D-brane masses and periods

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arXiv: 2203.09426

ANZGSF

April 1, 2022

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\begin{tikzcd}
\begin{array}{c}\mathcal{N}=2 \text{ supergravity} \\ \text{in } D=4\end{array} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \begin{array}{l}\text{Type II string theory} \\ \text{on } \mathbb{R}^{3,1}\times X \\ X \text{ Calabi-Yau 3-fold}\end{array} \arrow[d, leftrightarrow] \\
\begin{array}{l}\text{Arithmetic of} \\ \text{families of} \\ \text{Calabi-Yau 3-folds}\end{array} \arrow[r, leftrightarrow] & \begin{array}{l}\text{Geometry of} \\ \text{families of} \\ \text{Calabi-Yau 3-folds} \\ \text{and of their} \\ \text{parameter spaces}\end{array}
\end{tikzcd}
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Review of Calabi-Yau manifolds:

X compact, Kähler manifold of dimension n s.t. $Ric(g) = 0$ for a Kähler metric g.

Properties:

- Hodge decomposition: $H^n(X,\mathbb{C}) = \bigoplus_{k=0}^{n} H^{k,n-k}$
- $\dim_{\mathbb{C}} H^{n,0} = 1$, generator $\Omega \in \Omega^{n,0}_{cl}(X)$
- metric g can be deformed (keeping the Calabi-Yau condition) by changing the complex str. and the Kähler str independently

$\mathcal{M}$ = parameter space of cplx. str. def.

$\mathcal{M}$ is a Kähler manifold s.t. the Riemann tensor satisfies

$$R_{i\bar{j}k}^{\ \ \ \ l} = C_{ik}^{\ \ \bar{m}} \bar{C}_{\bar{j}\bar{m}}^{\ \ \ l} - G_{i\bar{j}} \delta_k^{\ l} - G_{k\bar{j}} \delta_i^{\ l}$$

where $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is the Kähler metric

$$K = -\log i \int_X \Omega \wedge \overline{\Omega}$$

C_{ijk} is holom., symmetric 3-tensor on $\mathcal{M}$

$\mathcal{M}$ is called special Kähler manifold

(Candelas, de la Ossa; Strominger '90
Freed '98)

Attractor CY manifolds:

Consider $N=2$ supersymmetric black hole solutions in IIB supergravity obtained compactifying IIB string theory on a Calabi-Yau 3fold X_z, $z \in \mathcal{M}$, and charge $\gamma \in H_3(X_z, \mathbb{Z})$.

$N=2$ supersymmetry algebra has a central charge

$$\{ Q_\alpha^I, Q_\beta^J \} = \varepsilon_{\alpha\beta} \varepsilon^{IJ} Z \qquad I,J=1,2$$

Bogomolnyi bound: $m \geq |Z|$

BPS state: $m = |Z|$, preserves half of of the supercharges.

We have

$$Z_\gamma(z) = e^{K/2} \int_\gamma \Omega_z$$

Ansatz for the black hole metric:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}d\underline{x}^2$$

r radial coord., vanishing at horizon.

(Ferrara, Kallosh, Strominger '95:)

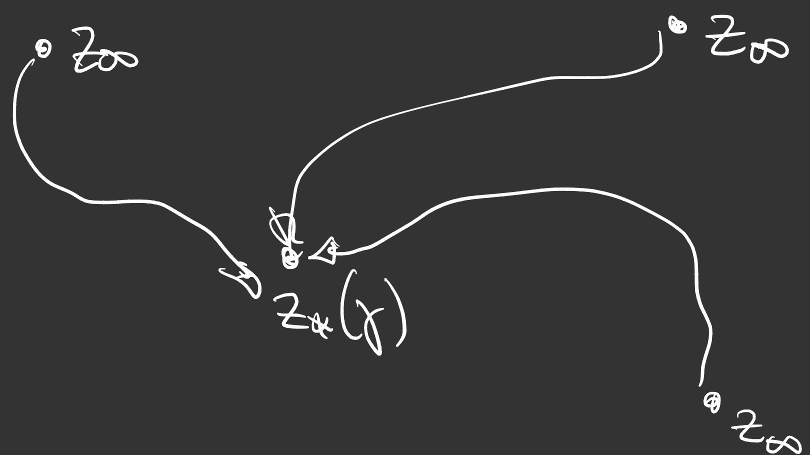
$$\frac{dU(\rho)}{d\rho} = -e^{U(\rho)}\,|Z_\gamma(z)| \qquad \rho = \frac{1}{r}$$

$$\frac{dz(\rho)}{d\rho} = -2e^{U(\rho)}\,G^{z\bar{z}}\partial_{\bar{z}}|Z_\gamma(z)|$$

with $U(\rho=0)=0$ (asymptotically flat)

horizon at $\rho=\infty$

$z(\rho)$ has a fixed point z_* which only depends on γ:



z_*: attractor point.

The endpoint z_* of the attractor flow is a minimum of $|Z_\gamma(z)|$, independent of the starting point z_∞.

(Near the horizon: metric is $AdS_2 \times S^2$
area $A = 4\pi |Z_\gamma(z_*)|^2$)

$\mathcal{M}$ is special Kähler $\Rightarrow$

arithmetically important condition:

Let $\Gamma \in H^3(X_z, \mathbb{Z})$ be dual to $\gamma \in H_3(X_z, \mathbb{Z})$

$$H^3(X_z, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

For $z = z_*$, $\Gamma \in H^{3,0} \oplus H^{0,3}$

$V = (H^{3,0} \oplus H^{0,3}) \cap H^3(X_{z_*}, \mathbb{R})$ is a real 2-plane spanned by $\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega$

$$V \cap H^3(X_z, \mathbb{Z}) = \begin{cases} \{0\} & \text{no attractor pt.} \\ \mathbb{Z} & \\ \mathbb{Z}^2 & \end{cases} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{attractor pt. of rk 1 or 2.}$$

(Moore '98)

$\mathrm{rk} = 0$ generic

$\mathrm{rk} = 1$ rare, e.g. $z_* =$ conifold pt. (singular CY)

$\mathrm{rk} = 2$ very rare, first example (Candelas, de la Ossa, van Straten '19).

Found using arithmetic:

Take X defined over $\mathbb{Q}$, i.e. $X = \{f=0\}$, f is a polynomial with coefficients in $\mathbb{Q}$ and counts solutions to $f=0$ in <u>finite</u> fields $\mathbb{F}_p$ for all primes p.

Bönisch, Klemm, S, Zagier '22:

values of $|Z_\gamma(z)| = |e^{K/2} \int_\gamma \Omega_z|$
(for the 14 hypergeometric cases)

- at attractor points $z_* \in \mathcal{M}$ are of the form (dim $\mathcal{M}=1$, rk 1 attractors at the conifold z_*)

$$\frac{w^\pm}{\sqrt{2w^+w^-}}$$

where $w^\pm$ are real periods.

(In fact, $w^\pm$ are periods of a modular form associated to X_{z_*} through counting points over finite fields)

- at attractor point of rank 2

$$\frac{aw^+ + bw^-}{\sqrt{Aw_+^2 + Bw_+w_- + Cw_-^2}} \qquad a,b,A,B,C \in \mathbb{Z}$$

What are periods?

$\mathbb{Z}$ integers

$\cap$

$\mathbb{R} \supset \mathbb{Q}$ rational numbers

$\cap$

$\overline{\mathbb{Q}}$ algebraic numbers = roots of a nonzero polynomial in 1 variable with rational coeff.
e.g. $\sqrt{2}$, $\sqrt{-3}$

$\cap$

$\mathcal{P}$ periods = values of integrals of algebraic functions with algebraic coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients (Kontsevich–Zagier '01)

$\cap$

$\mathbb{C}$ complex numbers

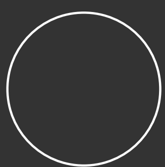
Examples: $\pi, \log 2, \Gamma(\frac{1}{3})^3, \zeta(3), \ldots$

Non-examples: e, γ

Not known: $\frac{1}{\pi}, \Gamma(\frac{1}{3}), \ldots$

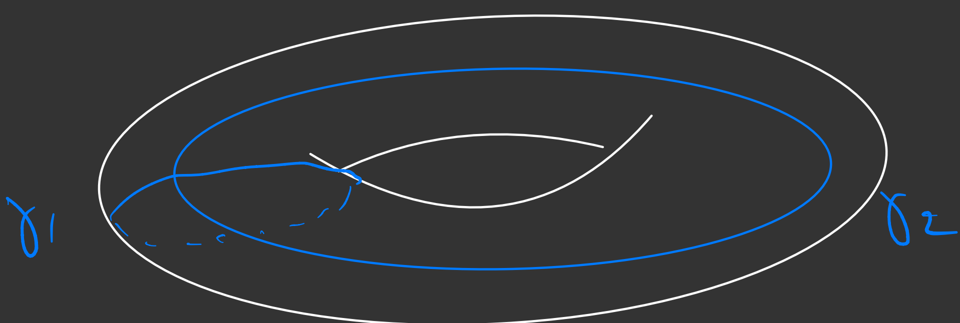
$$\pi = \iint_{x^2+y^2\leq 1} dx\,dy = 2\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$$= \int_{-1}^{1} \frac{dx}{y} \qquad y = \sqrt{1-x^2}$$



$\gamma \in H_1(S^1, \mathbb{Z}) = \mathbb{Z}$

$$\pi = \int_\gamma \omega \quad , \quad \omega = \frac{dx}{y}$$



$$E_\tau = \mathbb{C}/L_\tau \,, \quad L_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$$

$$a\tau^2 + b\tau + c = 0 \qquad a, b, c \in \mathbb{Z}.$$

$$\tau = \frac{-b + \sqrt{D}}{2a} \qquad\qquad D = b^2 - 4ac$$

$$\omega L_\tau \subset L_\tau \,, \quad \omega = \frac{b + \sqrt{D}}{2} \equiv \frac{D + \sqrt{D}}{2} \mod \mathbb{Z}$$