Spectral aspects of topological matter with an eye on strings

Guo Chuan Thiang

BICMR, Peking University

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Topological matter: what, why, how?

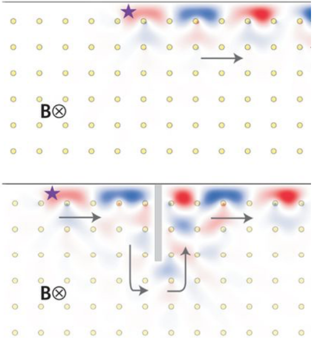
This talk: *Topological matter* (Nobel '16), based on string-math toolkit: Dualities, K-theory, index theory, gerbes, ...

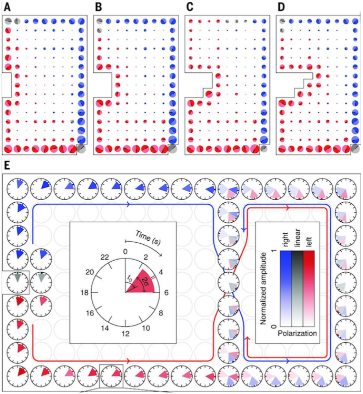
Key **experimental** feature of TM: unusual *spectral* properties which enjoy *topological protection* $\rightsquigarrow$ Index theory!

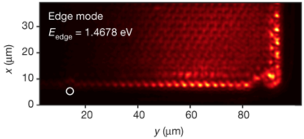
Desirable to understand why *certain* topological invariants compute certain "topological spectral quantities", *before* passing to "effectively topological theories".

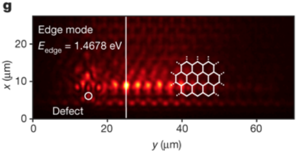
Abstract bulk topological invariants may be undetectable... until they are "transferred" to the boundary as "anomalies". This is the *bulk-boundary correspondence*.

Some experiments[1]: edge-following states









[1] Lu et at, Nature Photonics (2014); Süsstrunk, Huber, Science (2015); Klembt et at, Nature (2018)

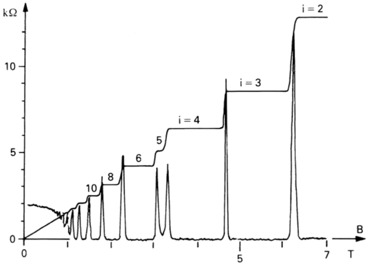
Landau quantisation revisited

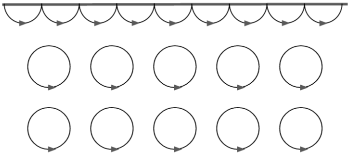
On Euclidean plane, Laplacian $-\nabla^2$ has spectrum $[0, \infty)$.

Landau gauge $A = x\,dy$ for uniform $\perp$ magnetic field $B = dx \wedge dy$.
'28-'30: Discretised spectrum for *Landau Hamiltonian*,

$$H_{\mathrm{Lan}} = -\nabla_A^2 = -\nabla^2 + 2ix\,\partial_y + x^2,$$

$\bullet_1 \quad \bullet_3 \quad \bullet_5 \quad \bullet_7 \quad \bullet_9 \quad \cdots$





Quantum Hall effect in 1980: Robust and universal!

Landau quantisation revisited

Topology enters (TKNN'82): Lattice symmetry $\mathbb{Z}^2 \to$ Fourier transform to quasi-momentum space ("magnetic Brillouin torus").

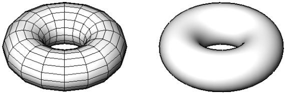
$\to$ Eigenstates below Fermi energy form a vector bundle over $\mathbb{T}^2$, whose Chern class equals Hall conductance. Each Landau level indeed has Chern $= 1$ (Kunz '87).

This relies on idealised Euclidean geometry, lattice symmetry, rational flux,...not truly universal!

Modern POVs: **Gapless edge states**

Index and T-duality [L+T:2009.07688, U. Bunke]

With a choice of lattice, we may reduce H_{Lan} to the *compact* torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, but compensate with moduli $\mathbb{T}^2$ of quasiperiodic boundary conditions.



T-duality

$\sqrt{H_{\mathrm{Lan}}} \rightsquigarrow$ family of (twisted) Diracs on T^2, parametrised by $\mathbb{T}^2$.

Landau levels $\leftrightarrow$ Dirac kernel $\leftrightarrow$ Atiyah–Singer families index!

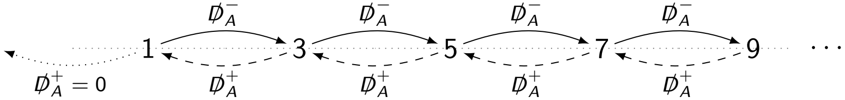
Actually, can dispense with (fictitious) lattice, by taking the so-called *coarse index*, introduced by Roe in '90s.

Landau levels revisited: supersymmetry

$$\not{D}_A = \begin{pmatrix} 0 & i\partial_x + (\partial_y - ix) \\ i\partial_x - (\partial_y - ix) & 0 \end{pmatrix}, \quad \not{D}_A^2 = \underbrace{\begin{pmatrix} H_{\mathrm{Lan}} - 1 & 0 \\ 0 & H_{\mathrm{Lan}} + 1 \end{pmatrix}}_{\geq 0}.$$

SUSY: $H_{\mathrm{Lan}} - 1$ and $H_{\mathrm{Lan}} + 1$ have same non-zero spectrum (gaps).

$H_{\mathrm{Lan}} - 1 \geq 0 \Leftrightarrow H_{\mathrm{Lan}} + 1 \geq 2 \Rightarrow (0, 2)$ is a gap of $H_{\mathrm{Lan}} \pm 1$.
$\Rightarrow (1, 3)$ is gap of H_{Lan}, thus $(2, 4)$ is gap of $H_{\mathrm{Lan}} \pm 1$
$\Rightarrow (3, 5)$ is gap of $H_{\mathrm{Lan}} \ldots$



Lowest **L**andau **L**evel is the kernel of $\not{D}_A^+$.

Dirac index

Normally, a Dirac operator on *compact* X has purely discrete spectrum. Its index is a *number*

$$\dim\ker(\not{D})^+ - \dim\operatorname{coker}(\not{D}^-) \in \mathbb{Z}.$$

Atiyah–Singer calculates this in terms of top. invariants of X.

For *non-compact* manifolds, Dirac has *continuous* spectrum, but mathematicians Roe–Higson show in '90s, how to define a *coarse index*, living in $K_*(C^*_{\mathrm{Roe}}(X))$.

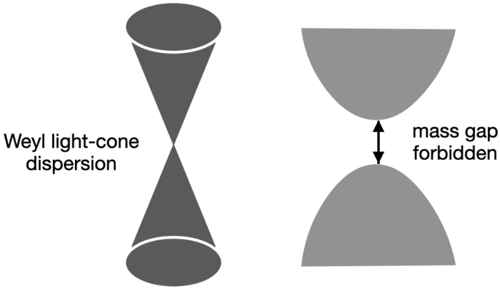
$C^*_{\mathrm{Roe}}(X) \ \sim$ "regularised noncommutative momentum space".

Coarse Dirac index

Quite abstract: can't isolate Dirac kernel and count it...?

Even dim: $\mathrm{Ind}(\not{D}_A)$ counts "zero-mode bundle over P", *after coupling to gauge field A to open gaps around* 0.

Odd-dim: $\mathrm{Ind}(\not{D})$ measures obstruction to opening spectral gaps.
$\sim$ chiral Dirac operators on $\mathbb{R}^{2n+1}$ are massless!



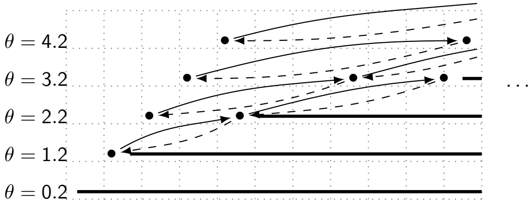
SUSY/Lichnerowicz

Spin and Riemannian geometry are related,

Dirac^2=Laplacian + curvature terms.

In 2D, $\mathfrak{spin}(2) \cong \mathfrak{u}(1)$. Riemann scalar and EM curvature interplay.

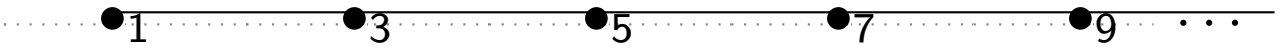
$$\text{Hyperbolic plane}: \not{D}^2_{\theta-\frac{1}{2}} = \begin{pmatrix} H_\theta - \theta & 0 \\ 0 & H_{\theta-1} + \theta - 1 \end{pmatrix} \geq 0$$



Hyperbolic Landau levels and ladder operators $\not{D}^{\pm}_{\theta-\frac{1}{2}}$.

Half-plane Landau spectrum?

Euclidean half-plane: Dirichlet spectrum of H_{Lan} equals[2] $[1, \infty)$.



- Other boundary geometries/conditions?
- Gap-filling in **hyperbolic** geometry?

$\theta = 3.2$

Exact spectral calculation is neither *possible* nor *required*.

Real question: "Is the boundary necessarily gapless/anomalous?"

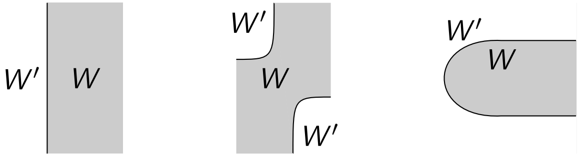
(Coarse) index theory efficiently answers YES to the above!

[2] De Bièvre, Pulé '02

Half-space Landau Hamiltonians are gapless [L+T:2009.07688]

One shows that Landau spectral projections cannot remain projections when operating on a *generic* half-space

Obstruction is due to Dirac kernel's coarse index localising onto the boundary ⇒ Edge spectra interpolate Landau levels!



Note: A randomly concocted "topological invariant" has no reason to localise to generally boundary, nor to have any spectral meaning!

Remark: TKNN Chern number does localise. . . to perfectly straight boundary (original K-theory proof of BBC).

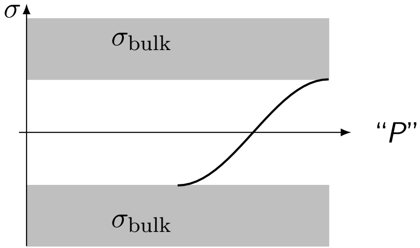
Spectral interpolation / spectral flow

"Boundary of Dirac is Dirac", or "dimensional reduction" is a useful heuristic, the gap-filling needs extra ingredients.

- Need to couple to A to open "non-trivial gaps" in spectrum (break T-symmetry!)
- Need Laplace-type Hamiltonian, bounded from below, to deduce the spectral gaps *above* topologically non-trivial spectra get filled.

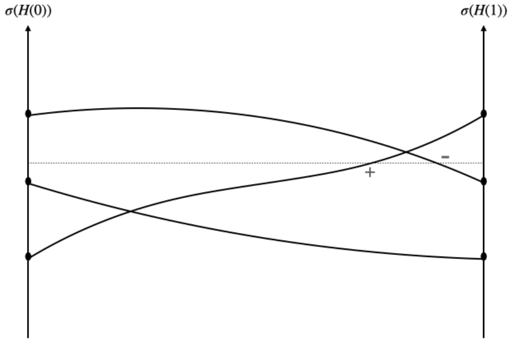
Existence of boundary states is nothing special.

But unbroken interpolation of bulk states *is* special!



Trivial spectral flow in finite dimensions

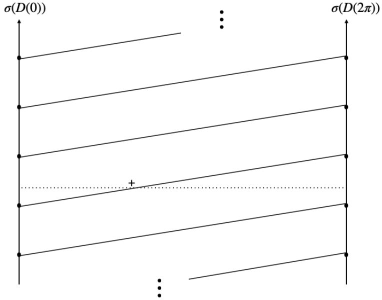
Let $H(t), t \in [0,1]$ be a loop of self-adjoint operators on finite-dimensional Hilbert space. Plot the eigenvalues:



"What goes up must come down" $\longrightarrow$ trivial spectral flow.

$\infty - \infty \in \mathbb{Z}$: spectral flow

Let $D = -i\frac{d}{d\theta}$ act on circle $S^1 = [0,1]/_{0\sim 1}$.
Then $\sigma(D) = 2\pi\mathbb{Z}$ with eigenfunctions $\psi_n(\theta) = e^{2\pi i n\theta}$

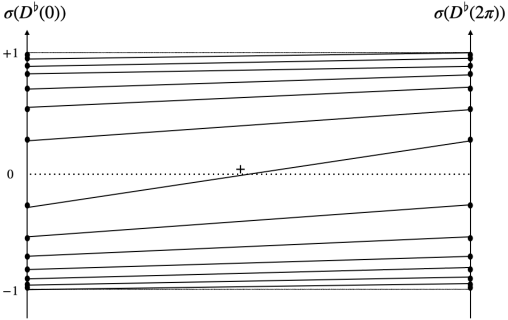


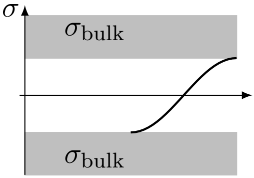
For $e^{ik} \in \mathrm{U}(1)$, twist $D(k) := D + k \rightsquigarrow \sigma(D(k)) = 2\pi\mathbb{Z} + k$.

Spectral flow possible due to "infinite Dirac sea". Partial Fourier-decomposition of unbroken spectrum of $-i\frac{d}{dx}$.

Spectral flow

Convert unbounded $D(k)$ to bounded $D^\flat(k) := \frac{D(k)}{\sqrt{1+D^2(k)}}$.





The flattened family $\{D^\flat(k)\}_{e^{ik}\in\mathrm{U}(1)}$ is a continuous loop of *bounded* self-adjoint *Fredholm* operators.

In solid-state physics, the infinite Dirac sea comes from "continuous spectral bands", *not* discrete spectrum $\to -\infty$!

Topology of self-adjoint Fredholm operators

Bounded Fredholm operator $F \Leftrightarrow$ 0-spectrum at most finite-multiplicity.

Fredholm index $\dim \ker(F) - \dim \ker(F^*)$ labels connected components of Fredholms.

Self-adjoint Fredholms are in the zero-index component, but they contain non-contractible loops!

Atiyah–Singer '69, showed that $\mathcal{F}_*^{\mathrm{sa}}$ has

$$\pi_n(\mathcal{F}_*^{\mathrm{sa}}) \cong \mathbb{Z}, \qquad n \text{ odd}.$$

$\pi_1(\mathcal{F}_*^{\mathrm{sa}}) \cong \mathbb{Z}$ is *topologically-protected spectral flow*, and can be computed by index theory.

"Higher" spectral flow

Q: What does $\pi_3(\mathcal{F}^{\mathrm{sa}}_*) \cong \mathbb{Z}$ mean (spectrally)?

An answer can be found by thinking about Weyl semimetals (chiral anomaly). Mathematically, one works directly with *unbounded* self-adjoint Fredholm operators $\mathcal{CF}^{\mathrm{sa}}$, equipped with *gap-topology*.

In 2000s, BLP established spectral flow for loops in $\mathcal{CF}^{\mathrm{sa}}$, and Joachim showed $\pi_{\mathrm{odd}}(\mathcal{CF}^{\mathrm{sa}}) \cong \mathbb{Z}$.

In fact, $\pi_1(\mathcal{CF}^{\mathrm{sa}}), \pi_3(\mathcal{CF}^{\mathrm{sa}}) \cong \mathbb{Z}$ are generated by "physically fundamental" examples!

Examples from *topological phases* typically have *noncompact manifolds-with-boundary*[3], having essential spectra.

[3] At least in "thermodynamic limit".

Dirac operators on half-line

Dirac/momentum operator $i\frac{d}{dz}$ on $[0,\infty)$ cannot be self-adjoint.

Reason: it is "unidirectional": no "reflection-at-boundary".

But $\not{D} = -i\frac{d}{dz} \oplus i\frac{d}{dz}$ can be made self-adjoint, by reflecting left-movers to right-movers.

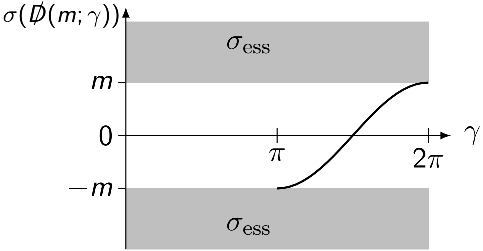
With mass term: $\not{D}(m;\gamma) = \begin{pmatrix} -i\frac{d}{dz} & me^{i\gamma} \\ me^{-i\gamma} & i\frac{d}{dz} \end{pmatrix}, \quad (m,\gamma) \in \widehat{\mathbb{R}}^2.$

For $m > 0$, essential (i.e. bulk) spectrum is

$$\sigma_{\mathrm{ess}}(\not{D}(m;\gamma)) = (-\infty,-m] \cup [m,\infty) \quad \rightsquigarrow \text{Fredholm!}$$

Loop of half-line Diracs generating $\pi_1(\mathcal{CF}^{\mathrm{sa}}) \cong \mathbb{Z}$

Additionally, $\not{D}(m;\gamma)$ has an eigenfunction $z \mapsto \begin{pmatrix}1\\1\end{pmatrix} e^{(m\sin\gamma)z}$ with eigenvalue $m\cos\gamma$. *Normalisable* only when $\gamma \in (\pi, 2\pi)$.



Conclusion: Massive half-line Dirac Hamiltonians are (continuously) parametrised by $(m,\gamma) \in \widehat{\mathbb{R}}^2 \setminus 0$. Each loop around origin has spectral flow $+1$: one zero mode contributor.

Physics: Weyl semimetals and Fermi arcs

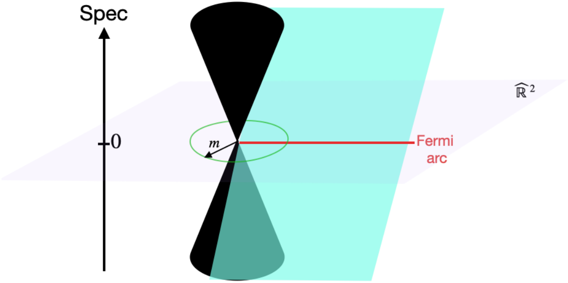
In 3D, L/R-handed Weyl Hamiltonian is $H^{\mathrm{Weyl}} = \mp i\nabla \cdot \boldsymbol{\sigma}$.
Fourier: energy-momentum dispersion $\sigma(H^{\mathrm{Weyl}})(\boldsymbol{p}) = \pm|\boldsymbol{p}|$.

Band crossing $\boldsymbol{p} = 0$ is a "U(1)" monopole for the bundle of negative eigenstates. There is a Dirac string to the anti-monopole at infinite momenum.

On a half-space $z \geq 0$, only p_x, p_y are conserved.

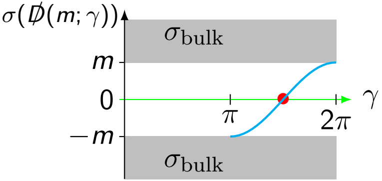
H^{Weyl} decomposes into family of half-line Diracs $\not{D}(m; \gamma)$, parametrised by $(m, \gamma) \leftrightarrow (p_x, p_y) \in \widehat{\mathbb{R}}^2$!

Weyl semimetals and Fermi arcs



Each radius m loop is basically the earlier spectral flow picture.

There is one zero-eigenvalue for each loop. Overall, there is a Fermi arc of zero-eigenvalues connecting origin to infinity.



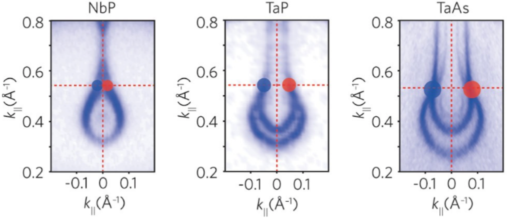
Topological Fermi arcs [T:2007.06193]

H_L^{Weyl} and H_R^{Weyl} occur together in "Weyl semimetals".
$\rightsquigarrow$ Fermi arc connecting their cone tips.

Fermi arc is holographic projection of bulk Dirac string (M+T '17).

In reality, $H_{L/R}^{\mathrm{Weyl}}$ is perturbed in some unknowable way, so Fermi arc is fuzzy and deformed,... *but never killed!*

Actual experimental Fermi arcs have exactly this feature!



Liu et al, Nature Materials (2016)

Fermi gerbe of Weyl semimetal [C+T: 2009.02064]

In 5D, the Weyl Hamiltonian is *TP*-invariant, thus quaternionic-linear in momentum space.

On half-space $z \geq 0$, H^{Weyl} decomposes into *quaternionic* half-line Dirac operators, parametrised by mass terms in $\mathbb{H} \cong \widehat{\mathbb{R}}^4$ rather than $\mathbb{C} \cong \widehat{\mathbb{R}}^2$:

$$\not{D}(m;\Gamma) = \begin{pmatrix} -i\frac{d}{dz} & m\Gamma \\ m\overline{\Gamma} & i\frac{d}{dz} \end{pmatrix}, \qquad \Gamma \in \mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong S^3.$$

We prove that the family $S^3 \ni \Gamma \mapsto \not{D}(m;\Gamma) \in \mathcal{CF}^{\mathrm{sa}}$ is topologically non-trivial!

Sketch: Construct a "Fermi gerbe" encoding how eigenvalues interpolate across the mass gap $(-m, m)$, compute non-trivial *Dixmier–Douady invariant*.

Hamiltonian anomaly and gerbes

Diracs on odd-dimensional *compact* manifolds, parametrised by gauge-equivalence classes of connections, is anomalous.

Roughly: Separation of spectra into $+$ and $-$ parts cannot be done globally, so Fock vacuum is ambiguous[4].

Our $\{\not{D}(m;\Gamma)\}_{\Gamma\in\mathrm{Sp}(1)}$ is similarly anomalous.

- For each $\lambda \in (-m, m)$, the region of $\mathrm{Sp}(1)$ without λ-eigenvalues does admit a sensible vacuum.
- For $-m < \lambda < \mu < +m$, transitioning between vacua $\lambda \to \mu$ involves determinant line bundle of eigenstates with energy within (λ, μ).
- Gerbe data comprises these "transition line bundles".

[4] Faddeev–Mickelsson–Carey–Murray (Segal)

Fermi gerbe of Weyl semimetal [C+T: 2009.02064]

We show that the DD-invariant of the Fermi gerbe for $\{\not{D}(m;\Gamma)\}_{\Gamma\in\mathrm{Sp}(1)}$, is the generator of $H^3(\mathrm{Sp}(1),\mathbb{Z})$.

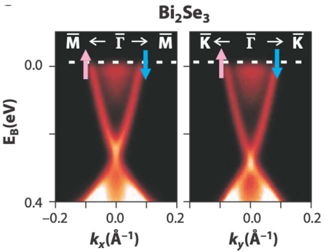
This means it represents $\pi_3(\mathcal{CF}^{\mathrm{sa}})\cong\mathbb{Z}\rightsquigarrow$ "higher" spectral flow.

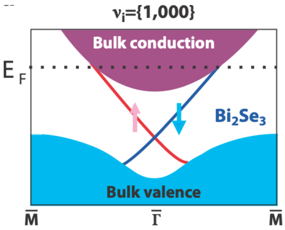
$\Rightarrow$ 5D half-space Weyl semimetal, and also "4D QHE" have topologically protected *Fermi surface* of boundary states.

Experimentally: In 3D, there exist T-invariant *topological insulators*, detected by gapless surface Dirac cones.

$\widetilde{KQ}^0(S^3,\tau_\pi)\cong\mathbb{Z}_2$.   Kane-Mele/Furuta-Kametani-Matsue-Minami

"Real" Fermi gerbe and topological insulators

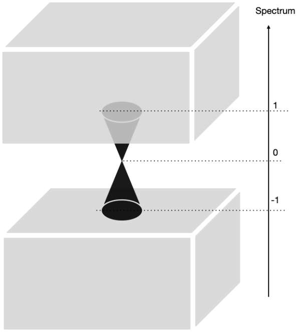




Discovery of Dirac cone edges state — Xia et al, '09

Direct proof of *topological* Dirac cones (with K. Gomi):
"Real" Fermi gerbe has DD-invariant in

$$\tilde{H}^3(S^2, \tau_\pi; \mathbb{Z}(1)) \cong \mathbb{Z}_2.$$



Stringy B-field interpretation?