Toroidal quantum groups

in gauge and string theories

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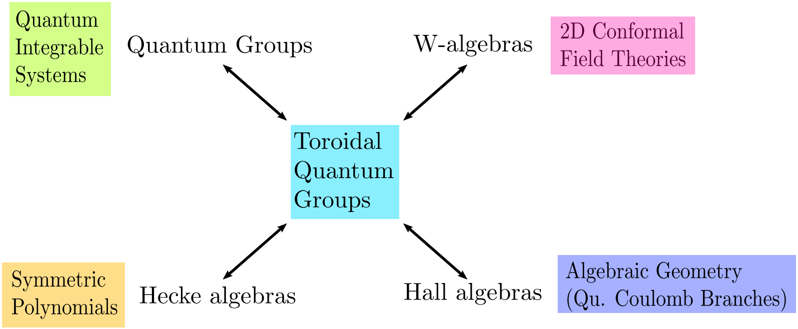
Why toroidal quantum groups?



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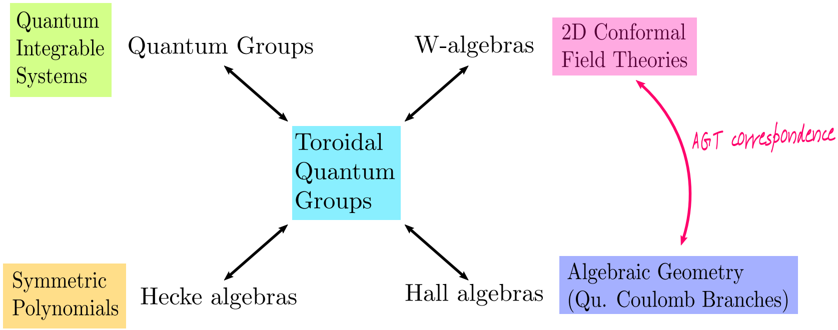

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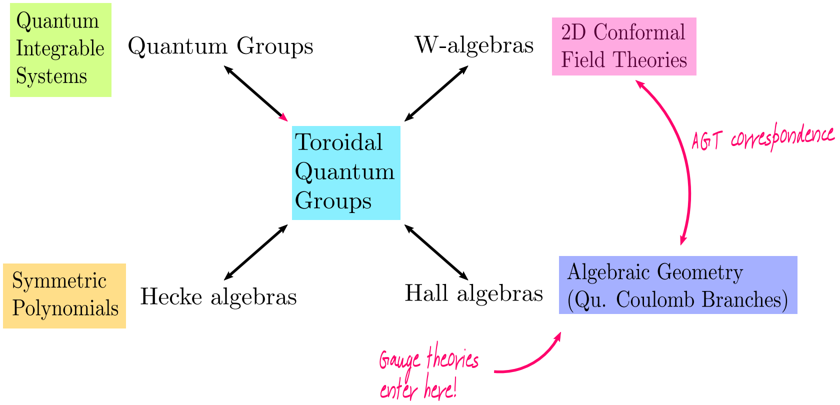

Where are toroidal quantum groups?

They appear in gauge and string theories in the context of:

- Proof of the **AGT correspondence**

  (between 4D $\mathcal{N} = 2$ SUSY gauge theories and 2D Conformal Field Theories).

  [Schiffmann, Vasserot 2012]

- Construction of the topological vertex of (refined) **topological string theories**.

  [Awata, Feigin, Shiraishi 2011]

- **BPS algebras** and Corner Vertex Operator algebras.

  [Gaiotto, Rapcak 2017] [Yamazaki, Li 2020]

Outline

I. Introduction

II. What are toroidal quantum groups?

III. AGT correspondence

IV. Topological strings and brane systems

V. Algebraic engineering

VI. Discussion

Deformation of Lie algebras?

From a Lie algebra $\mathfrak{g}$ with generators $e_\alpha,\ f_\alpha,\ h_\alpha$, define:

- Affinization: Deformation of the loop algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c$
  - ⇝ Generators $e_{\alpha,n} \sim e_\alpha \otimes t^n$, $f_{\alpha,n}$, $h_{\alpha,n}$ $\Rightarrow$ currents $e_\alpha(z) = \sum z^{-n} e_{\alpha,n}$.
- Quantization: $U_q(\mathfrak{g})$ is a deformation of the universal enveloping algebra $U(\mathfrak{g})$
  - ⇝ Generators e_α, f_α, $\psi_\alpha^\pm = q^{\pm h_\alpha}$ (new parameter q)

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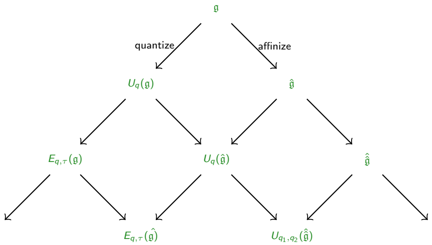
Repeating these procedures, we find the following picture:



Quantum affine algebras $U_q(\hat{\mathfrak{g}}_\Gamma)$ are generated by:

- Chevalley generators e_α, f_α, $\psi_\alpha^\pm$ with $\alpha \in \hat{\Gamma}$ **(Drinfeld-Jimbo presentation)**
- Drinfeld currents $e_\alpha(z)$, $f_\alpha(z)$, $\psi_\alpha^\pm(z)$ with $\alpha \in \Gamma$ **(Drinfeld presentation)**

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Example:

$\rightsquigarrow$ Quantum affine $\mathfrak{gl}_2$ generated by $e(z)$, $f(z)$, $\psi^\pm(z)$ (or $e_\alpha,\ f_\alpha,\ \psi_\alpha^\pm$ with $\alpha = 0, 1$)

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Remark: Same story for Yangians in the degenerate limit $q \simeq 1 + \hbar + \cdots$

Outline

Supersymmetric gauge theories

- We discuss here a class of gauge theories preserving **8 supercharges**:
  - 4D $\mathcal{N} = 2$ gauge theories on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$
  - 5D $\mathcal{N} = 1$ gauge theories on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1$

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- Field content:
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  - Hypermultiplets: quarks + superpartners

- The theory is fully determined by specifying:
  - the gauge group G ($\rightsquigarrow$ coupling g_{YM})
  - the representation R of the hypermultiplets ($\rightsquigarrow$ masses m_a)

$\rightsquigarrow$ We also need to specify the vacua $\langle\phi\rangle$ (Coulomb branch vevs)

- The partition function decomposes into factorized contributions: classical, one-loop and instantons

$$\mathcal{Z} = \mathcal{Z}_{\text{cl.}}\mathcal{Z}_{\text{1-loop}}\mathcal{Z}_{\text{inst.}}$$

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> the elementary particles, are all different from what they were in the false vacuum, and thus the observer is no longer capable of functioning biologically, or even chemically.) Since even 10^{-10} sec is considerably less than the response time of a single neuron, there is literally nothing to worry about; if a bubble is coming toward us, we shall never know what hit us.

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- When $G = U(N)$, the instanton contribution is a sum over N-tuples partitions $\boldsymbol{\lambda}$,

$$\mathcal{Z}_{\text{inst.}} = \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} N_{\boldsymbol{\lambda},\boldsymbol{\lambda}}^{-1}, \quad \mathfrak{q} \simeq e^{-1/g_{\text{YM}}^2}.$$

[Nekrasov 2001]

AGT correspondence

6D $\mathcal{N} = (2,0)$ gauge theory on $\mathbb{C}^2 \times \Sigma$

4D $\mathcal{N} = 2$ gauge theory on $\mathbb{C}^2$

$G = \bigotimes U(2)$

$G = \bigotimes U(N)$

2D Conformal Field Theory on Σ

Liouville CFT (Virasoro algebra)

A_{N-1} Toda CFT (W_N algebra)

AGT correspondence

[Alday, Gaiotto, Tachikawa 2009] [Wyllard 2009]

Focusing on $U(2)$/Liouville correspondence:

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⇝ The **instanton partition function** is related to the **four points conformal block**

$$\mathcal{Z}_{\text{inst.}} = \mathcal{N}_{\text{U(1)}} \times \langle V_{\alpha_1}(\infty) V_{\alpha_2}(1) V_{\alpha_3}(\mathfrak{q}) V_{\alpha_4}(0) \rangle \,.$$

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⇝ The partition function obtained as a norm of the **Gaiotto state**:

$$\mathcal{Z}_{\text{inst.}} = \langle G \| G \rangle \,, \quad \text{with} \quad L_1 \left| G \right\rangle = \mathfrak{q}^{1/2} \left| G \right\rangle \,, \quad L_{n>1} \left| G \right\rangle = 0.$$

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The CFT quantities are determined by the Virasoro symmetry

⇒ Where does it come from on the gauge side???

Instanton moduli space

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- $\mathfrak{t}$ also acts on the tangent space at the fixed point $T_{\boldsymbol{\lambda}}\mathcal{M}_k$.

The character $\chi_{\boldsymbol{\lambda},\boldsymbol{\lambda}}$ of this action produces the Nekrasov factor $N_{\boldsymbol{\lambda},\boldsymbol{\lambda}} = \mathbb{I}[\chi_{\boldsymbol{\lambda},\boldsymbol{\lambda}}]$.

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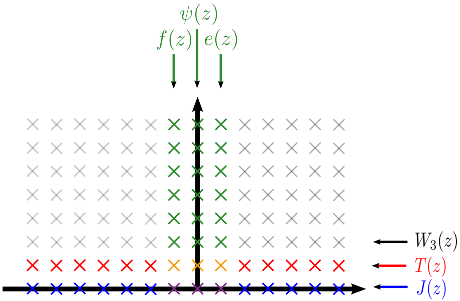
- It also defines an action of the **affine Yangian of $\mathfrak{gl}(1)$**,

$$e(z)\,|\boldsymbol{\lambda}\rangle\rangle = \sum_{\square \in A(\boldsymbol{\lambda})} \frac{1}{z - \phi_\square} \mathbb{I}[\chi_{\lambda+\square,\lambda} - \chi_{\lambda+\square,\lambda+\square}]\,|\boldsymbol{\lambda} + \square\rangle\rangle,$$

$$f(z)\,|\boldsymbol{\lambda}\rangle\rangle = \sum_{\square \in R(\boldsymbol{\lambda})} \frac{1}{z - \phi_\square} \mathbb{I}[\chi_{\lambda,\lambda-\square} - \chi_{\lambda-\square,\lambda-\square}]\,|\boldsymbol{\lambda} - \square\rangle\rangle.$$

$\rightsquigarrow$ The instanton partition function is covariant under this action!

- This algebra contains Virasoro and W_N subalgebras as “higher generators”:



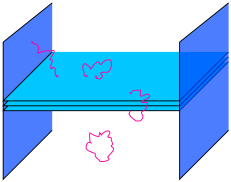
$\Rightarrow$ Algebraic proof of the AGT correspondence!

[Schiffmann, Vasserot 2012]

Outline

Brane systems

- Branes are higher dimensional versions of quantum strings, but much heavier!

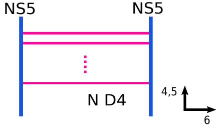


Low energy excitations given by **open strings** (gauge fields) while **closed strings** (gravity) can be decoupled.

Gauge theories describe the low-energy dynamics of brane systems.

- 4D $\mathcal{N} = 2$ gauge theories are obtained by suspending D4-branes between NS5-branes:

Pure $U(N)$ theory

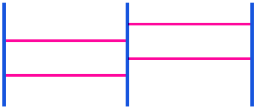


$\overbrace{\qquad\qquad\qquad}^{\mathbb{C}_{\varepsilon_1} \otimes \mathbb{C}_{\varepsilon_2}}$ (over columns 0–3)

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D4	x	x	x	x			x			

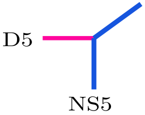
$U(2)$ with $N_f = 4$ fund. flavors

Pure $U(2) \times U(2)$



- 5D $\mathcal{N} = 1$ gauge theories are defined by (p, q)-branes:

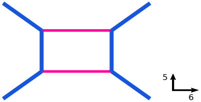
$\rightsquigarrow$ When D5 and NS5 meet, they form a bound state $=$ $(1, 1)$-brane



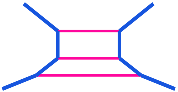
$\overbrace{\quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad}^{\mathbb{C}_{\varepsilon_1} \otimes \mathbb{C}_{\varepsilon_2} \otimes S^1}$

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
(p, q)	x	x	x	x	x	θ	θ			

Pure $U(2)$



Pure $U(3)$



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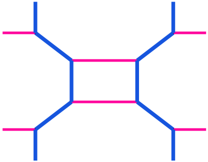
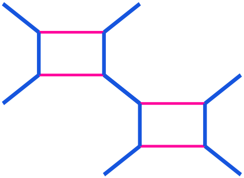


Pure $U(2) \times U(2)$



Topological vertex

- Partition functions can be obtained from the correspondence:

5D $\mathcal{N} = 1$ gauge theory		Top. strings on toric CY 3-fold
(p, q)-brane web	⟷	toric diagram
partition function $\mathcal{Z}_{\text{inst.}}$		string amplitude

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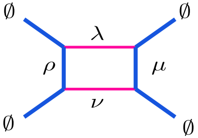
- Compute $\mathcal{Z}_{\text{inst}}$ through a diagrammatic technique using the **topological vertex**:



$$\mathcal{Z}_{\text{inst.}} \sim \sum_{\lambda,\mu,\nu,\rho} C_{\emptyset,\lambda,\rho} C_{\rho,\nu,\emptyset} C_{\nu,\mu,\emptyset} C_{\lambda,\emptyset,\mu}$$

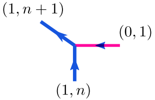
Quantum toroidal $\mathfrak{gl}(1)$

- The topological vertex gives the **matrix element** of interwiners between representations of the **quantum toroidal $\mathfrak{gl}(1)$ algebra** (Ding-Iohara-Miki algebra):

$$C_{\lambda,\mu,\nu} = \langle\lambda|\,\Phi\left(|\mu\rangle\!\rangle \otimes |\nu\rangle\right) = \left(\langle\!\langle\lambda| \otimes \langle\mu|\right)\Phi^*\,|\nu\rangle\,.$$

$$\Phi : \mathcal{F}_{0,1} \otimes \mathcal{F}_{1,n} \to \mathcal{F}_{1,n+1}$$

Fock space



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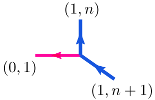
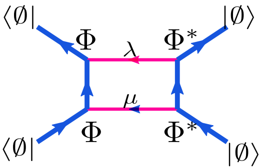


[Awata, Feigin, Shiraishi 2011]

- The calculation of top. strings amplitudes can be reformulated in this language.

⇝ It is even more efficient!!!



$$\mathcal{Z}_{\text{inst.}} = \left(\langle\emptyset| \otimes \langle\emptyset|\right) \sum_{\lambda,\mu} \left(\Phi_\lambda \Phi_\mu \otimes \Phi^*_\lambda \Phi^*_\mu\right) \left(|\emptyset\rangle \otimes |\emptyset\rangle\right)$$

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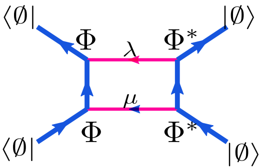

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- How do we find the interwiners?

⇝ Need to define the action on $\mathcal{F}_{0,1}$ (COHA) and $\mathcal{F}_{1,n}$ (vertex rep.)

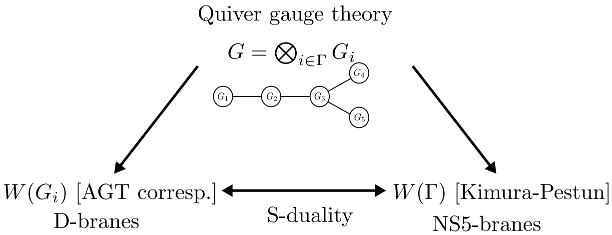
⇝ Solve the algebraic constraint by expanding on the instanton basis $|\lambda\rangle\rangle$.

⇝ Obtain Φ_λ and Φ^*_λ as vertex operators.

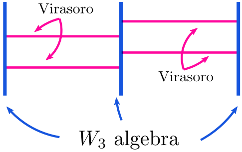
W-algebra

- The AGT-correspondence can also be understood in this framework.

⇝ In fact, two W-algebras are involved for a quiver gauge theory!!!



Pure $U(2) \times U(2)$



This observation provides another proof of the AGT corresp. for 5D gauge theories:

$\rightsquigarrow$ The KP algebra is obtained as $\mathcal{F}_{1,n_1} \otimes \cdots \otimes \mathcal{F}_{1,n_N} \simeq u(1) \otimes q - W_N$ **[Feigin et. al. 2009]**

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$\Rightarrow$ Combine both ingredients to derive the AGT correspondence!

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- This construction can be generalized to different brane systems and different observables.

$\rightsquigarrow$ It is the starting point for the algebraic engineering of gauge theories.

Outline

Algebraic Engineering

Goal: Realize SUSY gauge theories in the representation theory of a quantum group.

$\rightsquigarrow$ The **brane realization** leads to the construction of a **network of representations**.

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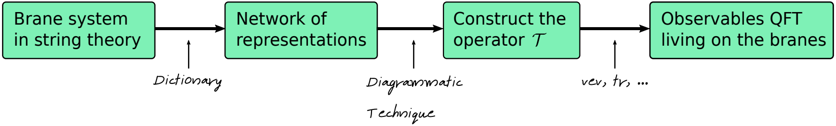

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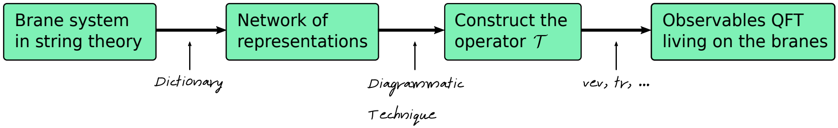

- Main results of the form:

$$\mathcal{Z} = \langle \mathcal{T} \rangle$$

Partition Function — Operator acting on repr. network

$$\chi(z) = \langle X(z) \mathcal{T} \rangle$$

qq-character (BPS loop observable) — Algebra current

Main results

- Over the last few years, this program has been applied successfully to several supersymmetry QFT and their brane systems:

Theories	Algebra	Reference
5D $\mathcal{N}=1$	quantum toroidal $\mathfrak{gl}(1)$	**[Awata, Feigin, Shiraishi 2011]**
5D $\mathcal{N}=1$ on $\mathbb{Z}_p$-orbifold	quantum toroidal $\mathfrak{gl}(p)$	**[Awata, Kanno, et. al. 2017]**
5D $\mathcal{N}=1$ on $\mathbb{Z}_p$-orbifold	new quantum toroidal algebras!	**[JEB, Jeong 2019]**
4D $\mathcal{N}=2$	affine Yangian $\mathfrak{gl}(1)$	**[JEB, Zhang 2018]**
6D $\mathcal{N}=(1,0)$	elliptic toroidal $\mathfrak{gl}(1)$	**[Foda, Zhu 2018]**
3D $\mathcal{N}=2^*$	quantum toroidal $\mathfrak{gl}(1)$	**[Zenkevich 2018]**
3D $\mathcal{N}=2$	quantum affine $\mathfrak{sl}(2)$	**[JEB 2107.10063]**

- Other results:

$\rightsquigarrow$ *D*-type quiver gauge theories **[JEB, Fukuda, Matsuo, Zhu 2017]**

$\rightsquigarrow$ qq-characters observables (generating function of Wilson loops)

[JEB, Fukuda, Harada, Matsuo, Zhu 2017]

Dictionary

- The choice of algebra reflects the background of the quantum field theory:
  - Factor $\mathbb{C}_\epsilon \rightsquigarrow$ **affinization**
  - Factor S^1, $S^1 \times S^1 \rightsquigarrow$ **trigonometric/elliptic** deformations
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- To each brane is associated a module:
  - D5/NS5 branes $\rightsquigarrow$ 2D boson Fock space $\mathcal{F}$ (equivalence $\mathcal{S}$-duality)
  - D3 $\rightsquigarrow$ $\text{Span}\{|k\rangle\rangle, k \geq 0\}$

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- Hypermultiplets can be introduced as shifted representations $\rightsquigarrow$ transverse D7-branes

[JEB 2107.10063]

Outline

Summary

Some of the advantages of the algebraic engineering are:

- Diagrammatic technique that simplifies actual calculations ("CFT methods")
- Local description of W-algebras
- Emphasize the integrable aspects of gauge theories (Bethe/gauge correspondence)
- Algebraic description of branes (e.g. crossings = R-matrix) **[Zenkevich 2018]**

What's next?

I. Address more complicated brane systems

- Generalized ADHM constructions: gauge origami, Magnificent four,...
  - ⇝ Need quantum groups of higher genus (more affinization)
- Beyond equivariant localization? Action on JK-residues?
  - ⇝ Analogy with massive integrable field theories **[Lukyanov 1995]**
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II. Understand how to combine different quantum groups

- Necessary to address general backgrounds, or understand surface defects.
- Recent progress in the context of VOA for 3- and 4-manifolds.

  [Feigin, Gukov 2018] [Cheng, Chun, Feigin, Ferrari, Gukov, Harrison, Passaro 2022]

  $\Rightarrow$ Similar gluing procedure for quantum groups?

III. Extend the derivation to more general observables

$\rightsquigarrow$ Exploit results and techniques from integrable systems

$\Rightarrow$ Need a better understanding of the role played by toroidal quantum groups in the Bethe/gauge correspondence.

$\rightsquigarrow$ E.g. Algebraic Bethe Ansatz **[Fukuda, Ohkubo, Shiraishi 2019]**.

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Thank you!

Bonus slides!

Quantum toroidal $\mathfrak{gl}(1)$ relations

$$[\psi^{\pm}(z),\psi^{\pm}(w)]=0,\quad \psi^{+}(z)\psi^{-}(w)=\frac{g(\hat{\gamma}z/w)}{g(\hat{\gamma}^{-1}z/w)}\psi^{-}(w)\psi^{+}(z),$$

$$\psi^{+}(z)x^{\pm}(w)=g(\hat{\gamma}^{\pm 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^{+}(z),$$

$$\psi^{-}(z)x^{\pm}(w)=g(\hat{\gamma}^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^{-}(z),$$

$$x^{\pm}(z)x^{\pm}(w)=g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z),$$

$$[x^{+}(z),x^{-}(w)]=\delta(\hat{\gamma}^{-1}z/w)\psi^{+}(\hat{\gamma}^{1/2}w)-\delta(\hat{\gamma}z/w)\psi^{-}(\hat{\gamma}^{-1/2}w),$$

$$\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\frac{z_2}{z_3}[x^{\pm}(z_1),[x^{\pm}(z_2),x^{\pm}(z_3)]]=0\quad\text{(Serre relations)},$$

with $$g(z)=\frac{(1-q_1z)(1-q_2z)(1-q_3z)}{(1-q_1^{-1}z)(1-q_2^{-1}z)(1-q_3^{-1}z)}.$$

Structure functions

Quantum affine $\mathfrak{sl}(n)$:

$$g_{\alpha,\alpha'}(z) = \begin{pmatrix} A(z) & B(z) & & & & \\ B(z) & A(z) & B(z) & & & \\ & B(z) & A(z) & B(z) & & \\ & & \ddots & \ddots & \ddots & \\ & & & B(z) & A(z) & B(z) \\ & & & & B(z) & A(z) \end{pmatrix}$$

with

$$A(z) = q^{-2}\frac{1-q^2 z}{1-q^{-2}z}, \quad B(z) = q\frac{1-q^{-1}z}{1-qz}$$

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with

$$A(z) = q_3^{-1}\frac{1-q_3 z}{1-q_3^{-1}z}, \quad B(z) = q_3^{1/2}\frac{1-q_1 z}{1-q_2^{-1}z}, \quad C(z) = q_3^{1/2}\frac{1-q_2 z}{1-q_1^{-1}z}$$

BACK

Bethe/gauge correspondence

$\mathcal{Z}_{\text{inst.}} \sim_{\varepsilon_1,\varepsilon_2\to 0} e^{\frac{1}{\varepsilon_1\varepsilon_2}\mathcal{F}}$ [Seiberg, Witten 1994]	$\longleftrightarrow$	SW curve	$\longleftrightarrow$	spectral curve		classical integrable system
$\mathcal{Z}_{\text{inst.}} \sim_{\varepsilon_1\to 0} e^{\frac{1}{\varepsilon_1\varepsilon_2}\mathcal{F}_{\hbar=\varepsilon_2}}$ [Nekrasov, Pestun, Shatashvili 2013]	$\longleftrightarrow$	qSW curve	$\longleftrightarrow$	Baxter TQ equ. T(z) poly.		quantum integrable system
$\mathcal{Z}_{\text{inst.}}$ at finite $\varepsilon_1,\varepsilon_2$ [Nekrasov 2015]	$\longleftrightarrow$	non-perturbative Schwinger-Dyson equations	$\longleftrightarrow$	qq-character $\chi(z)$ poly		quantum toroidal integrable system???

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- qq-character obtained as $\chi(z) = \langle X(z)\mathcal{T}\rangle$ in algebraic engineering.

$\rightsquigarrow$ Polynomiality follows from intertwining property of $\mathcal{T}$, $\langle X(z)\mathcal{T}\rangle = \langle \mathcal{T}X(z)\rangle$.

[JEB, Fukuda, Matsuo, Zhang 2017] (Gaiotto states **[JEB, Matsuo, Zhang 2015]**)

$\rightsquigarrow$ Interpretation as generating function of BPS loop observables **[H.C. Kim 2018]** BACK