T-duality - A pedagogical introduction

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ANZ Geometry, Strings and Fields seminar

Friday 13 Nov 2020

Contents

- Introduction to (topological) T-duality
- Spherical T-duality
- Non-isometric T-duality (*not covered*)

Based on collaborations with Mathai and many others (Evslin, Hannabuss, Sati, Wu, Klimčík, Bugden, Wright, ...)

Fourier Transform

Fourier series for $f : S^1 \to \mathbb{R}$

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-inx}\, dx$$

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)\, e^{inx}$$

Fourier transform for $f : \mathbb{R} \to \mathbb{R}$

$$\widehat{f}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(p)\, e^{ipx}\, dp$$

Fourier Transform - cont'd

More generally, for G a locally compact, abelian group, we have a Fourier transform $\mathcal{F} : \mathrm{Fun}(\mathrm{G}) \to \mathrm{Fun}(\widehat{\mathrm{G}})$

$$\widehat{f}(p) = \int_{\mathrm{G}} f(x)\, e^{-ipx}\ dx = \mathcal{F}(f)(p)$$
$$f(x) = \int_{\widehat{\mathrm{G}}} \widehat{f}(p)\, e^{ipx}\ dp$$

where

$$\widehat{\mathrm{G}} = \mathrm{Hom}(\mathrm{G}, \mathrm{U}(1)) = \mathrm{char}(\mathrm{G})$$

is the Pontryagin dual of G. I.e. a character is a U(1) valued function on G, satisfying $\chi(x+y) = \chi(x)\chi(y)$.
The characters form a locally compact, abelian group $\widehat{\mathrm{G}}$ under pointwise multiplication.

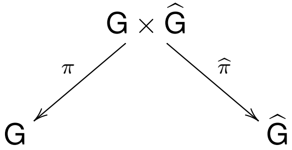
Fourier Transform - cont'd

$$G = S^1\,, \qquad \widehat{G} = \mathbb{Z}\,, \qquad e^{inx}$$
$$G = \mathbb{R}\,, \qquad \widehat{G} = \mathbb{R}\,, \qquad e^{ipx}$$

We can think of $\chi(x,p) = e^{ipx} \in \text{Fun}(G \times \widehat{G})$ as the 'universal' character.
Fourier transform expresses the fact that the characters of G span Fun(G).

Fourier Transform - cont'd

I.e. we have the following "correspondence"



$$\mathcal{F}f = \widehat{\pi}_*(\pi^*(f) \times \chi(x,p))$$

Fourier Transform - Geometric generalisations

T-duality is a geometric version of harmonic analysis, i.e. by replacing functions by geometric objects (such as bundles, sheaves, D-modules, ...) or, as an intermediate step, by topological characteristics associated to these objects (cohomology, K-theory, derived categories, ...).

Fourier-Mukai transform

Consider a manifold $P = M \times S^1$. By the Künneth theorem we have

$$H^\bullet(P) \cong H^\bullet(M) \otimes H^\bullet(S^1)$$

I.e.

$$H^n(P) \cong H^n(M) \oplus H^{n-1}(M)$$

We have a similar decomposition at the level of forms

$$\Omega^n(P)^{\text{inv}} \cong \Omega^n(M) \oplus \Omega^{n-1}(M)\,.$$

I.e. invariant degree n forms on P are of the form ω or $\omega \wedge d\theta$, where ω is an n, respectively $n-1$, form on M.

Consider $\widehat{P} = M \times \widehat{S}^1$. We have an isomorphism

$$\mathcal{F} \ : \ H^{\bar{i}}(P) \xrightarrow{\ \cong\ } H^{\overline{i+1}}(\widehat{P})$$

Fourier-Mukai transform - cont'd

where

$$H^{\bar{0}}(P) = \bigoplus_{i\geq 0} H^{2i}(P)\,, \quad H^{\bar{1}}(P) = \bigoplus_{i\geq 0} H^{2i+1}(P)\,,$$

Explicitly

$$\omega \ \mapsto \ d\widehat{\theta} \wedge \omega\,, \qquad d\theta \wedge \omega \ \mapsto \ \omega$$

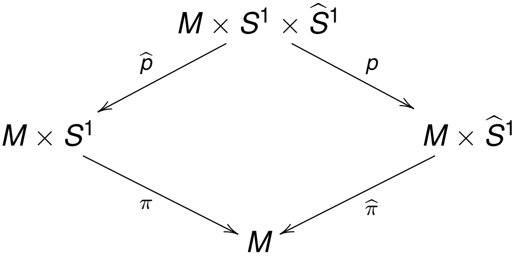
or

$$\mathcal{F}\Omega = \int_{S^1} (1 + d\theta \wedge d\widehat{\theta})\,\Omega = \int_{S^1} e^{d\theta\wedge d\widehat{\theta}}\,\Omega = \int_{S^1} e^{F}\,\Omega$$

Fourier-Mukai transform - cont'd

I.e. $\mathcal{F}$ is given by a correspondence

$$\mathcal{F}\Omega = p_* \left(\widehat{p}^* \, \Omega \wedge e^F \right)$$



Fourier-Mukai transform - cont'd

Once we recognize that $F = d\theta \wedge d\widehat{\theta}$ is the curvature of a canonical linebundle $\mathcal{P}$ (the Poincaré linebundle) over $S^1 \times \widehat{S}^1$, in fact $e^F = \mathrm{ch}(\mathcal{P})$, this immediately suggests a 'geometrization' in terms of vector bundles over P and $\widehat{P}$. $^{(*)}$

$$\mathcal{F}E = p_* \left(\widehat{p}^* E \otimes \mathcal{P}\right)$$

This gives rise to the so-called Fourier-Mukai transform

$$\mathcal{F} \; : \; K^i(P) \xrightarrow{\cong} K^{i+1}(\widehat{P})$$

which has many of the properties of the Fourier transform discussed earlier.

The discussion can be generalized to complexes of vector bundles (complexes of sheaves) and thus gives rise to a Fourier-Mukai correspondence between derived categories $D(P)$ and $D(\widehat{P})$.

T-duality - Closed string on $M \times S^1$

Closed strings on $M \times S^1$ are described by

$$X : \Sigma \to M \times S^1$$

where $\Sigma = \{(\sigma, \tau)\}$ is the closed string worldsheet.
Upon quantization, we find

- Momentum modes: $p = \frac{n}{R}$
- Winding modes: $X(0, \tau) \sim X(1, \tau) + mR$

$$E = \left(\frac{n}{R}\right)^2 + (mR)^2 + \text{osc. modes}$$

We have a duality $R \to 1/R$, such that ST on $M \times S^1$ is equivalent to ST on $M \times \widehat{S}^1$ (or a duality between IIA and IIB ST, for susy ST)

T-duality - Principal S^1-bundles

Suppose we have a pair (P, H), consisting of a principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & P \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

and a so-called H-flux H on P, a Čech 3-cocycle.

Topologically, P is classified by an element in $F \in H^2(M, \mathbb{Z})$ while H gives a class in $H^3(P, \mathbb{Z})$

T-duality - Principal S^1-bundles

The (topological) T-dual of (P, H) is given by the pair $(\widehat{P}, \widehat{H})$, where the principal S^1-bundle

$$\begin{array}{ccc} \widehat{S}^1 & \longrightarrow & \widehat{P} \\ & & \big\downarrow \widehat{\pi} \\ & & M \end{array}$$

and the dual H-flux $\widehat{H} \in H^3(\widehat{P}, \mathbb{Z})$, satisfy

$$\widehat{F} = \pi_* H \,, \qquad F = \widehat{\pi}_* \widehat{H}$$

where $\pi_* : H^3(P, \mathbb{Z}) \to H^2(M, \mathbb{Z})$, is the pushforward map ('integration over the S^1-fibre').

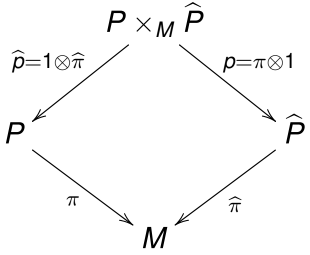
T-duality - Principal S^1-bundles

The ambiguity in the choice of $\widehat{H}$ is (almost) removed by requiring that

$$\widehat{p}^* H - p^* \widehat{H} \equiv 0 \qquad \in H^3(P \times_M \widehat{P}, \mathbb{Z})$$

where $P \times_M \widehat{P}$ is the correspondence space

$$P \times_M \widehat{P} = \{(x, \widehat{x}) \in P \times \widehat{P} \mid \pi(x) = \widehat{\pi}(\widehat{x})\}$$



T-duality - Principal S^1-bundles

Gysin sequences

$$\cdots \longrightarrow H^3(M) \xrightarrow{\pi^*} H^3(P) \xrightarrow{\pi_*} H^2(M) \xrightarrow{\cup F} H^4(M) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^3(M) \xrightarrow{\widehat{\pi}^*} H^3(\widehat{P}) \xrightarrow{\widehat{\pi}_*} H^2(M) \xrightarrow{\cup \widehat{F}} H^4(M) \longrightarrow \cdots$$

T-duality - Principal S^1-bundles

$$
\begin{array}{ccccccccccc}
0 & \xrightarrow{\cup \widehat{F}} & H^1(M) & \xrightarrow{\widehat{\pi}^*} & H^1(\widehat{P}) & \xrightarrow{\widehat{\pi}_*} & H^0(M) & \xrightarrow{\cup \widehat{F}} & H^2(M) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \cup F} & & \downarrow{\scriptstyle \cup F} & & \downarrow{\scriptstyle \cup \widehat{\pi}^* F} & & \downarrow{\scriptstyle \cup F} & & \downarrow{\scriptstyle \cup F} & & \\
H^1(M) & \xrightarrow{\cup \widehat{F}} & H^3(M) & \xrightarrow{\widehat{\pi}^*} & H^3(\widehat{P}) & \xrightarrow{\widehat{\pi}_*} & H^2(M) & \xrightarrow{\cup \widehat{F}} & H^4(M) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \\
H^1(P) & \xrightarrow{\cup \pi^* \widehat{F}} & H^3(P) & \xrightarrow{\widehat{p}^*} & H^3(P \times_M \widehat{P}) & \xrightarrow{\widehat{p}_*} & H^2(P) & \xrightarrow{\cup \pi^* \widehat{F}} & H^4(P) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\
H^0(M) & \xrightarrow{\cup \widehat{F}} & H^2(M) & \xrightarrow{\widehat{\pi}^*} & H^2(\widehat{P}) & \xrightarrow{\widehat{\pi}_*} & H^1(M) & \xrightarrow{\cup \widehat{F}} & H^3(M) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

T-duality - Examples

Consider principal S^1-bundles P over $M = S^2$, then

$$H^2(M, \mathbb{Z}) \cong \mathbb{Z}\,, \qquad H^3(P, \mathbb{Z}) \cong \mathbb{Z}$$

and we have, for example,

$$(S^2 \times S^1, 0) \longrightarrow (S^2 \times S^1, 0)$$

$$(S^2 \times S^1, 1) \longrightarrow (S^3, 0)$$

or more generally

$$(L_p, k) \longrightarrow (L_k, p)$$

where $L_p = S^3/\mathbb{Z}_p$ is the lens space.

T-duality - Twisted cohomology

Using $\Omega^k(P)^{\text{inv}} \cong \Omega^k(M) \oplus \Omega^{k-1}(M)$

$$F = dA\,, \qquad H = H_{(3)} + A \wedge H_{(2)}$$

we find

$$\widehat{F} = H_{(2)} = d\widehat{A}\,, \qquad \widehat{H} = H_{(3)} + \widehat{A} \wedge F$$

such that

$$\widehat{H} - H = \widehat{A} \wedge F - A \wedge \widehat{F} = d(A \wedge \widehat{A})\,.$$

Theorem

We have an isomorphism of ($\mathbb{Z}_2$-graded) differential complexes

$$T_* : \ (\Omega(P)^{inv}, d_H) \longrightarrow (\Omega(\widehat{P})^{inv}, d_{\widehat{H}})$$

where $d_H = d + H\wedge$.

Proof.

Define

$$T_*\omega = \int_{S^1} e^{A\wedge\widehat{A}}\,\omega$$

then

$$d_H\, T_* = T_*\, d_{\widehat{H}}\,.$$

□

and consequently, we have isomorphisms

$$T_* \;:\; H^{\bar{i}}(P,H) \xrightarrow{\;\cong\;} H^{\overline{i+1}}(\widehat{P},\widehat{H})$$

T-duality - Twisted cohomology

as well as

$$T_* \; : \; K^i(P, H) \xrightarrow{\;\cong\;} K^{i+1}(\widehat{P}, \widehat{H})$$

For example,

$$K^i(L_p, k) \cong \begin{cases} \mathbb{Z}_k & i = 0 \\ \mathbb{Z}_p & i = 1 \end{cases}$$

	String Theory $M_4 \times Y_6$	
 $\mathcal{N} = 1$ $\mathcal{N} = 2$ $\mathcal{N} = 3$	Complex manifold Kähler Calabi-Yau Hyper-Kähler	
	S^1 Strings $H \in \mathrm{H}^3(Y, \mathbb{Z})$ Mirror Symmetry / T-duality generalized geometry	
	$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$	

	String Theory $M_4 \times Y_6$	M-Theory / 11D SUGR $M_4 \times Y_7$
 $\mathcal{N} = 1$ $\mathcal{N} = 2$ $\mathcal{N} = 3$	Complex manifold Kähler Calabi-Yau Hyper-Kähler	Contact manifold Sasakian Sasaki-Einstein 3-Sasakian
	S^1 Strings $H \in \mathrm{H}^3(Y, \mathbb{Z})$ Mirror Symmetry / T-duality generalized geometry	S^3 2- and 5-branes $H \in \mathrm{H}^7(Y, \mathbb{Z})$ Spherical T-duality? M-geometry?
	$S^1 \longrightarrow S^3$ $\downarrow$ S^2	$S^3 \longrightarrow S^7$ $\downarrow$ S^4

Much of the above can be generalized to principal SU(2)-bundles:
Gysin sequence for principal SU(2)-bundles $\pi : P \to M$

$$\cdots \longrightarrow H^7(M) \xrightarrow{\pi^*} H^7(P) \xrightarrow{\pi_*} H^4(M) \xrightarrow{\cup c_2(P)} H^8(M) \longrightarrow \cdots$$

where

$$c_2(P) = \frac{1}{8\pi^2}\,\mathrm{Tr}(F \wedge F) \in H^4(M)$$

is (a de Rham representative of) the 2nd Chern class of P.
However, in this case,

$$[M, B\mathrm{SU}(2)] \longrightarrow H^4(M, \mathbb{Z})$$

is, in general, neither surjective nor injective.

Physical derivation of T-duality

A 2D non-linear sigma model describes maps X from a 2-dimensional surface ('worldsheet') Σ to an N-dimensional manifold M ('target'), equipped with additional structure

For example

$$S[X] = \frac{1}{2}\int_\Sigma G_{ij}(X)\, dX^i \wedge \star dX^j + B_{ij}(X)\, dX^i \wedge dX^j$$

Symmetries of sigma model

Given a set of vector fields $v_a(X) = v_a^i(X)\partial_i$ forming a Lie algebra $\mathfrak{g}$

$$[v_a, v_b] = C^c{}_{ab} v_c$$

Consider the infinitesimal transformations

$$\delta_\epsilon X^i = v_a^i(X)\,\epsilon^a$$

we have

$$\delta_\epsilon S = \int_\Sigma \epsilon^a \left((\mathcal{L}_{v_a} G)_{ij}\, dX^i \wedge \star dX^j + (\mathcal{L}_{v_a} B)_{ij}\, dX^i \wedge dX^j \right)$$

The sigma model action is invariant under these transformations if

$$\mathcal{L}_{v_a} G = 0\,, \qquad \mathcal{L}_{v_a} B = 0$$

If this is the case, we can *gauge* the model by promoting the global symmetry to a local one (i.e. take $\epsilon \in C^\infty(\Sigma, \mathfrak{g})$)

The gauged action

Introducing gauge fields $A \in \Omega^1(\Sigma, \mathfrak{g})$ the gauged action is given by

$$S[X, A] = \frac{1}{2} \int_\Sigma G_{ij}(X)\, DX^i \wedge \star DX^j + B_{ij}(X)\, DX^i \wedge DX^j$$

where

$$DX^i = dX^i - v_a^i A^a$$

are the covariant derivatives.

Gauge invariance

The gauged action $S[X, A]$ is invariant with respect to the following (local) gauge transformations:

$$\delta_\epsilon X^i = v^i_a \, \epsilon^a$$
$$\delta_\epsilon A = d\epsilon + [A, \epsilon] = (d\epsilon^a + C^a{}_{bc} \, A^b \, \epsilon^c) \, T_a$$

where T_a is a basis of $\mathfrak{g}$.

Now suppose we want the gauged sigma model to be equivalent to the the ungauged model. Then we need to 'fix the gauge'

Gauge fixing

Introduce the curvature $F \in \Omega^2(\Sigma, \mathfrak{g})$

$$F = dA + A \wedge A = (dA^a + \tfrac{1}{2} C^a{}_{bc} A^b \wedge A^c) T_a = F^a T_a$$

and an 'auxiliary field' $\widehat{X} \in C^\infty(\Sigma, \mathfrak{g}^*)$, with infinitesimal transformation rules

$$\delta_\epsilon F^a = C^a{}_{bc} F^b \epsilon^c$$
$$\delta_\epsilon \widehat{X}_a = -C^c{}_{ab} \widehat{X}_c \epsilon^b$$

Gauge fixing

Consider the action

$$S[X, A, \widehat{X}] = \frac{1}{2} \int_\Sigma \left(G_{ij}(X)\, DX^i \wedge \star DX^j + B_{ij}(X)\, DX^i \wedge DX^j \right) + \int_\Sigma \widehat{X}_a\, F^a$$

The equation of motion for $\widehat{X}_a$ gives $F^a = 0$.

To solve this equation we need to lift the action of $\mathfrak{g}$ to an action of the group G ($\mathfrak{g} = \mathrm{Lie}\,\mathrm{G}$)

Example: Group manifold

Let $g : \Sigma \to \mathrm{G}$

$$S[g] = \frac{1}{2}\int_\Sigma (g^{-1}dg \overset{\wedge}{,} *g^{-1}dg)_G$$

Invariant under left action of $h \in \mathrm{G}$

$$S[hg] = S[g]$$

while

$$S[gh] = \frac{1}{2}\int_\Sigma (\mathrm{Ad}(h^{-1})g^{-1}dg \overset{\wedge}{,} \mathrm{Ad}(h^{-1}) * g^{-1}dg)_G$$

So, invariant under right action of G if G is Ad-invariant (Killing form)

Example: Gauged

In that case we can gauge in the standard way, and obtain the gauged model (with F-term)

$$S[g, A, \widehat{X}] = \frac{1}{2}\int_\Sigma (g^{-1}Dg \overset{\wedge}{,} *g^{-1}Dg)_G + \int_\Sigma \langle \widehat{X}, F \rangle$$

where

$$\begin{aligned} g^{-1}Dg &= g^{-1}dg - A \\ F &= dA + A \wedge A \end{aligned}$$

and gauge symmetry, for $h \in \mathrm{G}$

$$\begin{aligned} g &\to gh \\ A &\to h^{-1}Ah + h^{-1}dh \\ \widehat{X} &\to \mathrm{Ad}^*(h^{-1})\widehat{X} \end{aligned}$$

Example: Gauged model

Solving $F = 0$ gives $A = -dkk^{-1}$ for $k \in C^\infty(\Sigma, \mathrm{G})$, and substituting

$$g^{-1}Dg \to g^{-1}dg + dkk^{-1} = k\big((gk)^{-1}d(gk)\big)k^{-1}$$

I.e.

$$S[g, A = -dkk^{-1}] = S[gk]$$

so after ‘fixing the gauge’ we recover the ungauged model.

On the other hand, first solving the equation of motion for A, and then fixing the gauge, gives dual model

$$\widehat{S}[\widehat{X}] = \frac{1}{2}\int_\Sigma \widehat{G}^{ab}(\widehat{X})\, d\widehat{X}_a \wedge \star d\widehat{X}_b$$

with dual ‘metric’

$$\widehat{G}^{-1}{}_{ab} = G_{ab} - C^c{}_{ab}\widehat{X}_c$$

Abelian T-duality

Suppose we have a $U(1)^N$ isometry $X^m \to X^m + \epsilon^m$, then the T-duality rules are given by the Buscher rules

$$\widehat{Q}_{ij} = \begin{pmatrix} \widehat{Q}_{\mu\nu} & \widehat{Q}_{\mu n} \\ \widehat{Q}_{m\nu} & \widehat{Q}_{mn} \end{pmatrix} = \begin{pmatrix} Q_{\mu\nu} - Q_{\mu m}(Q^{-1})^{mn} Q_{n\nu} & -Q_{\mu m}(Q^{-1})^{m}{}_{n} \\ (Q^{-1})_{m}{}^{n} Q_{n\nu} & (Q^{-1})_{mn} \end{pmatrix}$$

where $Q_{ij} = G_{ij} + B_{ij}$.

More explicitly, for a $U(1)$ isometry,

$$
\begin{aligned}
\widehat{G}_{\bullet\bullet} &= \frac{1}{G_{\bullet\bullet}} \\
\widehat{G}_{\bullet\mu} &= \frac{B_{\bullet\mu}}{G_{\bullet\bullet}} \\
\widehat{G}_{\mu\nu} &= G_{\mu\nu} - \frac{1}{G_{\bullet\bullet}}\left(G_{\bullet\mu}G_{\bullet\nu} - B_{\bullet\mu}B_{\bullet\nu}\right) \\
\widehat{B}_{\bullet\mu} &= \frac{G_{\bullet\mu}}{G_{\bullet\bullet}} \\
\widehat{B}_{\mu\nu} &= B_{\mu\nu} - \frac{1}{G_{\bullet\bullet}}\left(G_{\bullet\mu}B_{\bullet\nu} - G_{\bullet\nu}B_{\bullet\mu}\right)
\end{aligned}
$$

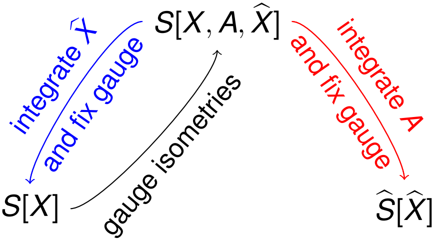
Abelian T-duality

Letting

$$G = G_{\mu\nu} dX^\mu \otimes dX^\nu + (d\theta + A_\mu dX^\mu) \otimes (d\theta + A_\nu dX^\nu)$$
$$B = \tfrac{1}{2} B_{\mu\nu} dX^\mu \wedge dX^\nu + B_{\mu\bullet} dX^\mu \wedge (d\theta + A_\nu dX^\nu)$$

gives back the T-duality rules for $H = dB$ and $F = dA$ as discussed before.

T-duality


$S[X, A, \widehat{X}]$
integrate $\widehat{X}$
and fix gauge
gauge isometries
integrate A
and fix gauge
$S[X]$
$\widehat{S}[\widehat{X}]$

THANKS