds for all large k and small ϵ.

We show below that for all large k and small ϵ,

$$h(t_k) < \overline{h}_k(0),\ u(t_k, r) \leq \overline{u}_k(0, r) \text{ for } r \in [0, h(t_k)]. \tag{4.19}$$

Since

$$h(t_k) - \tilde{h}_k(0) = \xi(t_k) - \hat{h} - \epsilon \to -\epsilon \text{ as } k \to \infty,$$

we have, in view of (4.16),

$$h(t_k) < \tilde{h}_k(0) < \overline{h}_k(0)$$

for all large k, say $k \geq k_1(\epsilon)$, and all small ϵ.

By Lemma 4.7,

$$\lim_{k\to\infty} \|u(t_k, \cdot) - \phi(\mu_0, \cdot - h(t_k))\|_{L^\infty([0,h(t_k)])} = 0.$$

Since

$$\mu(c_0^* - c_N t_k^{-1}) \to \mu_0,\ h(t_k) - \tilde{h}_k(0) + \epsilon \to 0 \text{ as } k \to \infty,$$

we deduce

$$\|u(t_k, \cdot) - \phi(\mu(c_0^* - c_N t_k^{-1}), \cdot - \tilde{h}_k(0) + \epsilon)\|_{L^\infty([0,h(t_k)])} \to 0 \text{ as } k \to \infty.$$

Therefore there exists $k_2(\epsilon) \geq k_1(\epsilon)$ such that for $k \geq k_2(\epsilon)$,

$$\begin{aligned} u(t_k, r) &\leq \phi(\mu(c_0^* - c_N t_k^{-1}), r - \tilde{h}_k(0) + \epsilon) + \epsilon \\ &< \phi(\mu(c_0^* - c_N t_k^{-1}), r - \tilde{h}_k(0)) + \epsilon = \overline{u}_k(0, r) \ \ (\forall r \in [0, h(t_k)]). \end{aligned}$$

Thus (4.19) holds for all small ϵ and $k \geq k_2(\epsilon)$. By enlarging $k_2(\epsilon)$ if necessary we may assume that (4.17) and (4.18) both hold for $k \geq k_2(\epsilon)$ and all small $\epsilon > 0$.

In view of (4.17), (4.18), (4.19) and the fact that $\overline{u}_r(t, 0) < 0,\ u_r(t, 0) = 0$, we can use a standard comparison argument to conclude that

$$h(t + t_k) \leq \overline{h}(t),\ u(t_k + t, r) \leq \overline{u}(t, r) \ \ (\forall t > 0,\ \forall r \in [0, h(t_k + t)])$$

for all small $\epsilon > 0$ and $k \geq k_2(\epsilon)$. It follows that

$$\begin{aligned} \xi(t + t_k) &= h(t + t_k) - \tilde{h}(t) + B\epsilon(1 - e^{-\alpha t}) + \hat{h} + \epsilon \\ &= h(t + t_k) - \overline{h}(t) - \left[\frac{\mu_0}{c_0^*} + o_{\epsilon,k}(1)\right] \epsilon e^{-\alpha t} + B\epsilon(1 - e^{-\alpha t}) + \hat{h} + \epsilon \\ &\leq -\left[\frac{\mu_0}{c_0^*} + o_{\epsilon,k}(1)\right] \epsilon e^{-\alpha t} + B\epsilon(1 - e^{-\alpha t}) + \hat{h} + \epsilon \\ &\to \hat{h} + (B + 1)\epsilon \ \text{ as } t \to \infty. \end{aligned}$$

Therefore

$$\limsup_{t\to\infty} \xi(t) \leq \hat{h} + (B + 1)\epsilon,$$

as we wanted. This completes the proof. □

5. Appendix: Further details for the proof of Lemma 4.6

For completeness, we give the detailed proof of the facts that for any given $\epsilon > 0$, there exists $n_1 = n_1(\epsilon)$ and $n_2 = n_2(\epsilon)$ such that

$$G(t) - R^* \geq -\epsilon \; (\forall t \geq s_{n_1}), \; G(t) - R_* \leq \epsilon \; (\forall t \geq \tilde{s}_{n_2}).$$

From the inequalities

$$\phi(r - R_*) \leq V(t,r) \leq \phi(r - R^*)$$

we have

$$|1 - V(t,r)| \leq Ce^{\beta r}$$

for some $C > 0$ and β >0. Therefore, for any $\varepsilon > 0$, there exists $K > 0$ and $T > 0$ such that

$$\sup_{r\in(-\infty,-K]} |V(\tilde{s}_n, r) - \phi(r - R_*)| < \varepsilon. \tag{5.1}$$

for $\tilde{s}_n > T$. Let $H(t) = G(t) + c_0^* t$, $W(t,r) = V(t, r - c_0^* t)$. (W,H) satisfies

$$\begin{cases} W_t - W_{rr} = f(W), t \in \mathbb{R}^1, r \leq H(t) \\ W(t, H(t)) = 0, H'(t) = -\mu_0 W_r(t, H(t)) \end{cases} \tag{5.2}$$

By Lemma 4.5 and (5.1), there exists $n_1 = n_1(\varepsilon)$ such that, for $n \geq n_1$,

$$G(\tilde{s}_n) \leq R_* + \varepsilon \tag{5.3}$$

$$V(\tilde{s}_n, r) \leq \phi(r - R_* - \varepsilon) + \varepsilon \quad \text{for} \quad r \leq R_*. \tag{5.4}$$

We note that we can find $N > 1$ independent of $\varepsilon > 0$ such that

$$\phi(r - R_* - \varepsilon) + \varepsilon \leq (1 + N\varepsilon)\phi(r - R_* - N\varepsilon) \quad \text{for} \quad r \leq R_* + \varepsilon. \tag{5.5}$$

Next we remark that for any $\delta \in (0, -f'(1))$ there exists $\eta > 0$ such that

$$\begin{cases} \delta \leq -f'(u) & \text{for} \quad 1 - \eta \leq u \leq 1 + \eta, \\ f(u) \geq 0 & \text{for} \quad 1 - \eta \leq u \leq 1. \end{cases}$$

Let us define an upper solution for problem (5.2) as follows:

$$\overline{H}(t) := R_* + N\varepsilon + c_0^* t + N\varepsilon\sigma(1 - e^{-\delta(t-\tilde{s}_n)})$$
$$\overline{W}(t,r) := (1 + N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi(r - \overline{H}(t))$$

Since $\lim_{r\to-\infty} \overline{W}(t,r) > 1$, there exists a smooth function $\overline{K}(t)$ of $t \geq \tilde{s}_n$ such that $\overline{K}(t) \to -\infty$ as $t \to \infty$ and $\overline{W}(t, \overline{K}(t)) > 1$. We will check that the triple $(\overline{W}, \overline{H}, \overline{K})$ is an upper solution for $t \geq \tilde{s}_n$, that is,

$$\overline{W}_t - \overline{W}_{rr} \geq f(\overline{W}) \text{ for } t > \tilde{s}_n, r \in [\overline{K}(t), \overline{H}(t)] \tag{5.6}$$

$$\overline{W}(t, \overline{K}(t)) \geq W(t, \overline{K}(t)) \text{ for } t \geq \tilde{s}_n, \tag{5.7}$$

$$\overline{W}(t, \overline{H}(t)) = 0, \overline{H}'(t) \geq -\mu_0 \overline{W}_r(t, \overline{H}(t)) \quad \text{for } t \geq \tilde{s}_n, \tag{5.8}$$

$$H(\tilde{s}_n) \leq \overline{H}(\tilde{s}_n), \; W(\tilde{s}_n, r) \leq \overline{W}(\tilde{s}_n, r) \quad \text{for} \quad r \in [\overline{K}(\tilde{s}_n), H(\tilde{s}_n)]. \tag{5.9}$$

From (5.3) we have

$$H(\tilde{s}_n) = G(\tilde{s}_n) + c_0^* \tilde{s}_n \leq R_* + N\varepsilon + c_0^* \tilde{s}_n = \overline{H}(\tilde{s}_n).$$

We also have, in view of (5.4),

$$\begin{aligned}\overline{W}(\tilde{s}_n, r) &= (1+N\varepsilon)\phi(r-\overline{H}(\tilde{s}_n))\\ &= (1+N\varepsilon)\phi(r-R_*-N\varepsilon-c_0^*\tilde{s}_n)\\ &\geq \phi(r-R_*-\varepsilon-c_0^*\tilde{s}_n)+\varepsilon\\ &\geq V(\tilde{s}_n, r-c_0^*\tilde{s}_n)=W(\tilde{s}_n, r)\end{aligned}$$

for $r \leq H(\tilde{s}_n)$. Thus (5.9) holds.

We next show (5.8). By definition $\overline{W}(t,\overline{H}(t))=0$ and direct calculation gives

$$\begin{aligned}&\overline{H}'(t)=c_0^*+N\varepsilon\sigma\delta e^{-\delta(t-\tilde{s}_n)},\\ &-\mu_0\overline{W}_r(t,\overline{H}(t))=c_0^*+N\varepsilon c_0^* e^{-\delta(t-\tilde{s}_n)}.\end{aligned}$$

Hence if we take $\sigma>0$ so that $c_0^* \leq \sigma\delta$ then

$$\overline{H}'(t) \geq -\mu_0\overline{W}_r(t,\overline{H}(t)).$$

This proves (5.8).

Since $W \leq 1$, by the definition of $\overline{K}(t)$, (5.7) clearly holds. Finally we show (5.6). Put $z=r-\overline{H}(t)$. Since

$$\begin{aligned}\overline{W}_t &= -\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)-(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\overline{H}'(t)\phi'(z)\\ &= -\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)-(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})(c_0^*+\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)})\phi'(z),\end{aligned}$$

and

$$\overline{W}_{rr}=(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi''(z),$$

we have

$$\begin{aligned}&\overline{W}_t-\overline{W}_{rr}-f(\overline{W})\\ =&-\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)-(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})(c_0^*+\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)})\phi'(z)\\ &\quad-(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi''(z)-f((1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi(z))\\ =&-\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)+(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\{-\phi''(z)-c_0^*\phi'(z)\}\\ &\quad-\sigma N\varepsilon\delta(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})e^{-\delta(t-\tilde{s}_n)}\phi'(z)\\ =&-\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)-\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z)\\ &\quad+(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})f(\phi(z))-f((1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi(z)).\end{aligned}$$

Now we consider the term $(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})f(\phi(z))-f((1+N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z))$. Denote

$$F(\xi,u):=(1+\xi)f(u)-f((1+\xi)u).$$

The mean value theorem yields

$$F(\xi,u)=\xi f(u)+f(u)-f((1+\xi)u)=\xi f(u)-\xi f'(u+\theta_{\xi,u}\xi u)u$$

for some $\theta_{\xi,u}\in(0,1)$. Since $\phi(z)\to 1$ as $z\to-\infty$, there exists $z_\eta<0$ such that $\phi(z)\geq 1-\eta$ for $z\leq z_\eta$.

For $r-\overline{H}(t)\le z_\eta$, we have

$$\begin{aligned}
&\overline{W}_t-\overline{W}_{rr}-f(\overline{W})\\
=&-\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)-\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z)+F(N\varepsilon e^{-\delta(t-\tilde{s}_n)},\phi(z))\\
=&-\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z)+N\varepsilon e^{-\delta(t-\tilde{s}_n)}f(\phi(z))\\
&+N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)\Big\{-f'\big(\phi(z)+\theta' N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)\big)-\delta\Big\}\\
\ge&\ 0,
\end{aligned}$$

where $\theta'=\theta'(t,z)\in(0,1)$. We note that by shrinking ε we can guarantee that $N\varepsilon<\eta$ and so $1+N\varepsilon e^{-\delta(t-\tilde{s}_n)}\le 1+\eta$ for $t\ge\tilde{s}_n$.

On the other hand for $z_\eta\le r-\overline{H}(t)\le 0$, we obtain

$$\begin{aligned}
&\overline{W}_t-\overline{W}_{rr}-f(\overline{W})\\
&=N\varepsilon e^{-\delta(t-\tilde{s}_n)}f(\phi(z))-\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z)\\
&\quad+N\varepsilon e^{-\delta(t-\tilde{s}_n)}\left\{-f'\big(\phi(z)+\theta' N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)\big)-\delta\right\}\phi(z)\\
&\ge N\varepsilon e^{-\delta(t-\tilde{s}_n)}\min_{0\le s\le 1}f(s)+\sigma\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}Q_\eta-N\varepsilon e^{-\delta(t-\tilde{s}_n)}\left(\max_{0\le s\le 1+\eta}f'(s)+\delta\right)\\
&=N\varepsilon e^{-\delta(t-\tilde{s}_n)}\left\{\min_{0\le s\le 1}f(s)-\max_{0\le s\le 1+\eta}f'(s)-\delta+\sigma\delta Q_\eta\right\}\\
&\ge 0,
\end{aligned}$$

where $Q_\eta:=\min_{z_\eta\le z\le 0}|\phi'(z)|>0$ provided that σ is large positive. Thus $\overline{W}_t-\overline{W}_{rr}-f(\overline{W})\ge 0$ for sufficiently large $\sigma>0$.

We may now apply the comparison principle to conclude that

$$W(t,r)\le\overline{W}(t,r),\quad H(t)\le\overline{H}(t)\ \text{ for } t\ge\tilde{s}_n \text{ and } r\in(\overline{K}(t),H(t)],$$

in particular

$$G(t)\le R_*+N\varepsilon(\sigma+1)$$

for $t\ge\tilde{s}_n$. By shrinking ε we obtain

$$G(t)\le R_*+\epsilon$$

for $t\ge\tilde{s}_n$ and $n\ge n_1$.

Next we show $G(t)\ge R^*-\epsilon$ for all large $t>0$. As in the construction of upper solution, for any $\varepsilon>0$, there exists $n_2=n_2(\varepsilon)$ such that, for $n\ge n_2$,

$$R^*-\varepsilon\le G(s_n), \tag{5.10}$$

$$\phi(r-R^*+\varepsilon)-\varepsilon\le V(s_n,r)\ \text{ for }\ r\le R^*-\varepsilon. \tag{5.11}$$

We note that we can find $N>1$ which does not depend on $\varepsilon>0$ such that

$$(1-N\varepsilon)\phi(r-R^*+N\varepsilon)\le\phi(r-R^*+\varepsilon)-\varepsilon\ \text{ for }\ r\le R^*-\varepsilon.$$

Now we define a lower solution as follows:

$$\begin{aligned}
&\underline{H}(t):=R^*-N\varepsilon+c_0^*t-N\varepsilon\sigma(1-e^{-\delta(t-s_n)}),\\
&\underline{W}(t,r):=(1-N\varepsilon e^{-\delta(t-s_n)})\phi(r-\underline{H}(t)).
\end{aligned}$$

Since $V(t,r) \geq \phi(r - R_*)$, there exists $C > 0$ and $\beta > 0$ such that V satisfies $V(t,r) \geq 1 - Ce^{\beta r}$ for all $r \leq 0$, that is, W satisfies

$$W(t,r) \geq 1 - Ce^{\beta(r - c_0^* t)}.$$

We fix $c > 0$ so that $\delta \leq \beta(c + c_0^*)$. By enlarging n we may assume that $C \leq N\varepsilon e^{\delta s_n}$. Let $\underline{K}(t) \equiv -ct$. We will check that the triple $(\underline{W}, \underline{H}, \underline{K})$ is a lower solution for $t \geq s_n$, that is,

$$\underline{W}_t - \underline{W}_{rr} \leq f(\underline{W}) \text{ for } t > s_n, r \in [\underline{K}(t), \underline{H}(t)] \tag{5.12}$$

$$\underline{W}(t, \underline{K}(t)) \leq W(t, \underline{K}(t)) \text{ for } t \geq s_n, \tag{5.13}$$

$$\underline{W}(t, \underline{H}(t)) = 0, \underline{H}'(t) \geq -\mu_0 \underline{W}_r(t, \underline{H}(t)) \quad \text{for } t \geq s_n, \tag{5.14}$$

$$\underline{H}(s_n) \leq H(s_n),\ W(s_n, r) \leq \underline{W}(s_n, r) \quad \text{for} \quad r \in [\underline{K}(s_n), H(s_n)]. \tag{5.15}$$

From (5.10) we have

$$\underline{H}(s_n) = R^* - N\varepsilon + c_0^* s_n \leq R^* - \varepsilon + c_0^* s_n \leq G(s_n) + c_0^* s_n = H(s_n)$$

We also have

$$\begin{aligned}
\underline{W}(s_n, r) &= (1 - N\varepsilon)\phi(r - \underline{H}(s_n)) \\
&= (1 - N\varepsilon)\phi(r - R^* + N\varepsilon - c_0^* s_n) \\
&\leq \phi(r - R^* + \varepsilon - c_0^* \tilde{s}_n) - \varepsilon \\
&\leq V(s_n, r - c_0^* s_n) = W(s_n, r)
\end{aligned}$$

for $r \leq \underline{H}(s_n)$. Hence (5.15) holds.

We next show (5.14). By definition $\underline{W}(t, \underline{H}(t)) = 0$, and direct calculation gives

$$\begin{aligned}
&\underline{H}'(t) = c_0^* - N\varepsilon\sigma\delta e^{-\delta(t - s_n)}, \\
&-\mu_0 \underline{W}_r(t, \underline{H}(t)) = c_0^* - N\varepsilon c_0^* e^{-\delta(t - s_n)}.
\end{aligned}$$

Hence if we take $\sigma > 0$ so that $c_0^* \leq \sigma\delta$ then

$$\underline{H}'(t) \leq -\mu_0 \underline{W}_r(t, \underline{H}(t)).$$

This proves (5.14).

For $t \geq s_n$, we have

$$\begin{aligned}
\underline{W}(t, \underline{K}(t)) = \underline{W}(t, -ct) &\leq (1 - N\varepsilon e^{-\delta(t - s_n)}) \\
&= 1 - N\varepsilon e^{\delta s_n} e^{-\delta t} \leq 1 - Ce^{-\delta t} \\
&\leq 1 - Ce^{-\beta(c + c_0^*)t} \leq W(t, -ct) = W(t, \underline{K}(t)).
\end{aligned}$$

Hence (5.13) holds.

Finally we show (5.12). Put $\zeta = r - \underline{H}(t)$. Since

$$\begin{aligned}
\underline{W}_t &= \delta N\varepsilon e^{-\delta(t - s_n)}\phi(z) - (1 - N\varepsilon e^{-\delta(t - s_n)})\underline{H}'(t)\phi'(\zeta) \\
&= \delta N\varepsilon e^{-\delta(t - s_n)}\phi(z) - (1 - N\varepsilon e^{-\delta(t - s_n)})(c_0^* - \sigma N\varepsilon\delta e^{-\delta(t - s_n)})\phi'(\zeta),
\end{aligned}$$

and

$$\underline{W}_{rr} = (1 - N\varepsilon e^{-\delta(t - s_n)})\phi''(\zeta),$$

we have

$$
\begin{aligned}
&\underline{W}_t - \underline{W}_{rr} - f(\underline{W})\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) - (1 - N\varepsilon e^{-\delta(t-s_n)})(c_0^* - \sigma N\varepsilon\delta e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\qquad - (1 - N\varepsilon e^{-\delta(t-s_n)})\phi''(\zeta) - f((1 - N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta))\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) + (1 - N\varepsilon e^{-\delta(t-s_n)})\{-\phi''(z) - c_0^*\phi'(\zeta)\}\\
&\quad + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\quad + (1 - N\varepsilon e^{-\delta(t-s_n)})f(\phi(\zeta)) - f((1 - N\varepsilon e^{-\delta(t-s_n)})\phi(\zeta))\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(z) + F(-N\varepsilon e^{-\delta(t-s_n)}, \phi(\zeta)).
\end{aligned}
$$

Since $\phi(\zeta) \to 1$ as $\zeta \to -\infty$, there exists $\zeta_\eta < 0$ such that $\phi(\zeta) \geq 1-\eta/2$ for $\zeta \leq \zeta_\eta$. For $r-\underline{H}(t) \leq \zeta_\eta$, we have

$$
\begin{aligned}
&\underline{W}_t - \underline{W}_{rr} - f(\underline{W})\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(z) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)}))\phi'(\zeta)\\
&\qquad - N\varepsilon e^{-\delta(t-s_n)}\Big\{f(\phi(\zeta)) - f'(\phi(\zeta) - \theta'' N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta))\phi(\zeta)\Big\}\\
&= - N\varepsilon e^{-\delta(t-s_n)}f(\phi(\zeta)) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\qquad + N\varepsilon e^{-\delta(t-s_n)}\Big\{f'(\phi(\zeta) - \theta'' N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta)) + \delta\Big\}\phi(\zeta)\\
&\leq 0,
\end{aligned}
$$

where $\theta'' = \theta''(t,z) \in (0,1)$. We note that by shrinking ε we can guarantee that $N\varepsilon < \eta/2$ and so

$$
1 \geq \phi(\zeta) - \theta'' N\varepsilon^{-\delta(t-s_n)}\phi(\zeta) \geq \phi(\zeta) - N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) \geq 1-\eta.
$$

On the other hand for $z_\eta \leq r - \overline{H}(t) \leq 0$ and $t \geq s_n$, we obtain

$$
\begin{aligned}
&\underline{W}_t - \underline{W}_{rr} - f(\underline{W})\\
&= - N\varepsilon e^{-\delta(t-s_n)}f(\phi(\zeta)) + \sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1 - N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(\zeta)\\
&\qquad + N\varepsilon e^{-\delta(t-s_n)}\left\{f'\big(\phi(\zeta) - \theta'' N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta)\big) + \delta\right\}\phi(\zeta)\\
&\leq - N\varepsilon e^{-\delta(t-s_n)}\min_{0\leq s\leq 1} f(s) + \sigma\delta N\varepsilon e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\qquad + N\varepsilon e^{-\delta(t-\tilde{s}_n)}\left(\max_{0\leq s\leq 1+\eta} f'(s) + \delta\right)\\
&\leq N\varepsilon e^{-\delta(t-s_n)}\left\{-\min_{0\leq s\leq 1} f(s) + \max_{0\leq s\leq 1+\eta} f'(s) + \delta - \sigma\delta\left(1-\frac{\eta}{2}\right)Q'_\eta\right\}\\
&\leq 0,
\end{aligned}
$$

by taking $\sigma > 0$ sufficiently large, where $Q'_\eta := \min_{\zeta_\eta\leq\zeta\leq 0}|\phi'(\zeta)| > 0$.

We may now apply the comparison principle to conclude that

$$
\underline{W}(t,r) \leq W(t,r), \quad \underline{H}(t) \leq H(t) \quad \text{for } t \geq s_n \text{ and } r \in (-ct, \underline{H}(t)],
$$

and in particular,

$$
R^* - N\varepsilon(\sigma+1) \leq G(t)
$$

for $t \geq s_n$. By shrinking ε we obtain

$$R^* - \epsilon \leq G(t)$$

for $t \geq s_n$ and $n \geq n_2$. □

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