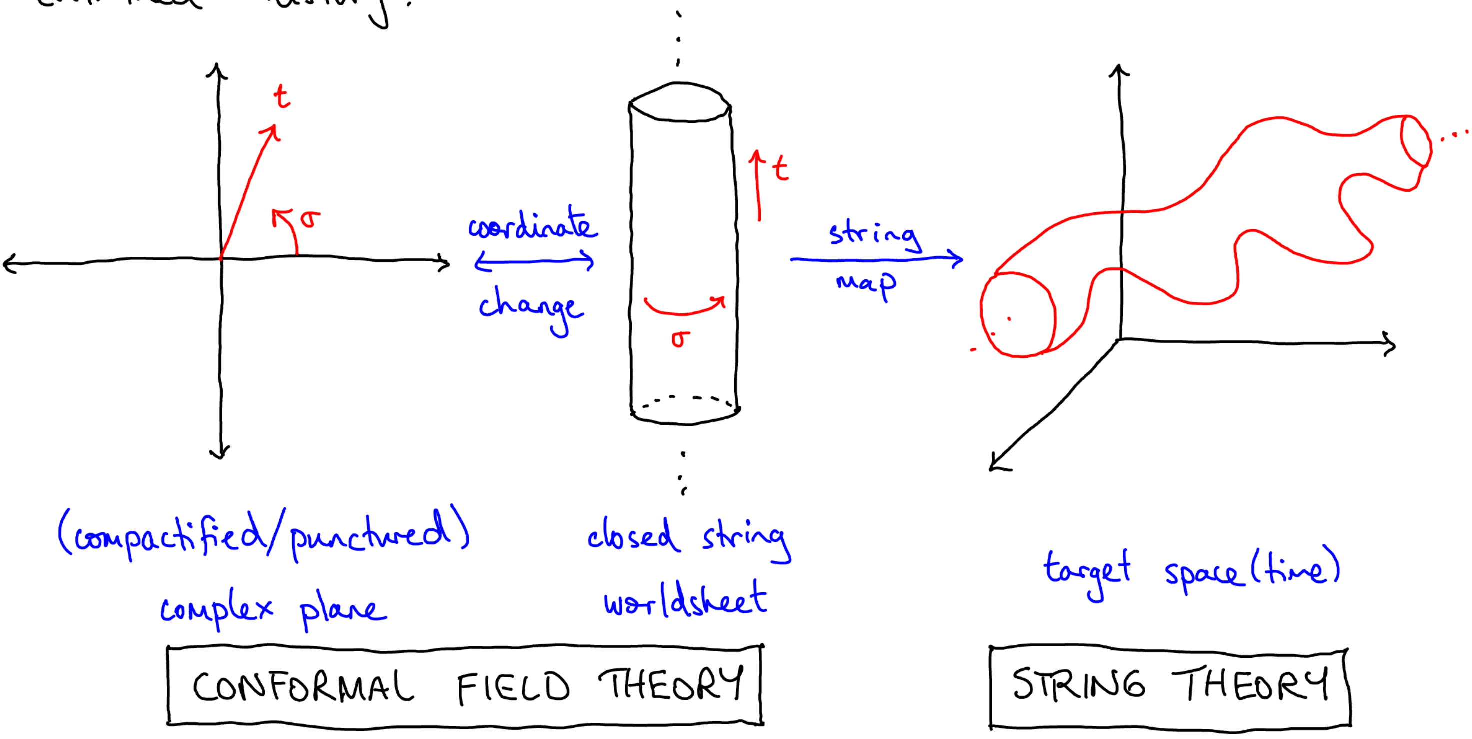
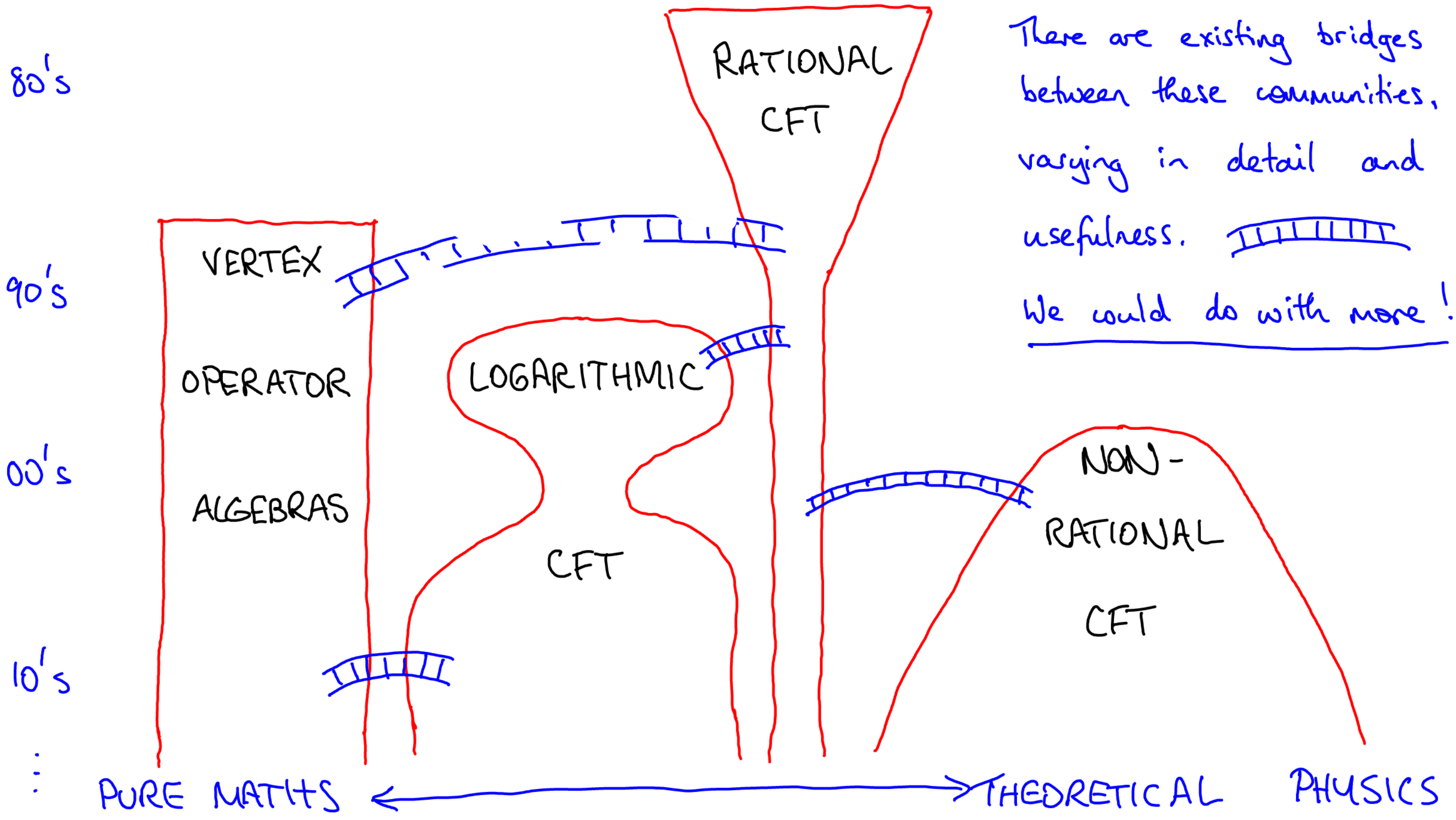
# The Nappi-Witten model as a logarithmic CFT (arXiv:2011.14453)

Conformal field theory (CFT) and string theory have a long entwined history!




One popular approach to string theory is to understand the CFT and impose gauge constraints (etc) on it. The first step, understanding the CFT, seems to have developed "parallel" communities:



Textbook string theory has target space $\mathbb{R}^d$ or "easy" compactifications like $\mathbb{T}^n \times \mathbb{R}^{d-n}$.

Generalisations in which the target space is a Lie (super) group are known as Wess-Zumino-Witten (WZW) models. The Nappi-Witten model is one such example. The Lie group $H_4$ is solvable, amounting to a 4-dim. extension of the Heisenberg gp.

The Lie algebra $\mathfrak{h}_4 = \mathrm{span}_{\mathbb{C}}\{\overbrace{E, F}^{\sim\text{raising/lowering}}, \overbrace{I, J}^{\sim\text{Cartan}}\}$ is specified by

$$[E,F] = I, \qquad [J,E] = E, \qquad [J,F] = -F, \qquad [I,J] = [I,E] = [I,F] = 0,$$

cf. the harmonic oscillator (and $\mathfrak{gl}(1|1)$).

- $\mathfrak{h}_4$ is solvable so the Killing form is degenerate $\rightarrow$ Sugawara construction of energy-momentum tensor fails.
- But $\mathfrak{h}_4$ admits a 2-parameter family of non-degenerate (invariant symm.) bilinear forms, so a generalised Sugawara construction applies $\Rightarrow$ conformal invariance. All have signature $(3,1)$ $\Rightarrow$ lorentzian.
- $\mathfrak{h}_4$ also has automorphisms showing that all invariant forms give isomorphic chiral algebras (vertex operator algebras). They also show that the level of the affinisation may be tuned to 1 (cf. free boson / fermion and $GL(1|1)$).
- The central charge is 4 (same as 4-dim. flat space model).

CLAIM: The Nappi-Witten model describes strings propagating in the background of a monochromatic plane wave.

Nappi & Witten show this by inserting a parametrisation of the group into the WZW action and extracting the metric (+ B-field + dilaton).

The spectrum of any CFT is organised into representations of the chiral algebra. Which representations appear defines the "type" of CFT:

| Rep theory | | Completely reducible? YES | Completely reducible? NO |
|---|---|---|---|
| Finitely many irreducibles? | YES | RATIONAL<br>eg. free fermion | LOG-RATIONAL<br>eg. symplectic fermions |
| | NO | NON-RATIONAL<br>eg. free boson | LOGARITHMIC<br>eg. symplectic bosons |

What type is the Nappi-Witten model?

① Vacuum module:

$$\begin{array}{ccc} & \mathbb{1} & \\ F & I, J & E \\ \ddots & \vdots & \ddots \end{array}$$

conf. dim.

I-charge $= 0$

[$I_0$ central]

J-charge

There are no null states, hence no constraints on the spectrum!

$\therefore$ Every Verma module of $\hat{\mathfrak{h}}_4$ is a representation of the chiral algebra.

$\therefore$ There are infinitely many irreducibles.

So, Nappi-Witten is either non-rational or logarithmic...

In fact, there are some reducible but indecomposable Verma modules,

eg. $\mathcal{V}_{0,0}$ $\quad \vdots \cdots \leftarrow SV \xleftarrow{F_0} SV \xleftarrow{F_0} |0,0\rangle \xrightarrow{E_{-1}} \ddots$

I-charge J-charge

Does this mean Nappi-Witten is logarithmic?

**Main result #1**

- The Verma module $\mathcal{V}_{i,j}$ is irreducible iff $i \notin \mathbb{Z}$.
- For $i \in \mathbb{Z}$, the maximal submodule of $\mathcal{V}_{i,j}$ is generated by a singular vector of J-charge $j+1$ $(i>0)$ or $j-1$ $(i \le 0)$ and conformal weight $\Delta_{i,j} + |i|$.

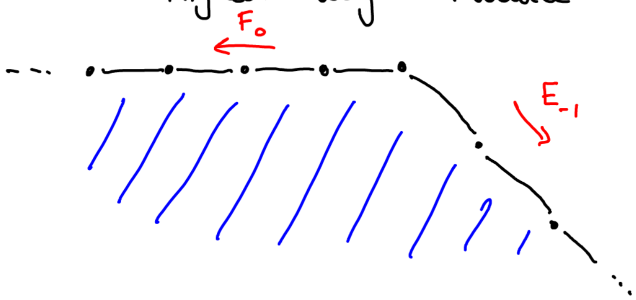
$$\therefore \mathrm{ch}[\mathcal{V}_{i,j}] = \frac{y^i z^{j+\frac{1}{2}} q^{\Delta_{i,j}}}{\sqrt{-1}\,\eta(q)\,\vartheta_1(z;q)} \quad \text{and} \quad \mathrm{ch}[\mathcal{L}_{i,j}] = \left(1 - z^{\mathrm{sgn}(i-\frac{1}{2})} q^{|i|}\right) \mathrm{ch}[\mathcal{V}_{i,j}].$$

In other CFTs with similar representation theories (symplectic bosons, fractional-level $sl_2$, $GL(1|1)$, ...), there is a set of "standard modules" whose characters give a "modular basis" for the space of all characters.

For Nappi-Witten, the standard modules aren't Vermas or their irreducible quotients, but are instead <u>relaxed</u> Verma modules $R_{i,j,\Delta}$ (and their images under "spectral flow").

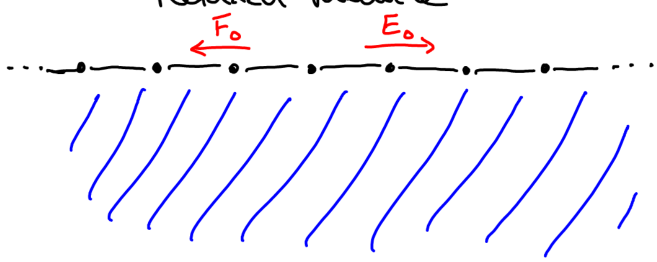
I-charge, J-charge, conf. dim.

Highest-weight module




- Conformal dimension $\Delta_{i,j}$ fixed by I- and J-charges.

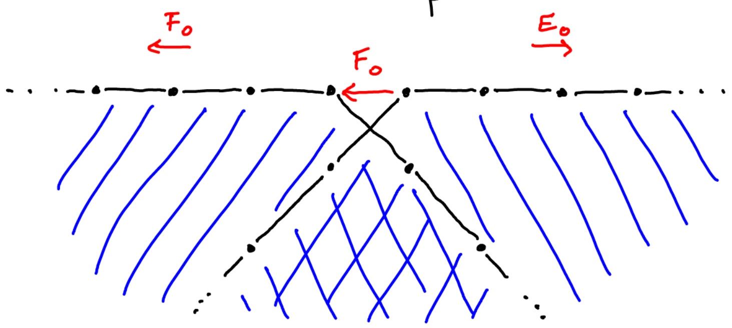
Relaxed module




- Conformal dimension $\Delta$ <u>independent</u> of I- and J-charges.

- Note that relaxed modules "outnumber" the highest-weight ones (3 free parameters vs just 2).

- Every family of relaxed modules (fixed $i, \Delta$; variable $j$) with $i \neq 0$ has reducible but indecomposable members, e.g.




- In this case, the relaxed modules <u>degenerate</u> into a highest-weight module and a conjugate highest-weight module.

- A relaxed Verma module may also have "relaxed singular vectors" so we need to consider irreducible relaxed modules $\mathcal{E}_{i,j,\Delta}$.

## Main result #2

- The relaxed Verma module $\mathcal{R}_{\overset{\text{I-charge}}{i},\overset{\text{J-charge}}{j},\overset{\text{conf. dim}}{\Delta}}$ is irreducible <u>iff</u> $i \notin \mathbb{Z}$ or $i = 0$ and $\Delta \neq 0$.
- For $i \in \mathbb{Z} \setminus \{0\}$, the maximal submodule of $\mathcal{R}_{i,j,\Delta}$ is generated by a relaxed singular vector of conf. dim. $\Delta + |i|$.

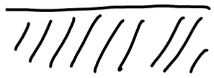
$$\therefore \ \operatorname{ch}[\mathcal{R}_{i,j,\Delta}] = \frac{y^i z^j q^\Delta}{\eta(q)^4} \sum_{n \in \mathbb{Z}} z^n \quad \text{and} \quad \operatorname{ch}[\mathcal{E}_{i,j,\Delta}] = (1 - q^{|i|}) \operatorname{ch}[\mathcal{R}_{i,j,\Delta}].$$

In fact, we have the following "degeneration pattern" of the relaxed Verma modules into irreducibles:

<u>$i \neq 0$</u>

$$\begin{tikzcd}
 & \mathcal{R}_{i,j,\Delta} \arrow[dl] \arrow[dr] & \\
\mathcal{V}_{i,j} \arrow[dr] & & \mathcal{E}_{i,j,\Delta} \arrow[dl] \\
 & \mathcal{L}_{i,j} &
\end{tikzcd}$$

$\mathcal{R}_{i,j,\Delta}$: irreducible for $i \notin \mathbb{Z}$ & $j \neq \Delta/i \bmod 1$



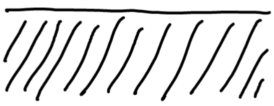
$\mathcal{V}_{i,j}$: irreducible for $i \notin \mathbb{Z}$ & $j = \Delta/i \bmod 1$

$\mathcal{E}_{i,j,\Delta}$: irreducible for $i \in \mathbb{Z}$ & $j \neq \Delta/i \bmod 1$

$\mathcal{L}_{i,j}$: irreducible for $i \in \mathbb{Z}$ & $j = \Delta/i \bmod 1$

<u>$i = 0$</u>

$\mathcal{R}_{0,j,\Delta}$: irreducible for $\Delta \neq 0$



$\mathcal{L}_{0,j}$: irreducible with $\Delta = 0$

## For the future

- This doesn't <u>demonstrate</u> logarithmic behaviour...
- Modularity of characters?
- Fusion $\Rightarrow$ logs??