# GENERAL INFORMATION

## Introduction

Welcome to MATH102. This booklet contains the important details of unit structure and administration for the unit MATH102 *Integral Calculus, Differential Equations and Introductory Statistics* as well as Assignment problems and Tuturial problems for Integral calculus and Differential equations with solutions.

## Enquiries

### Administrative and General Enquiries.

For all administrative matters and information in relation to your enrolment and candidature please write to, or see, the UNE Student Centre:

Student Centre
University of New England
Armidale NSW 2351
Phone: (02) 6773 4444
Fax: (02) 6773 4400
email: studentcentre@une.edu.au

### Academic Enquiries.

For academic enquiries concerning the content and assessment requirements of this unit, contact the unit coordinator:

Dr Gerd Schmalz
School of Mathematics, Statistics and Computer Science
University of New England
Armidale NSW 2351

Room B167, Booth Block

Phone: (02) 6773 3182

Fax: (02) 6773 3312

email: gerd@turing.une.edu.au (preferred!)

## Prerequisites

The prerequisites for doing this unit is a course on differential calculus, preferably MATH101. You should also have had some prior exposure to integral calculus such as that obtained at high school. If you have not seen integral calculus before you may still take this unit, but you can expect a heavy workload in the calculus section in order to be successful. Both the printed notes and reference book treat the subject from the beginning, assuming a working knowledge of differential calculus.

No prior knowledge of statistics is assumed for this unit.

What are the options if you took MATH101 in first semester and did not pass? There are several:

(a) Continue on with MATH102, taking into account the above remarks.

(b) Re-enrol in MATH101. This unit is available in external mode in second semester. You may enrol in external mode even if you are an internal student.

(c) Enrol in MATH123 Foundation Mathematics which is available in summer semester in external mode. It is also available in first semester each year.

The decision is yours and you should contact the Student Center if you wish to change your enrolment.

## Topics

### Integral Calculus:

Measuring and Integration, the definite integral, logarithms, exponentials and inverse trigonometric functions, techniques of integration, applications of the definite integral, curves defined parametrically, MacLaurin and Taylor expansions, power series, improper integrals, Lebesgue and Stieltjes integral.

**Differential Equations:**

separable equations, linear first order equations, linear second order differential equations with constant coefficients, applications of differential equations.

**Introductory statistics:**

Probability, Random variables, normal distribution, Poisson and Gamma distribution.

## Reference Book

The recommended reference book for the Integral Calculus and Differential Equations component is *Calculus* by Howard Anton, Wiley. Please note that the 7th and 8th editions have Bivens and Davis as co-authors with Anton. The new 8th edition exists in two versions: Early transcendentals (preferred version) and Late transcendentals. It may be purchased through the United Campus Bookshop: phone: (02) 6772 3468, fax: (02) 6772 3469. There are also second hand copies of the 5th or 6th edition around as this book has been prescribed as a text in previous years and is also used in the follow up unit PMTH212. The text is extremely thorough and is designed for students taking calculus for the first time. If it is some time since you have done integral calculus, it is recommended that you spend some time studying the material on antiderivatives in 7.1, 7.2, 7.3, 7.4, 7.5 of Anton 6th edn. (6.1, 6.2, 6.3, 6.4, 6.5 of Anton 7th or 8th edn.) before beginning on the notes.

## Web Page

The course material and a Bulletin Board for questions and discussion are available at http://mcs.une.edu.au/∼math102/. Actual unit information will be disseminated by email and through that website. Therefore it is recommended that you submit an email address that you check regularly and that you visit the unit web-site regularly.

## Assessment

The assessment will be based on your performance in the examination (70%) and assignments (30%).

In order to pass the unit you must achieve a minimum of 40% on the examination and a final overall scaled mark of at least 50. In order to obtain a credit in the unit you must achieve a minimum of 60% on the examination and a final overall scaled mark of at least 65. In order to obtain a distinction in the unit you must obtain a final overall scaled mark of at least 75. In order to obtain a high distinction in the unit you must obtain a final overall scaled mark of at least 85.

## Examination

There will be a three-hour examination in the November examination period. The examination will be worth 70% of the assessment marks. The whole semester's work is examinable. Copies of some past papers are available from the library.

The examination will consist, for the most part, of problems analogous to those in the assignments and lecture notes. It is essential that you work through all these exercises.

Pocket calculators will be allowed in the examination. You will be allowed to take 5 pages of handwritten notes into the exam. No printed material or photocopies.

## Assignments

There are eleven assignments for the unit, and you are expected to hand in solutions to each assignment. Sample solutions will be returned with your assignment. It is important that you hand in the assignments at the nominated times, or at least make contact with us to let us know if there is some problem.

The assignments contribute 30% towards the assessment.

## Plagiarism

Students are warned to read the statement in the Faculty's Undergraduate and Postgraduate Handbooks regarding the University's Policy on Plagiarism. Full details of the Policy on Plagiarism are available in the UNE Handbook and at the following web site:

http://www.une.edu.au/rmo/policies/polACAD.html

## Library Services for External Students

The UNE Libraries have an extensive collection of books, journal articles and online resources. You can borrow books, ask for photocopies of articles and exam papers, or request librarians to perform a subject search on your behalf. All your questions about these library services are answered at:
http://www.une.edu.au/library/external/index.htm.

### Making requests

There are a number of ways for your to tell UNE Libraries what you want:

- Telephone the External Students Library Helpline on 02 6773 3124
- Fax request forms to 02 6773 3273
- Use the online forms available from External Students page at <http://www.une.edu.au/library/external/index.htm>
- Email the message "OCS email forms please" (no subject) to offcamp@pobox.une.edu.au to receive email request forms. This is the easiest way to make multiple requests.
- Mail your requests to:

  Document Services
  Dixson Library
  PO Box U246
  UNE, NSW, 2351

There is no charge for the loan of items, although you are responsible for return postage. Photocopy requests provided electronically or posted to you are free for reasonable quantities. Subject searches and sets of examination papers are free.

**Online resources**

There are many online resources available to staff and students from UNE Libraries page at <http://www.une.edu.au/library>. They include the:

- **catalogue**. This shows the books, journals, Reserve Room items, audiovisual materials and other resources held in the UNE Libraries collection. In particular, it includes journal titles you can access electronically.

- **e-resources**. These valuable resources include journal indexes, links to useful websites, subject guides and web search tools. They are listed by School to make it easy for you to investigate what's most relevant to your discipline. *Journal indexes* enables you to search by topic for references to articles. A number of UNE Libraries' indexes include full-text, so you can read or print the articles straight away. The *web search tools* provide links to the most popular search engines.

- **eSKILLS UNE**. This is a series of lessons showing you how to find and evaluate information, write essays and reference assignments correctly.

Access to some e-resources is restricted to UNE staff and students, so you will need to register online for a UNE username and password. Further advise is available at <http://www.une.edu.au/library/external/elecres> .

**Borrowing from other University libraries**

UNE students can apply for reciprocal (in person) borrowing rights at all other Australian University libraries. There is usually a small fee for this service. For full details on how to apply in the various states of Australia, go to <http://www.une.edu.au/library/external/unison.htm>.

**More information**

For additional information, telephone the External Students Library helpline on 02 6773 3124 or view the External page at <http://www.une.edu.au/library/external/index.htm>.

## Assignment Schedule

| Assignment | Latest date to be submitted/ posted by |
|---|---|
| 1 | 6 August |
| 2 | 13 August |
| 3 | 20 August |
| 4 | 27 August |
| 5 | 3 September |
| 6 | 10 September |
| 7 | 2 October |
| 8 | 10 October |
| 9 | 17 October |
| 10 | 22 October |
| 11 | 29 October |

**REQUESTS FOR AN EXTENSION OF TIME FOR SUBMITTING AN ASSIGNMENT ARE TO BE ADDRESSED TO THE UNIT COORDINATOR.**

## Timetable

There are three lectures per week on these topics throughout the semester.

The topics below correspond to the Chapter headings in the printed notes for Integral Calculus and Differential Equations. Relevant reading material from Anton is also shown.

| Week | Dates | Topics as listed in printed notes | References to Anton (7,8th edn) | References to Anton (6th edn) |
|---|---|---|---|---|
| 1 | 23 July to 27 July | 1. Counting and Measuring<br>2. Probability | 6.1, 6.2,<br>- | 7.1, 7.2<br>- |

**Material for Assignment 1 has been completed**

| Week | Dates | Topics as listed in printed notes | References to Anton (7,8th edn) | References to Anton (6th edn) |
|---|---|---|---|---|
| 2 | 30 July to 3 August | 3. The definite Integral<br>4. Properties of the Integral<br>5. The Integral as a Function of the Upper Limit of Integration<br>6. Primitives | 6.1, 6.2,<br>6.4 - 6.6 | 7.1, 7.2<br>7.4 - 7.6 |

**Material for Assignment 2 has been completed**

| | | | | |
|---|---|---|---|---|
| 3 | 6 August to 10 August | 6. The Substitution Rule<br>7. Applications of the Integral | 6.3, 6.8<br>7.1-7.4, 7.6 | 7.3, 7.8,<br>8.1-8.4, 8.6 |
| 4 | 13 August to 17 August | 8. The Logarithm<br>9. The Exponential and Power Functions<br>10. The Inverse Trigonometric Functions | 6.9<br>4.4 | 7.9<br>4.5 |

**Material for Assignment 3 has been completed**

| | | | | |
|---|---|---|---|---|
| 5 | 20 August to 24 August | 11. Integration by Parts<br>12. Further Integration | 8.2<br>8.1, 8.3-8.5 | 9.2<br>9.1, 9.3-9.5 |

**Material for Assignment 4,5 has been completed**

| | | | | |
|---|---|---|---|---|
| 6 | 27 August to 31 August | 13. Curves Defined Parametrically<br>14. Polynomial Approximation | 11.1- 11.4<br>10.8, 10.9 | 12.1- 12.4<br>11.5, 11.9 |

**Material for Assignment 6 has been completed**

| | | | | |
|---|---|---|---|---|
| 7 | 3 September to 7 September | 14. Polynomial Approximation (cont.)<br>15. Improper Integrals | 10.10<br>8.7, 8.8 | 10.11<br>9.7, 9.8 |

**Material for Assignment 7 has been completed**

| | | | | |
|---|---|---|---|---|
| 8 | 10 September to 14 September | 16. Lebesgue and Stieltjes Integrals<br>17. The Gamma function<br>18. Random variables | | |

**Material for Assignment 8 has been completed**

MID SEMESTER BREAK 16 September to 2 October

| | | | | |
|---|---|---|---|---|
| 9 | 2 October to 5 October | 19. Distributions | | |

**Material for Assignment 9 has been completed**

| | | | | |
|---|---|---|---|---|
| 10 | 8 October to 12 October | 18. First Order Differential Equations | 9.1, 9.3 | 10.1, 10.3 |

**Material for Assignment 10 has been completed**

| | | | | |
|---|---|---|---|---|
| 11 | 15 October to 19 October | 19. Second Order Differential Equations | 9.4 | 10.3 |
| 12 | 22 October to 26 October | 19. (continued) Second Order Differential Equations | 9.4 | |

**Material for Assignment 11 has been completed**

| | | | | |
|---|---|---|---|---|
| 13 | 29 October to 2 November | Revision | | |

| | |
|---|---|
| **Examination Period** | **7 November — 21 November** |

# ASSIGNMENT PROBLEMS

## ASSIGNMENT 1

**Question 1.** Using axioms (S0)-(S2) on compatibility of the measure with the set operations from the Lecture Notes show that $m(E) \leq m(F)$ for measurable sets $E$, $F$ with $E \subset F$. (Hint: Represent $F$ as a union $F = E \cup (F \setminus E)$.)

**Question 2** A guided missile has five distinct sections through which a signal must pass if the missile is to operate properly. Each of the individual sections has two circuits through which the signal may pass, at least one of which must function if the signal is to traverse that section. The probability that any single circuit will fail is 0.1.

(a) Calculate the probability that the signal passes through *any* section.

(b) Calculate the probability that a signal passes through *all* sections, thus allowing the missile to function.

**Question 3** Suppose a student who is about to take a multiple choice test has only learned 40% of the material covered by the exam. Thus, there is a 40% chance that she will know the answer to the question. However if she does not know the answer to a question, she still has a 20% chance of getting the right answer by guessing.

(a) If we choose a question at random from the exam, what is the probability that she will get it right?

(b) If we know that she correctly answered a question in the exam, what is the probability that she learned the material covered in the question?

# Assignment 2

**Question 1.** Evaluate $\sum_{k=1}^{6} f(x_k)$ where $x_k = \frac{k}{2}$ and $f(x) = \sin \pi x$.

**Question 2.** (a) Let $f(x) = 1 - x^2$, $0 \leq x \leq 1$. Find the smallest and the biggest Riemann sum for $f$ on $[0, 1]$ with partition $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$.

(b) This Riemann sum is a (rough) approximation to the integral $\int_0^1 (1 - x^2) dx$. What is the exact value of this integral?

**Question 3.** Find the following indefinite integrals

(a) $\int (4x^4 - 3x^2 + 5x)\, dx$

(b) $\int x \sqrt[3]{x}\, dx$

(c) $\int \dfrac{dx}{x\sqrt{x}}$

(d) $\int (1 - 3x)(1 + 3x)\, dx$.

**Question 4.** Differentiate $\sqrt{a^2 - x^2}$ with respect to $x$. Hence find

$$(a) \quad \int \frac{x\, dx}{\sqrt{a^2 - x^2}} \qquad \text{and} \qquad (b) \quad \int_0^2 \frac{x\, dx}{\sqrt{4 - x^2}}.$$

**Question 5.** Find the equation of the curve through $(4, 2)$ whose tangent (for $x > 0$) forms an angle with the $x$-axis, the tangent of which is equal to $\frac{1}{\sqrt{x}}$. Sketch the curve for $x > 0$.

**Question 6.** The velocity $v$ of a body moving in a straight line is given by the equation $v = 3t^2 + 2t$. Find the distance $s$ in terms of $t$ given that $s = 10$ when $t = 2$. (Remember that $v = \frac{ds}{dt}$.)

# ASSIGNMENT 3

**Question 1.** Evaluate

$$(a) \quad \int x\sqrt{3x+8}\,dx \qquad \text{by substituting} \quad u = 3x+8,$$

and

$$(b) \quad \int \frac{x\,dx}{a^2 - x^2} \qquad \text{by substituting} \quad u = a^2 - x^2.$$

**Question 2.** Evaluate the integrals

$$(a) \quad \int x^2\sqrt{x^3-1}\,dx$$

and

$$(b) \quad \int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx.$$

**Question 3.** Find the area of the region bounded by $y = x^3 - x$, the segment of the $x$-axis between $-1$ and $\frac{1}{2}$. Include a rough sketch of the region in your working.

**Question 4.** Find the length of the arc of the curve $y = x^{3/2}$ from the point $(1,1)$ to the point $(4,8)$.

**Question 5.** State the natural domains (values for which the formula makes sense) for each of the following functions:

(a) $\ln(\sin^2 x)$

(b) $\ln(\tan x)$

(c) $\ln\sqrt{x^2+2}$.

Find the derivative with respect to $x$ in each case.

**Question 6.** Assume that $\ln 10 \approx 2.3026$ to 4 places. Evaluate the following (without using the ln button on your calculator.)

$$(i)\quad \ln 100, \quad (ii)\quad \ln(0.1), \quad (iii)\quad \ln 1000, \quad (iv)\quad \ln(10\,\mathrm{e}).$$

Using the approximation $2^{10} = 1024 \approx 1000 = 10^3$, find $\ln 2$, approximately.

**Question 7.** Evaluate

$$\int_1^3 \left( \frac{1}{2x} - 2\,\mathrm{e}^{-3x} \right) dx.$$

**Question 8.** Evaluate

$$(a)\quad \int x\,\mathrm{e}^{-x^2}\,dx$$

$$(b)\quad \int \mathrm{e}^x \sqrt{\mathrm{e}^x + 1}\,dx.$$

**Question 9.** Evaluate

$$(a)\quad \int \frac{x}{1+x^2}\,dx$$

$$(b)\quad \int \tan x\,dx$$

$$(c)\quad \int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx.$$

# ASSIGNMENT 4

**Question 1.** Evaluate

$$(a) \quad \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$$
$$(b) \quad \int_0^{1} \frac{ds}{3+s^2}$$

**Question 2.** Evaluate

$$(a) \quad \int \frac{x^2\,dx}{1+x^6} \quad \text{by substituting } u = x^3$$
$$(b) \quad \int \frac{x\,dx}{\sqrt{1-x^4}} \quad \text{by a suitable substitution}$$
$$(c) \quad \int_0^{\pi/2} \frac{\sin\theta}{1+\sin^2\theta}\,d\theta.$$

**Question 3.**

$$(a) \quad \text{Find } \int x\sin x\,dx \quad \text{by integration by parts.}$$
$$(b) \quad \text{Find } \int x^2\cos x\,dx.$$

Hint: integrate by parts and use the result in (a).

**Question 4.** Find $\int \tan^{-1} x\,dx$.

Hint: Let $u = \tan^{-1} x$ and $\frac{dv}{dx} = 1$.

**Question 5.** Let $I_n(t) = \int_0^t x^n\,\mathrm{e}^{-x}\,dx$. Use integration by parts to show that

$$I_n(t) = -t^n\,\mathrm{e}^{-t} + nI_{n-1}(t).$$

Use the result to evaluate $\int_0^2 x^3\,\mathrm{e}^{-x}\,dx$.

**Question 6.** Find any local maximum or minimum value of $y = x^x$ $(x > 0)$. Find where the function is increasing and decreasing, and sketch the graph of the function. What happens near $x = 0$?

**Question 7.** The region $R$ is bounded by the curve $y = 2\,\mathrm{e}^{-x}$, the lines $x = 1, x = 2$, and the $x$-axis.

(a) Sketch the region $R$ and find its area.

(b) Find the volume of the region obtained by rotating $R$ around the $x$-axis.

(c) Find the volume of the region obtained by rotating $R$ around the $y$-axis.

# ASSIGNMENT 5

**Question 1.**

$$(a) \quad \int \frac{dx}{(x-1)(x-2)} \qquad (b) \quad \int \frac{dx}{x(x-1)(x-2)}.$$

**Question 2.** Express in partial fractions and integrate with respect to $x$:

$$(a) \quad \frac{4x+3}{(x+2)^2} \qquad (b) \quad \frac{x-3}{3x^2+2x-5}.$$

**Question 3.** Find

$$(a) \quad \int \frac{3+x^2}{1+x^2}\,dx \qquad (b) \quad \int \frac{dx}{x+x^3}.$$

**Question 4.** Integrate with respect to $x$:

$$(a) \quad \frac{1}{\sqrt{x^2+2x+26}} \qquad (b) \quad \frac{1}{\sqrt{-x^2-2x+24}} \qquad (c) \quad \frac{1}{\sqrt{x^2+2x+1}}.$$

(Hint for (a): You may quote the result $\int \frac{du}{\sqrt{1+u^2}} = \ln(u + \sqrt{1+u^2}) + C$ - see notes.)

**Question 5.** Evaluate $\int_0^1 \sin\sqrt{z}\, dz$. (Hint: Start off with a substitution $u = \sqrt{z}$.)

**Question 6.** Sketch the graph of $y = |x-2|$ and find $\int_0^3 |x-2|\, dx$.

**Question 7.** Classify the following functions as odd or even and in each case find $\int_{-1}^1 f(x)\, dx$.

$$(a) \quad f(x) = x^5 \cos x \qquad (b) \quad f(x) = x^6 \qquad (c) \quad f(x) = (1+x^4)\tan x.$$

# Assignment 6

**Question 1.** Sketch roughly the following curves, and put names to them:

$$(a) \quad x^2 + 2y = 4 \qquad (b) \quad x^2 + 2y^2 = 4 \qquad (c) \quad x + 2y^2 = 4.$$

**Question 2.** Find the length of the curve $x = at^2$, $y = at^3$ between the points $(0,0)$ and $(a,a)$.

**Question 3.**

(a) A curve is given in polar coordinates by $r = 2 - 2\sin\theta$. Sketch its graph in the $xy$-plane.

(b) Find the area enclosed by this curve.

**Question 4.**

(a) Express the equation

$$(x^2 + y^2)^2 + 2(x^2 + y^2) = 4x^2$$

in polar coordinates and then sketch the graph of the equation in the $xy$-plane.

(b) Find the area of one loop of this curve.

**Question 5.** Use the MacLaurin polynomial of degree three for $e^x$ to find $e^{-0.1}$ approximately.

**Question 6.** Find the MacLaurin series for $\cos x$.

**Question 7.** Use geometric series to

(a) Express the repeating decimal $0.5959\ldots$ as a rational number.

(b) Express $\dfrac{1}{1+x^4}$ as a series in $x$ valid for $|x| < 1$.

(c) Express $\displaystyle\int_0^x \frac{dx}{1+x^4}$ as a series in $x$ valid for $|x| < 1$.

**Question 8.** Write down the MacLaurin's series for $e^x$. Hence derive series expansions for

(a) $\dfrac{x}{e^x}$ (Hint: write it as $x\,e^{-x}$.)

(b) $e^{-x^2}$

(c) $\displaystyle\int e^{-x^2}\,dx$

# ASSIGNMENT 7

**Question 1.** Write down the binomial expansion for $(1+x)^k$. Hence derive series expansions for

(a) $\dfrac{1}{\sqrt{1-x^2}}$.

(b) $\sin^{-1} x = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$.

**Question 2.** Write out the series expansions for $\ln(1+x)$ and $\ln(1-x)$. Hence derive a series expansion for $\frac{1}{2}\ln\frac{1+x}{1-x}$. Using a suitable choice for $x$, use this result to evaluate $\ln 3$ to 3 decimal places.

**Question 3.** The Taylor expansion of a function $f$ about the point $x_0$ is (see notes):

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

Find the Taylor expansion of $f(x) = 3x^4 - x^3 + 10x - 7$ about $x_0 = 1$.

**Question 4.** Find, when they exist, the values of the following integrals:

$$(a)\quad \int_1^\infty \frac{dx}{x\sqrt{x}} \qquad (b)\quad \int_0^1 \frac{dx}{\sqrt{x}} \qquad (c)\quad \int_0^\infty \frac{dx}{9+x^2}.$$

**Question 5.** (a) Find a distribution function $F$ and its density $\rho$ for the function $y = f(x) = \sqrt[3]{x}$ defined on $[0,8]$.

(b) Compute $\int_0^2 y\rho(y)dy$ and $\int_0^8 f(x)\,dx$. Compare the results.

**Question 6.** Show that the Stieltjes integral $\int_{-1}^{1} x^3\, d|x|$ equals $\frac{1}{2}$.

**Question 7.** Show that

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}}\, \mathrm{e}^{-t}\, dt = \sqrt{\pi}.$$

Hint. Use the substitution $t = u^2$ and $\int_0^\infty \mathrm{e}^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$.

# ASSIGNMENT 8

**Question 1.** The following data give the number of man–hours ($Y$) required to complete a job as a function of units already completed ($X$).

| X | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Y | 40 | 34 | 30 | 24 | 22 | 20 |

(a) Plot the data.

(b) Calculate the sample covariance and the correlation.

(c) Are the two variables related? Give reasons!

**Question 2.** A discrete random variable $X$ has the following probability distribution :

$$P(X = x) = 1/3,\ x = 0, 1, 2.$$

Calculate

(a) $E(X)$

(b) $V(X)$

(c) $V(X^2)$

Show all working.

**Question 3** The random variable$X$ has density

$$f(x) = \frac{1}{2}e^{-x/2},\ x > 0$$

(a) Find a number $x_0$ such that

$$P(X > x_0) = 1/2$$

(b) Interpret your result.

# ASSIGNMENT 9

**Question 1.** The random variable $X$ follows the distribution given by the density

$$f(x) = cxe^{-x^2/2},\ x > 0$$

(a) Determine $c$.

(b) Find the mean of $X$.

**Question 2.** The random variable $X$ follows the Poisson distribution given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(a) Verify that the given function is a probability distribution function, by showing that

$$\sum_x P(X = x) = 1$$

(b) If $\lambda = 1$, verify that $E(X) = 1$.

(c) When $\lambda$ is very small, the term $P(X = 0)$ dominates all other terms, and the distribution is right skewed. Find the value of $\lambda$ for which

$$P(X = 0) > P(X = 1) + P(X = 2) + \ldots$$

**Question 3.**

The data in Table 1 shows the number of fires that occurred per day in 1979 in a town. The expected frequencies have been calculated on the basis of a Poisson distribution with mean 0.9, as calculated from the given data.

| Number of fires | Number of days | Expected |
|---:|---:|---:|
| 0 | 151 | 148.4 |
| 1 | 118 | 133.6 |
| 2 | 77 | 60.1 |
| 3 or more | 19 | 22.9 |
| total (days) | 365 | 365 |

Table 1: Observed and expected frequencies for fire data

(a) Verify the value for the mean.

(b) Verify the expected frequency for no fires per day.

# Assignment 10

**Question 1.** Solve (that is, find all solutions of) the differential equation

$$\frac{dy}{dx} = \frac{xy}{1+x^2}.$$

**Question 2.** Solve the initial value problem

$$y^2\frac{dy}{dx} = \cos 2x, \quad y(0) = 1.$$

**Question 3.** Solve

$$x\frac{dy}{dx} - 2y = x + 1 \text{ where } x > 0.$$

**Question 4.** Solve the initial value problem

$$\frac{dy}{dx} + y - x = 0, \quad y(0) = 1.$$

**Question 5.** Tests on a fossil show that 85% of its carbon-14 has decayed. Estimate the age of the fossil, assuming a half-life of 5750 years for carbon-14.

**Question 6.** The fish population in a lake is found to grow according to the logistic differential equation

$$\frac{dP}{dt} = 0.0001\,P\,(10000 - P), \quad (P > 0)$$

where $t$ is the time in years and $P = P(t)$ is the population at time $t$. Initially the population is 1000.

(a) Solve this differential equation, and express $P$ in terms of $t$.

(b) What is the population after 5 years?

(c) How long does it take for the population to reach 7500?

# ASSIGNMENT 11

Solve the following differential equations. Where applicable, find the solution which satisfies the given initial values.

**Question 1.**

$$2y'' + y' + y = 0.$$

**Question 2.**

$$y'' + 10y' + 25y = 0.$$

**Question 3.**

$$y'' + 2y' + 5y = 0, \text{ where } y(0) = 0, \quad y'(0) = 1.$$

**Question 4.**

$$y'' + 7y' - 8y = \mathrm{e}^x\,.$$

**Question 5.**

$$y'' - y' - 2y = \cos x - 5\sin x.$$

**Question 6.** When a mass $M$ is set vibrating along the $y$-axis on the end of a spring, the distance $y$ at time $t$ of the mass from the equilibrium position satisfies the equation

$$M\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

where $c$ is a constant depending on the frictional force, and $k$ is a constant depending on the spring and the force of gravity.

(a) Solve this equation for the case when $M = 4$, $c = 4$, and $k = 101$. Assume the initial conditions $y(0) = 10$, and $y'(0) = 0$.

(b) Put the solution in cosine form.

(c) What is the period of the oscillation? (This is the time taken for the bobbing mass to complete one cycle.)

(d) What is the frequency of the oscillation? (This is the number of oscillations per second.)

# Calculus Tutorials

## Tutorial 1

1. Using axioms (S0)-(S2) on compatibility of the measure with the set operations from the Lecture Notes show that

   (a) $m(A \cup B) = m(A) + m(B)$ if $A \cap B = \emptyset$.

   (b) $m(A \setminus B) = m(A) - m(B)$ if $B \subset A$.

2. Evaluate

$$\text{(a)} \quad \sum_{k=1}^{3} k^3 \qquad \text{(b)} \quad \sum_{j=2}^{6} (3j-1) \qquad \text{(c)} \quad \sum_{i=-4}^{1} (i^2 - i) \qquad \text{(d)} \quad \sum_{n=0}^{5} 1.$$

3. Express in sigma notation

$$\text{(a)} \quad 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 49 \cdot 50 \qquad \text{(b)} \quad 1 - 3 + 5 - 7 + 9 - 11.$$

4. Evaluate

$$\text{(a)} \quad \sum_{k=1}^{n} n \qquad \text{(b)} \quad \sum_{i=0}^{0} (-3) \qquad \text{(c)} \quad \sum_{k=1}^{n} kx \qquad \text{(d)} \quad \sum_{k=m}^{n} c \quad (n \geq m).$$

5. Express $\sum_{k=4}^{18} k(k-3)$ in sigma notation with

   (a) $k = 0$ as the lower limit of summation

(b) $k = 5$ as the lower limit of summation.

6. Evaluate the Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$, where

$$f(x) = 4 - x^2; \quad a = -3,\ b = 4; \quad n = 4;$$

$$\Delta x_1 = 1,\ \Delta x_2 = 2,\ \Delta x_3 = 1,\ \Delta x_4 = 3;$$
$$x_1^* = -\frac{5}{2},\ x_2^* = -1,\ x_3^* = \frac{1}{4},\ x_4^* = 3.$$

7. Find the smallest and largest values that the Riemann sum

$$\sum_{k=1}^{3} f(x_k^*)\Delta x_k$$

can have on the interval $[0, 4]$ if $f(x) = x^2 - 3x + 4$, and

$$\Delta x_1 = 1,\ \Delta x_2 = 2,\ \Delta x_3 = 1.$$

# Tutorial 2

1. Evaluate

$$\text{(a)}\ \int x^3\sqrt{x}\ dx \qquad \text{(b)}\ \int \sec x(\sec x + \tan x)\ dx.$$

Check your answer by differentiating.

2. Find an equation for the curve passing through the point $(-3, 0)$ whose tangent at each point $(x, y)$ has slope $2x + 1$.

3. Evaluate

$$\text{(a)}\ \int_4^9 2y\sqrt{y}\ dy \qquad \text{(b)}\ \int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right) dx \qquad \text{(c)}\ \int_0^2 |2x - 3|\ dx.$$

4. Find the total area between the curve $y = x^2 - 3x - 10$ and the interval $[-3, 8]$.

5. Find the average value of $\sin x$ over the interval $[0, \pi]$.

6. Let

$$F(x) = \int_1^x (t^3 + 1)\, dt.$$

(a) Use the Second Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check your answer to part (a) by integrating and then differentiating.

7. Prove that the function

$$F(x) = \int_0^x \frac{1}{1+t^2}\, dt + \int_0^{1/x} \frac{1}{1+t^2}\, dt$$

is constant on the interval $(0, \infty)$.

Hint. Differentiate with respect to $x$.

# Tutorial 3

1. Evaluate by making the indicated substitutions:

(a) $\displaystyle\int \cot x \csc^2 x\, dx, \quad u = \cot x;$

(b) $\displaystyle\int x^2 \sqrt{1+x}\, dx, \quad u = 1 + x.$

2. Evaluate $\displaystyle\int t\sqrt{7t^2 + 12}\, dt.$

3. Evaluate

$$\text{(a)} \quad \int_0^1 \frac{du}{\sqrt{3u+1}} \qquad \text{(b)} \quad \int_0^{\pi/2} \sin^2 3x \cos 3x\, dx.$$

4. For positive integers $m$ and $n$, show that

$$\int_0^1 x^m (1-x)^n\, dx = \int_0^1 x^n (1-x)^m\, dx.$$

[Hint: use a substitution.]

5. Sketch the region enclosed by the curves $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$ and find its area.

6. Find the volume of rotation about the $x$-axis of the region enclosed by the curves $y = x^2$, $x = 0$, $x = 2$, $y = 0$.

7. Find the volume of rotation about the $y$-axis of the region enclosed by the curves $y = x^2$, $x = 1$, $x = 2$, $y = 0$.

8. Find the arc length of the curve $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2}$ between $x = 2$ and $x = 3$.

# Tutorial 4

1. Simplify

$$\text{(a)} \quad \log_2 16 \qquad \text{(b)} \quad \log_2 \frac{1}{32} \qquad \text{(c)} \quad \log_4 4 \qquad \text{(d)} \quad \log_9 3.$$

2. Solve for $x$:

$$\text{(a)} \quad \ln \frac{1}{x} = -2 \qquad \text{(b)} \quad e^{-2x} - 3e^{-x} = -2.$$

3. Simplify and state the values of $x$ for which the simplification is valid:

$$\text{(a)} \quad e^{-\ln x} \qquad \text{(b)} \quad e^{\ln x^2} \qquad \text{(c)} \quad \ln e^{-x^2} \qquad \text{(d)} \quad \ln \frac{1}{e^x}.$$

4. Evaluate

$$\text{(a)} \quad \int \cot x \, dx \qquad \text{(b)} \quad \int \frac{dx}{x \ln x} \qquad \text{(c)} \quad \int_{-1}^{0} \frac{x \, dx}{x^2 + 5} \qquad \text{(d)} \quad \int_{1}^{4} \frac{dx}{\sqrt{x}(1 + \sqrt{x})}.$$

5. For $y = x^{(e^x)}$, find $\dfrac{dy}{dx}$ by logarithmic differentiation.

6. Evaluate

$$(a) \quad \int e^{\sin x} \cos x \; dx \qquad (b) \quad \int \pi^{\sin x} \cos x \; dx \qquad (c) \quad \int_{-\ln 3}^{\ln 3} \frac{e^x \; dx}{e^x + 4}.$$

# Tutorial 5

1. Find the exact value of

$$(a) \quad \sin^{-1} \frac{\sqrt{3}}{2} \qquad (b) \quad \cos^{-1} \frac{1}{2} \qquad (c) \quad \tan^{-1} 1.$$

2. Find $\frac{dy}{dx}$, where

$$(a) \quad y = \sin^{-1} \frac{1}{x} \qquad (b) \quad y = \cos^{-1} \cos x \qquad (c) \quad y = \tan^{-1} \frac{1-x}{1+x}.$$

3. Evaluate

$$(a) \int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}} \quad (b) \int \frac{dx}{1+16x^2} \quad (c) \int_1^3 \frac{dx}{\sqrt{x}(x+1)} \quad (d) \int \frac{dx}{x\sqrt{1-(\ln x)^2}}.$$

4. Use integration by parts to evaluate

$$(a) \quad \int x e^{-x} \; dx \qquad (b) \quad \int x^2 \cos x \; dx \qquad (c) \quad \int_1^e x^2 \ln x \; dx.$$

5. Use a reduction formula to evaluate

$$(a) \quad \int \sin^3 x \; dx \qquad (b) \quad \int_0^{\pi/4} \sin^4 x \; dx.$$

6. Suppose $f$ is a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^{1} x f''(x) \; dx = f'(1) + f'(-1) + f(-1) - f(1).$$

# Tutorial 6

1. Complete the square and evaluate

(a) $\displaystyle\int \frac{dx}{\sqrt{8+2x-x^2}}$

(b)* $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+10}}$.

2. Decompose into partial fractions and evaluate

(a) $\displaystyle\int \frac{dx}{x^2+3x-4}$

(b) $\displaystyle\int \frac{x^3\ dx}{x^2-3x+2}$

(c) $\displaystyle\int \frac{x^2\ dx}{(x+2)^3}$

(d) $\displaystyle\int \frac{dx}{x^4-16}$

(e) $\displaystyle\int \frac{x^4+6x^3+10x^2+x}{x^2+6x+10}\ dx.$

# Tutorial 7

1. Express the equation $x^2+y^2-6y=0$ in polar coordinates.

2. Sketch and name the following curves (given in polar coordinates):

(a) $\theta = \frac{\pi}{6}$ (b) $r=5$ (c) $r=-6\cos\theta$ (d) $r=4-4\cos\theta$.

3. Find the area of

(a) the region in the first quadrant enclosed by the first loop of the spiral $r=\theta$, $(\theta \geq 0)$, and the lines $\theta=\pi/6$ and $\theta=\pi/3$;

(b) the region common to the circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

4. Find the arc length of the curves (described parametrically)

   (a) $x = 4t + 3$, $y = 3t - 2$, $0 \le t \le 2$.

   (b) $x = (1+t)^2$, $y = (1+t)^3$, $0 \le t \le 1$.

5. Find the arc-length of the spiral described in polar coordinates by the equation

$$r = a\theta$$

for $\theta_0 \le \theta \le \theta_1$.

# Tutorial 8

1. Find the fourth degree Maclaurin polynomials for the functions

$$\text{(a)} \quad e^{-2x} \qquad \text{(b)} \quad \tan x \qquad \text{(c)} \quad xe^x.$$

2. Find the third degree Taylor polynomial for $e^x$ about $x = 1$.

3. Find the Maclaurin series for $\dfrac{1}{1+x}$ and express your answer with sigma notation.

4. Find the Taylor series for $\dfrac{1}{x}$ about $x = -1$ and express your answer with sigma notation.

5. Use the Maclaurin series for $e^x$ to approximate $\sqrt{e}$ to four decimal places.

6. Derive the Maclaurin series for $\dfrac{1}{(1+x)^2}$ by differentiating an appropriate Maclaurin series term by term.

7. Use any method to find the first four nonzero terms in the Maclaurin series for $e^{-x^2}\cos x$.

# Tutorial 9

1. Evaluate the integrals that converge:

   (a) $\displaystyle\int_0^\infty e^{-x}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$

   (c) $\displaystyle\int_0^{\pi/2} \tan x\,dx$

   (d) $\displaystyle\int_0^\infty \frac{dx}{x^2}.$

2. Use the Weierstrass M-test to show that the improper integral

$$\int_1^\infty \frac{\sin x}{x^2}\,dx$$

converges.

3. (a) Find a distribution function $F$ and its density $\rho$ for the function $y = f(x) = \sin x$ defined on $[0, \frac{\pi}{2}]$.

   (b) Compute $\int_0^1 y\rho(y)dy$ and $\int_0^{\pi/2} f(x)\,dx$. Compare the results.

4. Show that the Stieltjes integral $\int_{-1}^1 x\,d|x|$ equals 1.

# Tutorial 10

Solve the given separable differential equation. Where convenient, express the solution explicitly as a function of x.

1. $\frac{dy}{dx} = \frac{y}{x}$

2. $\sqrt{1+x^2}\,y' + x(1+y) = 0$

3. $\mathrm{e}^{-y}\sin x - y'\cos^2 x = 0$

4. Solve the initial-value problem

$$y^2 t\frac{dy}{dt} - t + 1 = 0, \quad y(1) = 3 \quad (t > 0).$$

5. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that a sample weighs 10 mg initially.

   (a) Find a formula for the amount that will remain after $t$ days.

   (b) How much will remain after 10 weeks?

Solve the given first-order linear differential equation. Where convenient, express the solution explicitly as a function of $x$.

6.

$$\frac{dy}{dx} + 3y = \mathrm{e}^{-2x}$$

7.

$$x^2\, y' + 3xy + 2x^6 = 0 \qquad (x > 0)$$

8. Solve the initial-value problem

$$\frac{dy}{dx} - xy = x, \qquad y(0) = 3.$$

9. Find an equation of the curve in the $xy$-plane that passes through the point $(1, 1)$ and has slope $= \frac{y^2}{2\sqrt{x}}$.

# Tutorial 11

1. Verify that $c_1\,\mathrm{e}^{2x} + c_2\,\mathrm{e}^{-x}$ is a solution of $y'' - y' - 2y = 0$ by substituting the function into the equation.

2. Find the general solution of

$$y'' + 3y' - 4y = 0.$$

3. Find the general solution of
$$y'' - 2y' + y = 0.$$

4. Find the general solution of
$$y'' + 5y = 0.$$

5. Find the general solution of
$$y'' - 4y' + 13y = 0.$$

6. Solve the initial value problem
$$y'' + 2y' - 3y = 0, \qquad y(0) = 1, \qquad y'(0) = 5.$$

7. Solve the initial value problem
$$y'' + 4y' + 5y = 0, \qquad y(0) = -3, \qquad y'(0) = 0.$$

# Tutorial 12

1. Use the method of undetermined coefficients to find the general solution of the differential equation
$$y'' + 6y' + 5y = 2\,\mathrm{e}^{3x}\,.$$

2. Use the method of undetermined coefficients to find the general solution of the differential equation
$$y'' + 3y' - 4y = 5\,\mathrm{e}^{7x}\,.$$

3. Use the method of undetermined coefficients to find the general solution of the differential equation
$$y'' - 9y' + 20y = -3\,\mathrm{e}^{5x}\,.$$

4. Use the method of undetermined coefficients to find the general solution of the differential equation
$$y'' + y' - 12y = 4x^2.$$

# Tutorial Solutions

## Tutorial 1

1. (a) From (S3) $m(A \cup B) = m(A) + m(B) - m(A \cap B)$. But $A \cap B = \emptyset$ and due to (S0) $m(A \cap B) = m(\emptyset) = 0$.

   (b) If $B \subset A$ then $A = (A \setminus B) \cup B$ and $B \cap (A \setminus B) = \emptyset$. From (a) then $m(A) = m(B) + m(A \setminus B)$. Thus, $m(A \setminus B) = m(A) - m(B)$.

2. (a) $1^3 + 2^3 + 3^3 = 36$

   (b) $3(2) - 1 + 3(3) - 1 + \ldots + 3(6) - 1 = 5 + 8 + 11 + 14 + 17 = 55$

   (c) $(16 + 4) + (9 + 3) + (4 + 2) + (1 + 1) + (0 + 0) + (1 - 1) = 40$

   (d) $1 + 1 + 1 + 1 + 1 + 1 = 6$

3. (a) $\displaystyle\sum_{k=1}^{49} k(k+1)$

   (b) $1 - 3 + 5 - 7 + 9 - 11$

$$\begin{aligned}
&= (2(0)+1) - (2(1)+1) + \ldots - (2(5)+1) \\
&= (-1)^0(2(0)+1) + (-1)^1(2(1)+1) + \ldots + (-1)^5(2(5)+1) \\
&= \sum_{k=0}^{5}(-1)^k(2k+1) \\
&= \sum_{k=1}^{6}(-1)^{k+1}(2k-1) \qquad \text{(alternatively)}
\end{aligned}$$

4. (a) $\underbrace{n + n + \ldots + n}_{n} = n \times n = n^2$

   (b) -3

   (c) $1x + 2x + \ldots + nx = (1 + 2 + \ldots + n)x = \frac{1}{2}n(n+1)x$ (for $n \geq 1$)

   (d) $\underbrace{c + c + \ldots + c}_{n-m+1} = c(n - m + 1)$

5. (a) Put $k' = k - 4$, so $k = k' + 4$. When $k = 4$, $k' = 0$; when $k = 18$, $k' = 14$. Hence,

$$\sum_{k=4}^{18} k(k-3) = \sum_{k'=0}^{14}(k'+4)(k'+4-3) = \sum_{k=0}^{14}(k+4)(k+1),$$

(since $k$ and $k'$ are dummy variables).

(b) As above, putting $k' = k + 1$.

6.

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta x_k &= \sum_{k=1}^{4}(4-(x_k^*)^2)\Delta x_k \\ &= (4-(-\frac{5}{2})^2)(1) + (4-(-1)^2)(2) + (4-(\frac{1}{4})^2)(1) \\ &\quad +(4-(3)^2)(3) \\ &= -\frac{9}{4} + 6 + \frac{63}{16} - 15 \\ &= -\frac{117}{16}\end{aligned}$$

6. $f(x) = x^2 - 3x + 4$ has a stationary point when $f'(x) = 2x - 3 = 0$, i.e. when $x = \frac{3}{2}$. Since $f$ and $f'$ are continuous, maximum and minimum values of $f$ on the intervals $[0, 1], [1, 3]$ and $[3, 4]$ will occur at the endpoints of these intervals or at the stationary point. Observing that

$$f(0) = 4, \qquad f(1) = 2, \qquad f(\frac{3}{2}) = \frac{7}{4}, \qquad f(3) = 4, \qquad f(4) = 8:$$

the smallest value for the Riemann sum occurs when we take $x_1^* = 1$, $x_2^* = \frac{3}{2}$, $x_3^* = 3$, giving

$$\sum_{k=1}^{3} f(x_k^*)\Delta x_k = (2)(1) + (\frac{7}{4})(2) + (4)(1) = \frac{19}{2};$$

the largest value for the Riemann sum occurs when we take $x_1^* = 0$, $x_2^* = 3$, $x_3^* = 4$, giving

$$\sum_{k=1}^{3} f(x_k^*)\Delta x_k = (4)(1) + (4)(2) + (8)(1) = 20.$$

## Tutorial 2

1. (a) $\displaystyle\int x^3\sqrt{x}\, dx = \int x^3 x^{\frac{1}{2}} dx = \int x^{\frac{7}{2}} dx = \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + C = \frac{2x^{\frac{9}{2}}}{9} + C$

(b) $\displaystyle\int \sec x(\sec x + \tan x)\, dx = \int \sec^2 x + \sec x \tan x\, dx = \tan x + \sec x + C$

2. $\frac{dy}{dx} = 2x+1$, so $y = \int 2x+1\, dx = x^2+x+C$. When $x = -3$, $y = 0$, so $0 = (-3)^2+(-3)+C$, which gives $C = -6$. Hence the equation of the curve is $y = x^2 + x - 6$.

3. (a) $\displaystyle\int_4^9 2y\sqrt{y}\ dy = \int_4^9 2y^{\frac{3}{2}}\ dy = \left[\frac{4}{5}y^{\frac{5}{2}}\right]_4^9 = \frac{4}{5}(9^{\frac{5}{2}} - 4^{\frac{5}{2}}) = \frac{844}{5}$

(b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(x + \frac{2}{\sin^2 x}\right)\ dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (x + 2\csc^2 x)\ dx = \left[\frac{x^2}{2} - 2\cot x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$
$= (\frac{\pi^2}{8} - 2\cot\frac{\pi}{2}) - (\frac{\pi^2}{72} - 2\cot\frac{\pi}{6}) = \frac{\pi^2}{9} + 2\sqrt{3}$

(c) Note that

$$|2x-3| = \begin{cases} 3-2x, & x \le \frac{3}{2}, \\ 2x-3, & x \ge \frac{3}{2}. \end{cases}$$

Thus $\displaystyle\int_0^2 |2x-3|\ dx = \int_0^{\frac{3}{2}} (3-2x)\ dx + \int_{\frac{3}{2}}^2 (2x-3)\ dx$
$= [3x - x^2]_0^{\frac{3}{2}} + [x^2 - 3x]_{\frac{3}{2}}^2 = (\frac{9}{2} - \frac{9}{4}) + ((4-6) - (\frac{9}{4} - \frac{9}{2})) = \frac{5}{2}.$

4. The required area is $\displaystyle\int_{-3}^8 |x^2 - 3x - 10|\ dx$. Now

$$x^2 - 3x - 10 = (x-5)(x+2) \begin{cases} \ge 0, & x \le -2, \\ \le 0, & -2 \le x \le 5, \\ \ge 0, & x \ge 5. \end{cases}$$

So the above integral becomes

$$\begin{aligned} & \int_{-3}^{-2} (x^2 - 3x - 10)\ dx - \int_{-2}^{5} (x^2 - 3x - 10)\ dx + \int_5^8 (x^2 - 3x - 10)\ dx \\ = & \left[\frac{x^3}{3} - \frac{3x^2}{2} - 10x\right]_{-3}^{-2} - \left[\frac{x^3}{3} - \frac{3x^2}{2} - 10x\right]_{-2}^{5} + \left[\frac{x^3}{3} - \frac{3x^2}{2} - 10x\right]_5^8 \\ = & \frac{23}{6} - (-\frac{343}{6}) + \frac{243}{6} \\ = & \frac{203}{2}. \end{aligned}$$

5. $\displaystyle\frac{1}{\pi - 0}\int_0^\pi \sin x\ dx = \frac{1}{\pi}\left[-\cos x\right]_0^\pi = \frac{2}{\pi}$

6. (a) $F'(x) = x^3 + 1$
(b) $F(x) = \left[\frac{1}{4}t^4 + t\right]_1^x = \frac{1}{4}x^4 + x - \frac{5}{4}$; $F'(x) = x^3 + 1$.

7. To prove that $F(x)$ is constant, differentiate to get

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt + \frac{d}{dx}\int_0^{\frac{1}{x}} \frac{1}{1+t^2}\,dt \\ &= \frac{1}{1+x^2} + \frac{1}{1+(\frac{1}{x})^2}\frac{d}{dx}(\frac{1}{x}) \\ &= \frac{1}{1+x^2} + \frac{x^2}{1+x^2}\cdot\frac{-1}{x^2} \\ &= 0. \end{aligned}$$

So $F(x)$ is constant on the interval $(0,\infty)$ (noting that the functions $\frac{1}{x}$ and $\frac{1}{1+x^2}$ are continuous on the same interval).

## Tutorial 3

1. (a) $u = \cot x$, $\frac{du}{dx} = -\csc^2 x$, i.e. $du = -\csc^2 x\ dx$. Hence

$$\int \cot x \csc^2 x\ dx = -\int u\ du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cot^2 x + C.$$

(b) $u = 1+x$ (so $x = u-1$), $du = dx$, hence $\int x^2\sqrt{1+x}\ dx =$ $\int (u-1)^2\sqrt{u}\ du = \int (u^2-2u+1)u^{\frac{1}{2}}\ du = \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}})du =$ $\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + C.$

2. Put $u = 7t^2+12$, $du = 14t\ dt$,

$$\int t\sqrt{7t^2+12}\ dt = \frac{1}{14}\int \sqrt{u}\ du = \frac{1}{14}\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{21}(7t^2+12)^{\frac{3}{2}} + C.$$

3. (a) Put $v = 3u+1$, $dv = 3du$. When $u = 0$, $v = 1$. When $u = 1$, $v = 4$. Hence

$$\int_0^1 \frac{du}{\sqrt{3u+1}} = \frac{1}{3}\int_1^4 \frac{dv}{\sqrt{v}} = \frac{1}{3}\left[2v^{\frac{1}{2}}\right]_1^4 = \frac{2}{3}(2-1) = \frac{2}{3}.$$

(b) Put $u = \sin 3x$, $du = 3\cos 3x\ dx$. $x = 0 \to u = 0$, $x = \frac{\pi}{2} \to u = -1$. So

$$\int_0^{\frac{\pi}{2}} \sin^2 3x \cos 3x\ dx = \frac{1}{3}\int_0^{-1} u^2\ du = \frac{1}{3}\left[\frac{1}{3}u^3\right]_0^{-1} = \frac{1}{9}(-1-0) = -\frac{1}{9}.$$

4. Put $u = 1 - x$ (so $x = 1 - u$). $du = -dx$, $x = 0 \to u = 1$, $x = 1 \to u = 0$, so

$$\int_0^1 x^m(1-x)^n\ dx = -\int_1^0 (1-u)^m u^n\ du = \int_0^1 (1-u)^m u^n\ du = \int_0^1 x^n(1-x)^m\ dx,$$

since $u$ and $x$ are dummy variables.

5. $\displaystyle\int_{\frac{1}{4}}^1 \sqrt{x} - x^2\ dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_{\frac{1}{4}}^1 = (\frac{2}{3} - \frac{1}{3}) - (\frac{2}{3}\cdot\frac{1}{8} - \frac{1}{3}\cdot\frac{1}{64}) = \frac{49}{192}$

6. $\displaystyle\pi\int_0^2 y^2\ dx = \pi\int_0^2 x^4\ dx = \pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}$

7. $\displaystyle 2\pi\int_1^2 xy\ dx = 2\pi\int_1^2 x.x^2\ dx = 2\pi\left[\frac{1}{4}x^4\right]_1^2 = \frac{\pi}{2}(16-1) = \frac{15\pi}{2}$

8. $\displaystyle\frac{dy}{dx} = \frac{x^3}{4} - \frac{1}{x^3}.$

$$L = \int_2^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \int_2^3 \sqrt{1 + \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2}dx = \int_2^3 \sqrt{\frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6}}dx$$
$$= \int_2^3 \frac{x^3}{4} + \frac{1}{x^3}\ dx = \left[\frac{x^4}{16} - \frac{1}{2x^2}\right]_2^3 = \frac{81}{16} - \frac{1}{18} - 1 + \frac{1}{8}$$

## Tutorial 4

1. (a) $\log_2 16 = \log_2 2^4 = 4\log_2 2 = 4(1) = 4$
(b) $\log_2 \frac{1}{32} = \log_2 2^{-5} = -5$
(c) $\log_4 4 = 1$
(d) $\log_9 3 = \log_9 9^{\frac{1}{2}} = \frac{1}{2}$

2. (a) $\ln\frac{1}{x} = -2 \Rightarrow -\ln x = -2 \Rightarrow \ln x = 2 \Rightarrow e^{\ln x} = e^2 \Rightarrow x = e^2$
(b) $e^{-2x} - 3e^{-x} + 2 = 0$ is a quadratic equation in $e^{-x} \Rightarrow (e^{-x}-2)(e^{-x}-1) = 0 \Rightarrow e^{-x} = 2$ or $e^{-x} = 1 \Rightarrow x = \ln\frac{1}{2}$ or $x = 0$.

3. (a) $e^{-\ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$, for $x > 0$.
(b) $e^{\ln x^2} = x^2$, for $x^2 > 0 \Leftrightarrow x \neq 0$.
(c) $\ln(e^{-x^2}) = -x^2$, for all real $x$.
(d) $\ln(\frac{1}{e^x}) = \ln e^{-x} = -x$, for all real $x$.

4. (a) Put $u = \sin x, du = \cos x\, dx$, $\int \cot x\, dx = \int \frac{\cos x\ dx}{\sin x} = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$.

(b) Put $u = \ln x, du = \frac{1}{x}dx$, $\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$.

(c) Let $u = x^2+5, du = 2x\ dx, x = -1 \to u = 6, x = 0 \to u = 5$.
$\displaystyle\int_{-1}^{0} \frac{x\ dx}{x^2+5} = \frac{1}{2}\int_6^5 \frac{du}{u} = \frac{1}{2}[\ln u]_6^5 = \frac{1}{2}(\ln 5 - \ln 6) = \ln\sqrt{\frac{5}{6}}$.

(d) Let $u = 1+\sqrt{x}, du = \frac{dx}{2\sqrt{x}}, x = 1 \to u = 2, x = 4 \to u = 3$.
$\displaystyle\int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\int_2^3 \frac{du}{u} = 2[\ln u]_2^3 = 2(\ln 3 - \ln 2) = \ln\frac{9}{4}$.

5.

$$\begin{aligned}
y &= x^{(e^x)} \\
\ln y &= e^x \ln x \\
\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}e^x \ln x \\
\frac{dy}{dx} &= ye^x(\ln x + \frac{1}{x}) \\
&= x^{(e^x)}e^x(\ln x + \frac{1}{x})
\end{aligned}$$

6. (a) Put $u = \sin x, du = \cos x\ dx$, $\int e^{\sin x}\cos x\ dx = \int e^u\ du = e^u + C = e^{\sin x} + C$.

(b) Put $u = \sin x, du = \cos x\ dx$, $\displaystyle\int \pi^{\sin x}\cos x\ dx = \int \pi^u\ du = \int e^{u\ln\pi}\ du =$
$\displaystyle\frac{e^{u\ln\pi}}{\ln\pi} + C = \frac{\pi^u}{\ln\pi} + C = \frac{\pi^{\sin x}}{\ln\pi} + C$.

(c) Let $u = e^x + 4, du = e^x\ dx, x = -\ln 3 \to u = \frac{13}{3}, x = \ln 3 \to u = 7$.
$\displaystyle\int_{-\ln 3}^{\ln 3} \frac{e^x\ dx}{e^x+4} = \int_{\frac{13}{3}}^{7} \frac{du}{u} = [\ln u]_{\frac{13}{3}}^{7} = \ln 7 - \ln\frac{13}{3} = \ln\frac{21}{13}$.

## Tutorial 5

1. (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$

2. (a) $\displaystyle\frac{dy}{dx} = \frac{1}{\sqrt{1-(\frac{1}{x})^2}} \cdot \frac{-1}{x^2} = \frac{-1}{\frac{x^2}{|x|}\sqrt{x^2-1}} = \frac{-1}{|x|\sqrt{x^2-1}}$

(b) $\displaystyle\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 x}}(-\sin x) = \frac{\sin x}{|\sin x|}$

(c) $\dfrac{dy}{dx} = \dfrac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \dfrac{-(1+x)-(1-x)}{(1+x)^2} = \dfrac{-2}{(1+x)^2+(1-x)^2} = \dfrac{-1}{1+x^2}$

3. (a) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

(b) $\displaystyle\int \frac{dx}{1+16x^2} \overset{u=4x}{=} \frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1} u + C = \frac{1}{4}\tan^{-1}(4x) + C$

(c) $\displaystyle\int_1^3 \frac{dx}{\sqrt{x}(x+1)} \overset{u=\sqrt{x}}{=} 2\int_1^{\sqrt{3}} \frac{du}{u^2+1} = 2\left[\tan^{-1} u\right]_1^{\sqrt{3}} = 2(\frac{\pi}{3} - \frac{\pi}{4}) = \frac{\pi}{6}$

(d) $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}} \overset{u=\ln x}{=} \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\ln x) + C$

4. (a) Put $u = x,\ u' = 1;\ v' = e^{-x},\ v = -e^{-x}$.
$\displaystyle\int xe^{-x}dx = -xe^{-x} - \int -e^{-x} = -xe^{-x} - e^{-x} + C.$

(b) Put $u = x^2,\ u' = 2x;\ v' = \cos x,\ v = \sin x$.
$\displaystyle\int x^2\cos x\ dx = x^2\sin x - \int 2x\sin x\ dx.$
Now put $u = x,\ u' = 1;\ v' = \sin x,\ v = -\cos x$ and the integral becomes
$\displaystyle x^2\sin x - 2\left(-x\cos x - \int -\cos x\ dx\right) = x^2\sin x + 2x\cos x - 2\sin x + C.$

(c) Put $u = \ln x,\ u' = \dfrac{1}{x};\ v' = x^2,\ v = \dfrac{x^3}{3}$. $\displaystyle\int_1^e x^2\ln x\ dx =$
$\displaystyle\left[\frac{x^3\ln x}{3}\right]_1^e - \int_1^e \frac{1}{x}\frac{x^3}{3}dx = \left[\frac{x^3\ln x}{3}\right]_1^e - \int_1^e \frac{x^2}{3}dx = \left[\frac{x^3\ln x}{3} - \frac{x^3}{9}\right]_1^e = \frac{2e^3+1}{9}.$

5. (a) $\displaystyle\int \sin^3 x\ dx = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\ dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C$

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^4 x\ dx = \left[-\frac{1}{4}\sin^3 x\cos x\right]_0^{\frac{\pi}{4}} + \frac{3}{4}\int_0^{\frac{\pi}{4}} \sin^2 x\ dx =$
$\displaystyle\left[-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\cdot\frac{-1}{2}\sin x\cos x\right]_0^{\frac{\pi}{4}} + \frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} dx =$
$\displaystyle\left[-\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x\right]_0^{\frac{\pi}{4}} = \frac{3\pi}{32} - \frac{1}{4}$

6. Put $u = x,\ u' = 1;\ v' = f'',\ v = f'$.
$\displaystyle\int_{-1}^1 xf''(x)dx = [xf'(x)]_{-1}^1 + \int_{-1}^1 f'(x)dx = f'(1) - (-1)f'(-1) + [f(x)]_{-1}^1$
$= f'(1) + f'(-1) + f(1) - f'(-1).$

# Tutorial 6

1. (a) $\displaystyle\int \frac{dx}{\sqrt{8+2x-x^2}} = \int \frac{dx}{\sqrt{9-(x-1)^2}} = \sin^{-1}\left(\frac{x-1}{3}\right) + C.$

(b) $\displaystyle I = \int \frac{dx}{\sqrt{x^2-6x+10}} = \int \frac{dx}{\sqrt{(x-3)^2+1}} \overset{u=x-3}{=} \int \frac{du}{\sqrt{u^2+1}}.$ Now put $u = \tan\theta$, $du = \sec^2\theta\ d\theta$, (noting that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $\sec\theta > 0$) to get

$$I = \int \frac{\sec^2\theta\ d\theta}{\sqrt{\tan^2\theta+1}} = \int \frac{\sec^2\theta\ d\theta}{\sqrt{\sec^2\theta}} = \int \frac{\sec^2\theta\ d\theta}{|\sec\theta|} = \int \sec\theta\ d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta\ d\theta}{\cos^2\theta}$$

$= \displaystyle\int \frac{\cos\theta\ d\theta}{1-\sin^2\theta}$. Now put $v = \sin\theta, dv = \cos\theta\ d\theta$, and we have

$$I = \int \frac{dv}{1-v^2} = \frac{1}{2}\ln\left|\frac{1+v}{1-v}\right| + C$$

after integrating by partial fractions. Back-substituting and simplifying, we get

$$I = \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin\tan^{-1}u}{1-\sin\tan^{-1}u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\frac{u}{\sqrt{1+u^2}}}{1-\frac{u}{\sqrt{1+u^2}}}\right| + C =$$

$$\frac{1}{2}\ln\left|\frac{\sqrt{1+u^2}+u}{\sqrt{1+u^2}-u}\right| + C = \frac{1}{2}\ln\left|\frac{(\sqrt{1+u^2}+u)^2}{1+u^2-u^2}\right| + C = \ln\left|\sqrt{1+u^2}+u\right| + C$$

$$= \ln\left|\sqrt{1+(x-3)^2}+x-3\right| + C.$$

Of course, this answer could have been achieved more easily with the use of integrating formulae, but it is worthwhile seeing how such formulae are unnecessary if one perseveres with appropriate substitutions.

2. (a) $\displaystyle\frac{1}{x^2+3x-4} = \frac{1}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$, where $1 = A(x+4) + B(x-1)$.
Put $x = 1$: $1 = 5A \Rightarrow A = \frac{1}{5}$; put $x = -4$: $1 = -5B \Rightarrow B = -\frac{1}{5}$.

$$\int \frac{dx}{x^2+3x-4} = \frac{1}{5}\int\left(\frac{1}{x-1} - \frac{1}{x+4}\right)dx = \frac{1}{5}\ln\left|\frac{x-1}{x+4}\right| + C.$$

(b) $\displaystyle\frac{x^3}{x^2-3x+2} = x+3+\frac{7x-6}{(x-1)(x-2)} = x+3+\frac{A}{x-1}+\frac{B}{x-2}$,
where $7x-6 = A(x-2)+B(x-1)$. Put $x = 1 \Rightarrow A = -1$, put $x = 2 \Rightarrow B = 8$.

$$\int \frac{x^3dx}{x^2-3x+2} = \int\left(x+3+\frac{8}{x-2}-\frac{1}{x-1}\right)dx = \frac{1}{2}x^2+3x+8\ln|x-2|-\ln|x-1|+C.$$

(c) $\dfrac{x^2}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$ where $x^2 = A(x+2)^2 + B(x+2) + C$. Equating coefficients of $x^2 \Rightarrow A = 1$. Put $x = 2 \Rightarrow C = 4$. Put $x = 0 \Rightarrow 0 = 4A + 2B + C \Rightarrow B = -4$.

$$\int \frac{x^2 dx}{(x+2)^3} = \int \left( \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{4}{(x+2)^3} \right) dx = \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C.$$

(d) $\dfrac{1}{x^4 - 16} = \dfrac{1}{(x-2)(x+2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+4}$, where $1 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$. Put $x = 2 \Rightarrow A = \frac{1}{32}$. Put $x = -2 \Rightarrow B = -\frac{1}{32}$. Put $x = 0 \Rightarrow 1 = 8A - 8B - 4D \Rightarrow D = -\frac{1}{8}$. Equating coefficients of $x^3 \Rightarrow 0 = A + B + C \Rightarrow C = 0$.

$$\int \frac{dx}{x^4 - 16} = \frac{1}{32} \int \left( \frac{1}{x-2} - \frac{1}{x+2} \right) dx - \frac{1}{8} \int \frac{dx}{x^2+4} = \frac{1}{32} \ln \left| \frac{x-2}{x+2} \right| - \frac{1}{16} \tan^{-1} \frac{x}{2} + C.$$

(e)

$$\int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10} dx = \int \left( x^2 + \frac{x}{x^2 + 6x + 10} \right) dx = \frac{x^3}{3} + \int \frac{x}{(x+3)^2 + 1} dx$$

$$\overset{u=x+3}{=} \frac{x^3}{3} + \int \frac{u-3}{u^2+1} du = \frac{x^3}{3} + \frac{1}{2} \int \frac{2u\, du}{u^2+1} - 3 \int \frac{du}{u^2+1}$$

$$= \frac{x^3}{3} + \frac{1}{2} \ln(u^2+1) - 3\tan^{-1} u + C = \frac{x^3}{3} + \frac{1}{2} \ln((x+3)^2+1) - 3\tan^{-1}(x+3) + C.$$

## Tutorial 7

1.

$$\begin{aligned} x^2 + y^2 - 6y &= 0 \\ r^2(\cos^2\theta + \sin^2\theta) - 6r\sin\theta &= 0 \\ r^2 &= 6r\sin\theta \\ r &= 6\sin\theta \end{aligned}$$

2. (a) Line through the origin.

(b) Circle centre origin radius 5.

(c) $r = -6\cos\theta \Rightarrow r^2 = -6r\cos\theta \Rightarrow x^2+y^2 = -6x \Rightarrow (x+3)^2+y^2 = 9$, upon completing the square, which is a circle centre (-3,0), radius 3.

(d) Cardioid.

3. (a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}r^2 d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\theta^2 d\theta = \left[\frac{\theta^3}{6}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{7\pi^3}{1296}$

(b) The two curves intersect when $3\cos\theta = 1+\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$. By symmetry about the $x$-axis, the area is

$$\begin{aligned}
A &= 2\left[\int_0^{\frac{\pi}{3}} \frac{1}{2}(1+\cos\theta)^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(3\cos\theta^2)d\theta\right] \\
&= \int_0^{\frac{\pi}{3}} (1+2\cos\theta+\cos^2\theta)d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9\cos^2\theta d\theta \\
&= \int_0^{\frac{\pi}{3}} (\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta)d\theta + \frac{9}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\cos 2\theta+1)d\theta \\
&= \left[\frac{3\theta}{2}+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{3}} + \frac{9}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}-0-0-0\right] + \frac{9}{2}\left[0+\frac{\pi}{2}-\frac{\sqrt{3}}{4}-\frac{\pi}{3}\right] \\
&= \frac{5\pi}{4}.
\end{aligned}$$

4. (a) $\displaystyle\int_0^2 \sqrt{4^2+3^2}dt = \int_0^2 5dt = 10$

(b)

$$\begin{aligned}
\int_0^1 \sqrt{(2(1+t))^2+(3(1+t)^2)^2}dt &= \int_0^1 (1+t)\sqrt{4+9(1+t)^2}dt \\
&= \left[\frac{(4+9(1+t)^2)^{\frac{3}{2}}}{18}\cdot\frac{2}{3}\right]_0^1 \\
&= \frac{1}{27}(40\sqrt{40}-13\sqrt{13})
\end{aligned}$$

5. We have $r = a\theta$ and $r' = a$. Hence

$$\ell = \int_{\theta_1}^{\theta_2} \sqrt{a^2\theta^2+a^2}\,d\theta = |a|\int_{\theta_1}^{\theta_2} \sqrt{\theta^2+1}\,d\theta.$$

Substitute $\theta = \sinh t$, $d\theta = \cosh t dt$. Then

$$\ell = \int_{\sinh^{-1}\theta_1}^{\sinh^{-1}\theta_2} \cosh^2 t dt.$$

Use $\cosh^2 t = \frac{\cosh 2t+1}{2}$.

$$\begin{aligned}\ell &= \int_{\sinh^{-1}\theta_1}^{\sinh^{-1}\theta_2} \frac{\cosh 2t+1}{2} dt = \left[\frac{\sinh 2t}{4} + \frac{t}{2}\right]_{\sinh^{-1}\theta_1}^{\sinh^{-1}\theta_2} \\ &= \frac{\sinh(2\sinh^{-1}\theta_2) - \sinh(2\sinh^{-1}\theta_1)}{4} + \frac{\sinh^{-1}\theta_2 - \sinh^{-1}\theta_1}{2}.\end{aligned}$$

# Tutorial 8

1.

$$\begin{aligned}(a)\qquad f(x) &= e^{-2x} & f(0) &= 1 \\ f'(x) &= -2e^{-2x} & f'(0) &= -2 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= (-2)^n e^{-2x} & f^{(n)}(0) &= (-2)^n\end{aligned}$$

$$S_4(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4$$

$$\begin{aligned}(b)\qquad f(x) &= \tan x & f(0) &= 0 \\ f'(x) &= \sec^2 x & f'(0) &= 1 \\ f''(x) &= 2\sec^2 x \tan x & f''(0) &= 0 \\ f'''(x) &= 4\sec^2 x \tan^2 x + 2\sec^4 x & f'''(0) &= 2 \\ f^{(4)}(x) &= \cdots & f^{(4)}(0) &= 0\end{aligned}$$

$$S_4(x) = x + \frac{1}{3}x^3$$

(c)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$
$$xe^x = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \cdots$$
$$S_4(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}$$

2. $f^{(n)}(x) = e^x$, so $f^{(n)}(1) = e$ for all $n$.

$$S_3(x) = e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{6}$$

3.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1, \text{ so}$$
$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \text{for } |x| < 1.$$

4.

$$\frac{1}{x} = \frac{-1}{1-(x+1)} = -\sum_{n=0}^{\infty} (x+1)^n$$

5. For $e^x$ we have the Maclaurin polynomial $S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ with remainder term

$$R_n(x) = \frac{f^{(n+1)}(\epsilon) x^{n+1}}{(n+1)!}$$

for some $\epsilon$ between 0 and $x$. Now $f^{(n+1)}(\epsilon) = e^\epsilon \le e^{\frac{1}{2}} \le 2$ for $0 \le \epsilon \le \frac{1}{2}$. So

$$\left| R_n(\frac{1}{2}) \right| \le \frac{2(\frac{1}{2})^{n+1}}{(n+1)!} = \frac{1}{2^n (n+1)!} \le 0.00005 \quad \text{for } n = 5, \text{ hence}$$

$$\begin{aligned} e^{\frac{1}{2}} &\simeq 1 + \frac{1}{2} + \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})^3}{6} + \frac{(\frac{1}{2})^4}{24} + \frac{(\frac{1}{2})^5}{120} \\ &\simeq 1.6487 \quad \text{to 4 decimal places.} \end{aligned}$$

6.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Differentiate both sides of this equation to get

$$\frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1},$$

hence

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n-1} n x^{n-1} = \sum_{n=0}^{\infty}(-1)^n (n+1) x^n.$$

7.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots$$

$$\begin{aligned} e^{-x^2}\cos x &= (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \ldots)(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots) \\ &= 1 + (-1 - \frac{1}{2})x^2 + (\frac{1}{2} + \frac{1}{2} + \frac{1}{24})x^4 + (-\frac{1}{6} - \frac{1}{4} - \frac{1}{24} - \frac{1}{720})x^6 + \ldots \\ &= 1 - \frac{3x^2}{2} + \frac{25x^4}{24} - \frac{331x^6}{720} + \ldots \end{aligned}$$

(keeping only terms of degree 6 or less.)

## Tutorial 9

1. (a) $\displaystyle\int_0^\infty e^{-x}dx = \lim_{R\to\infty}\left(\left[-e^{-x}\right]_0^R = e^0 - e^{-R}\right) = 1 - 0 = 1$

(b) $\displaystyle\lim_{R\to\infty}\int_1^R \frac{dx}{\sqrt{x}} = \lim_{R\to\infty}\left[2x^{\frac{1}{2}}\right]_1^R = \lim_{R\to\infty} 2\sqrt{R} - 2$ which diverges to $\infty$.

(c) $\displaystyle\lim_{r\to\frac{\pi}{2}^-}\int_o^r \tan x \; dx = \lim_{r\to\frac{\pi}{2}^-}[-\ln|\cos x|]_0^r = 0 - \lim_{r\to\frac{\pi}{2}^-}\ln|\cos r| = -\lim_{s\to 0}\ln|s|$ which diverges to $\infty$.

(d) $\displaystyle\lim_{\substack{r\to 0^+\\ R\to\infty}}\int_r^R \frac{dx}{x^2} = \lim_{\substack{r\to 0^+\\ R\to\infty}}\left[-\frac{1}{x}\right]_r^R = \lim_{r\to 0^+}\frac{1}{r} - \lim_{R\to\infty}\frac{1}{R} = \lim_{r\to 0^+}\frac{1}{r} - 0$ which diverges to $\infty$.

2. Since $|\sin x| \leq 1$ we have $\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$. Since $\int_1^\infty \frac{dx}{x^2}$ converges, the integral $\int_1^\infty \frac{\sin x\, dx}{x^2}$ must converge as well.

3. (a) The distribution function $F(y) = \text{mes}\{0 \le \sin x < y\} = \text{mes}\{0 \le x < \sin^{-1} y\} = \sin^{-1} y$ for $0 \le y \le \frac{\pi}{2}$, it vanishes for $y < 0$ and equals 1 for $y > \frac{\pi}{2}$. Therefore the density $\rho(y) = F'(y) = \frac{1}{\sqrt{1-x^2}}$.

$$(b) \qquad \int_0^1 \frac{y\,dy}{\sqrt{1-y^2}} = -\frac{1}{2}\int_0^1 \frac{-2y}{\sqrt{1-y^2}}\,dy$$

We substitute $1 - y^2 = t$. Then the lower bound becomes 1 and the upper bound becomes 0. Hence

$$\int_0^1 \frac{y\,dy}{\sqrt{1-y^2}} = -\frac{1}{2}\int_1^0 \frac{du}{\sqrt{u}}\,dy = -\frac{1}{2}[2\sqrt{u}]_1^0 = 1.$$

On the other hand $\int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = 1$. The results coincide.

4. Let $x_0 = -1 < x_1 < \cdots < x_N = 1$ be a partition of $[-1, 1]$ such that non of the $x_i = 0$. Then there is exactly one subinterval $[x_{i-1}, x_i]$ that contains 0.

We have

$$\int_1^1 xd|x| \approx \sum_{k=1}^{i-1} x_{k-1}(|x_k| - |x_{k-1}|) + x_{i-1}(|x_i| - |x_{i-1}|) + \sum_{k=i+1}^{N} x_{k-1}(|x_k| - |x_{k-1}|)$$

Now, $|x_k| = -x_k$ for $k \le i-1$ and $|x_k| = x_k$ for $k \ge i$. Hence

$$\int_1^1 xd|x| \approx -\sum_{k=1}^{i-1} x_{k-1}(x_k - x_{k-1}) + x_{i-1}(-x_i - x_{i-1}) + \sum_{k=i+1}^{N} x_{k-1}(x_k - x_{k-1})$$

As the partition becomes finer the first sum tends to $-\int_{-1}^0 x\,dx = \frac{1}{2}$, the term $x_{i-1}(-x_i - x_{i-1})$ tends to 0 and the last sum tends to $\int_0^1 x\,dx = \frac{1}{2}$. It follows

$$\int_1^1 xd|x| = 1.$$

# Tutorial 10

1.

$$\frac{dy}{y} = \frac{dx}{x} \text{ so } \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\begin{aligned}
\ln|y| &= \ln|x| + c \\
|y| &= e^{\ln|x|+c} = |x|\,e^c \\
y &= \pm\, e^c\, x = Kx, \text{ where } K \text{ is a constant.}
\end{aligned}$$

2.

$$\begin{aligned}
\sqrt{1+x^2}\,\frac{dy}{dx} &= -x(1+y) \\
\int \frac{dy}{1+y} &= -\int \frac{x\,dx}{\sqrt{1+x^2}} \\
\ln|1+y| &= -\sqrt{1+x^2} + c \\
|1+y| &= e^c\ e^{-\sqrt{1+x^2}} \\
1+y &= K\ e^{-\sqrt{1+x^2}} \\
y &= -1 + K\ e^{-\sqrt{1+x^2}}, \text{ where } K \text{ is a constant.}
\end{aligned}$$

3.

$$\begin{aligned}
\frac{dy}{dx}\cos^2 x &= e^{-y}\sin x \\
\int e^y\,dy &= \int \frac{\sin x}{\cos^2 x}dx \\
e^y &= \frac{1}{\cos x} + c = \sec x + c \\
y &= \ln(\sec x + c)
\end{aligned}$$

4.

$$\begin{aligned}
y^2\frac{dy}{dt} &= 1 - \frac{1}{t} \\
\int y^2\,dy &= \int \left(1 - \frac{1}{t}\right) \\
\frac{y^3}{3} &= t - \ln t + c
\end{aligned}$$

When $x = 1$, $y = 3$ so $9 = 1 + c$. Hence $c = 8$ and

$$\begin{aligned}
y^3 &= 3t - 3\ln t + 24 \\
y &= \sqrt[3]{3t - 3\ln t + 24}.
\end{aligned}$$

5. Let $y$ be the number of grams left after $t$ days. Then

$$y' = ky, \quad y(0) = 10.$$

which has solution $y = 10\,e^{kt}$. After 140 days, $y = 5$, so $5 = 10\,e^{140k}$, whence $k = -\ln 2/140$. Hence, $y = 10\,e^{-(\ln 2)t/140}$. When $t = 70$, $y = 10\,e^{-(\ln 2)70/140} = 10\,e^{-(\ln 2)/2} \approx 7.07$ (mg).

6. Here $p(x) = 3$, so $\rho = e^{\int p(x)dx} = e^{3x}$. Thus $\frac{dy}{dx}\,e^{3x} + 3y\,e^{3x} = e^{3x}\,e^{-2x} = e^{x}$, $(y\,e^{3x})' = e^{x}$, whence $y\,e^{3x} = e^{x} + c$. $y = e^{-2x} + c\,e^{-3x}$.

7. Here $p(x) = 3/x$, so $\rho = e^{\int p(x)dx} = e^{3\ln x} = x^3$. On multiplying by $\rho$ we get $x^3y' + 3x^2y = -2x^7$, $(x^3y)' = -2x^7$, whence $x^3y = -\frac{x^8}{4} + c$. $y = -x^5/4 + c/x^3$.

8. $\rho = e^{-\int x\,dx} = e^{-x^2/2}$. So

$$\begin{aligned}
e^{-x^2/2}\frac{dy}{dx} - x\,e^{-x^2/2}\,y &= x\,e^{-x^2/2}\\
(e^{-x^2/2}\,y)' &= x\,e^{-x^2/2}\\
e^{-x^2/2}\,y &= -\,e^{-x^2/2} + c\\
y &= -1 + c\,e^{x^2/2}.
\end{aligned}$$

When $x = 0$, $y = 3$, so $3 = -1 + c\,e^0 = -1 + c$. Hence $c = 4$ and $y = -1 + 4\,e^{x^2/2}$.

9.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y^2}{2\sqrt{x}}\\
\int \frac{dy}{y^2} &= \int \frac{dx}{2\sqrt{x}}\\
\frac{-1}{y} &= \sqrt{x} + c
\end{aligned}$$

When $x = 1$, $y = 1$, so $-1 = 1 + c$ and $c = -2$. Hence

$$-\frac{1}{y} = \sqrt{x} - 2 \quad \text{and} \quad y = \frac{1}{2 - \sqrt{x}}.$$

# Tutorial 11

1. $y = c_1\,\mathrm{e}^{2x} + c_2\,\mathrm{e}^{-x}$, $y' = 2c_1\,\mathrm{e}^{2x} - c_2\,\mathrm{e}^{-x}$, $y'' = 4c_1\,\mathrm{e}^{2x} + c_2\,\mathrm{e}^{-x}$. Hence

$$\begin{aligned} y'' - y' - 2y &= (4c_1\,\mathrm{e}^{2x} + c_2\,\mathrm{e}^{-x}) - (2c_1\,\mathrm{e}^{2x} - c_2\,\mathrm{e}^{-x}) - 2(c_1\,\mathrm{e}^{2x} + c_2\,\mathrm{e}^{-x}) \\ &= c_1\,\mathrm{e}^{2x}(4 - 2 - 2) + c_2\,\mathrm{e}^{-x}(1 + 1 - 2) \end{aligned}$$

Hence $y$ is a solution of the given d.e.

2. The auxiliary equation is $m^2 + 3m - 4 = 0$. Hence $(m - 4)(m - 1) = 0$ and $m_1 = 4$, $m_2 = 1$.

$$y = c_1\,\mathrm{e}^{-4x} + c_2\,\mathrm{e}^{x}.$$

3. The auxiliary equation is $m^2 - 2m + 1 = 0$. Hence $(m - 1)^2 = 0$ and $m_1 = m_2 = 1$.

$$y = c_1\,\mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x}.$$

4. The auxiliary equation is $m^2 + 5 = 0$. Hence $m = \pm\,\mathrm{i}\,\sqrt{5}$. $u = 0$, $v = 5$.

$$y = c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x).$$

5. The auxiliary equation is $m^2 - 4m + 13 = 0$. Hence $m = 2 \pm 3\,\mathrm{i}$. $u = 3$, $v = 3$.

$$y = \mathrm{e}^{2x}(c_1 \cos(3x) + c_2 \sin(3x)).$$

6. The auxiliary equation is $m^2 + 2m - 3 = 0$. Hence $(m + 3)(m - 1) = 0$ and $m_1 = -3$, $m_2 = 1$.

$$y = c_1\,\mathrm{e}^{-3x} + c_2\,\mathrm{e}^{x}.$$

Now, $y' = -3c_1\,\mathrm{e}^{-3x} + c_2\,\mathrm{e}^{x}$. The conditions $y(0) = 1$, $y'(0) = 5$ give

$$\begin{aligned} c_1 + c_2 &= 1 \\ -3c_1 + c_2 &= 5. \end{aligned}$$

Subtracting the first of these equations form the second gives $-4c_1 = 4$, whence $c_1 = -1$ and $c_2 = 2$.

$$y == -\,\mathrm{e}^{-3x} + 2\,\mathrm{e}^{x}.$$

7. The auxiliary equation is $m^2 + 4m + 5 = 0$. Hence $m = -2 \pm \mathrm{i}$. $u = -2$, $v = 1$.

$$y = \mathrm{e}^{-2x}(c_1 \cos x + c_2 \sin x).$$

Now, $y' = -2\,\mathrm{e}^{-2x}(c_1 \cos x + c_2 \sin x) + \mathrm{e}^{-2x}(-c_1 \sin x + c_2 \cos x)$. The conditions $y(0) = -3$, $y'(0) = 0$ give $c_1 = -3$ and $-2c_1 + c_2 = 0$, whence $c_2 = -6$.

$$y = -3\,\mathrm{e}^{-2x}(\cos x + 2 \sin x).$$

# Tutorial 12

1. To get $y_1$ and $y_2$ we solve $y'' + 6y' + 5y = 0$. The auxiliary equation is $m^2 + 6m + 5 = 0$. Hence $(m + 5)(m + 1) = 0$ and $m_1 = -5$ and $m_2 = -1$. $y_1 = \mathrm{e}^{-5x}$, $y_2 = \mathrm{e}^{-x}$. Since the RHS is $\mathrm{e}^{3x}$ multiplied by constant, we look for a solution which is also $\mathrm{e}^{3x}$ multiplied by constant, i.e. of the form $y = A\,\mathrm{e}^{3x}$. Substituting into $y'' + 6y' + 5y = 2\,\mathrm{e}^{3x}$ gives

$$(9A\,\mathrm{e}^{3x}) + 6(3A\,\mathrm{e}^{3x}) + 5(A\,\mathrm{e}^{3x}) = 2\,\mathrm{e}^{3x}\,.$$

That is $32A\,\mathrm{e}^{3x} = 2\,\mathrm{e}^{3x}$. Hence $A = 1/16$. Our particular solution is $y_p = \mathrm{e}^{3x}/16$. The general solution is $y = c_1\,\mathrm{e}^{-5x} + c_2\,\mathrm{e}^{-x} + \mathrm{e}^{3x}/16$.

2. To get $y_1$ and $y_2$ we solve $y'' + 3y' - 4y = 0$. The auxiliary equation is $m^2 + 3m - 4 = 0$. Hence $(m + 4)(m - 1) = 0$ and $m_1 = 4$ and $m_2 = 1$. $y_1 = \mathrm{e}^{x}$, $y_2 = \mathrm{e}^{-4x}$. Since the RHS is $\mathrm{e}^{7x}$ multiplied by constant, we look for a solution of the form $y = A\,\mathrm{e}^{7x}$. Substituting into $y'' + 3y' - 4y = 5\,\mathrm{e}^{7x}$ gives

$$(49A\,\mathrm{e}^{7x}) + 3(7A\,\mathrm{e}^{7x}) - 4(A\,\mathrm{e}^{7x}) = 5\,\mathrm{e}^{7x}\,.$$

That is $66A\,\mathrm{e}^{7x} = 5\,\mathrm{e}^{7x}$. Hence $A = 5/66$. Our particular solution is $y_p = 5\,\mathrm{e}^{7x}/66$. The general solution is $y = c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-4x} + 5\,\mathrm{e}^{7x}/66$.

3. To get $y_1$ and $y_2$ we solve $y'' - 9y' + 20y = 0$. The auxiliary equation is $m^2 - 9m + 20 = 0$. Hence $(m - 4)(m - 5) = 0$ and $m_1 = 4$ and $m_2 = 5$. $y_1 = \mathrm{e}^{4x}$, $y_2 = \mathrm{e}^{5x}$. Since the RHS is $\mathrm{e}^{5x}$ multiplied by constant and this is a solution of the homogeneous equation, we need to use the modification rule. Thus we look for a solution of the form $y = Ax\,\mathrm{e}^{5x}$. Then

$y' = A\,\mathrm{e}^{5x} + 5Ax\,\mathrm{e}^{5x}$, $y'' = 10A\,\mathrm{e}^{5x} + 25Ax\,\mathrm{e}^{5x}$. Substituting into $y'' - 9y' + 20y = -3\,\mathrm{e}^{5x}$ gives

$$(10A\,\mathrm{e}^{5x} + 25Ax\,\mathrm{e}^{5x}) - 9(A\,\mathrm{e}^{5x} + 5Ax\,\mathrm{e}^{5x}) + 20(Ax\,\mathrm{e}^{5x}) = -3\,\mathrm{e}^{5x}\,.$$

That is $A\,\mathrm{e}^{5x} = -3\,\mathrm{e}^{5x}$. Hence $A = -3$. Our particular solution is $y_p = -3x\,\mathrm{e}^{5x}$. The general solution is $y = c_1\,\mathrm{e}^{4x} + c_2\,\mathrm{e}^{5x} - 3x\,\mathrm{e}^{5x}$.

4. To get $y_1$ and $y_2$ we solve $y'' + y' - 12y = 0$. The auxiliary equation is $m^2 + m - 12 = 0$. Hence $(m+4)(m-3) = 0$ and $m_1 = 3$ and $m_2 = -4$. $y_1 = \mathrm{e}^{3x}$, $y_2 = \mathrm{e}^{-4x}$. Since the RHS is a polynomial of degree two, we look for a solution of the same form $y = A_0 + A_1x + A_2x^2$. Then $y' = A_1 + 2A_2x$ and $y'' = 2A_2$. Substituting into $y'' + y' - 12y = 4x^2$ gives

$$2A_2 + (A_1 + 2A_2x) - 12(A_0 + A_1x + A_2x^2) = 4x^2.$$

That is $(2A_2 + A_1 - 12A_0) + (2A_2 - 12A_1)x - 12A_2x^2 = 4x^2$. Now we equate coefficients of like powers: $-12A_2 = 4$, so $A_2 = -1/3$; $2A_2 - 12A_1 = 0$, so $A_1 = -1/18$; $2A_2 + A_1 - 12A_0 = 0$, so $A_0 = -13/216$. Our particular solution is $y_p = -13/216 - x/18 - x^2/3$. The general solution is $y = c_1\,\mathrm{e}^{3x} + c_2\,\mathrm{e}^{-4x} - 13/216 - x/18 - x^2/3$.