Karnaugh Maps

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September 5, 2013

Outline

Motivation

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  - the axiom $a + a' = 1$ and occasionally
  - the property $a + a = a$.
- One of the fundamental algorithms in the field of CAD, Karnaugh maps are used for many small design problems and constitute the starting point for many other algorithms.
- The ideea is to visualise adjacency in Boolean space by using a projection of an n-dimensional hypercube onto two-dimensional rectangle such that adjacent points in the hypercube remain adjacent in the projection.

Grid Layout

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  - For example, $wx'y'z'$ and $wx'yz'$ are adjacent (notice y and y').
- Since a Boolean expression has a canonical form, we draw a *grid* or table such that *all possible* standard products have a unique position or box in the grid.
  - For example, Boolean expression $f(x, y, z)$ may be associated with the grid

	xy	xy'	$x'y'$	$x'y$
z				
z'				

in which the terms on the edges of the grid serve as labels.

Grid Layout

- More specifically, a special case of Boolean expression $xy + y'z$, for instance, has the canonical form

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  which can be represented by
  - placing a "1" in the corresponding box in the above grid for each standard product term in the canonical form,
  - leaving the rest of the boxes (if any) empty or filled with "0".
- The finished grid here, called a **Karnaugh map**, reads

	xy	xy'	$x'y'$	$x'y$
z	1	1	1	
z'	1			

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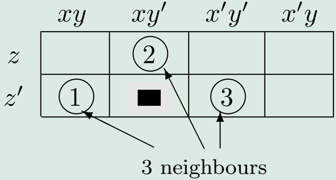
  - each standard product term is uniquely represented by a box,
  - adjacent boxes (vertically or horizontally or in other *legal* directions) represent adjacent product terms,
  - each box has exactly n neighbours (adjacent product terms) if the Boolean expression has exactly n variables.

Example

1. For Boolean expressions with variables x, y and z,



and

	$y'x'$	$y'x$	yx	yx'
z'				
z				

are both valid grids, and there are other possible valid grids as well.

Example

2. For a Boolean expression with 4 variables w, x, y and z, a typical valid grid would be

	wx	wx'	$w'x'$	$w'x$
yz				
yz'		②		
$y'z'$	①	■	③	
$y'z$		④		

where ①, ②, ③, and ④ are the 4 neighbours of the shaded box.

Example

3. For a Boolean expression with 5 variables v, w, x, y and z, the following grid

	vwx	vwx'	$vw'x'$	$vw'x$	$v'w'x$	$v'w'x'$	$v'wx'$	$v'wx$
yz			④					
yz'								
$y'z'$			②					
$y'z$		①	■	③		⑤		

is a typical grid and shaded box has 5 neighbours ①, ②, ..., ⑤.

Block

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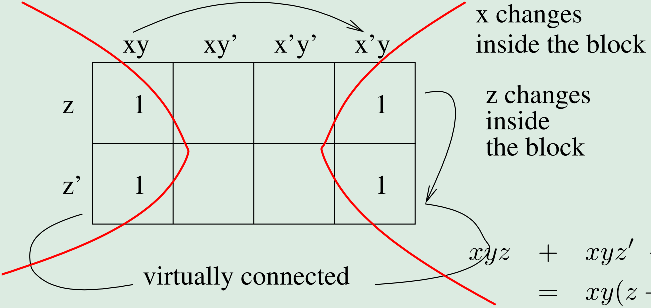
- In fact, we'll reserve the word **block** exclusively for this sense.
  - A block of 2^M boxes in a Boolean expression of N variables can be represented by a single product term with $N - M$ literals.
  - If a variable changes when we move inside a block, the literals related to the variable will not exist in the reduced single product term.

Example

4. The circled area in the Karnaugh map is a 2×2 block representing

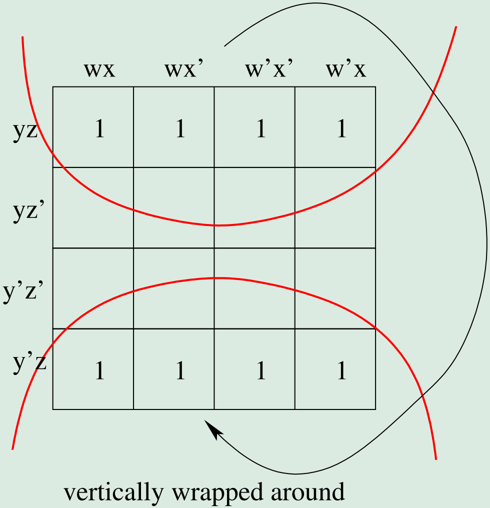


$$\begin{aligned} xyz \quad + \quad xyz' \; + \; x'yz \; + \; x'yz' \\ = \quad xy(z+z') + x'y(z+z') \\ = \quad xy + x'y = (x+x')y = y \end{aligned}$$

In other words, the block of 4 is simplified to a single product term y (with $1 = N - M = 3 - 2$ literals)

Example

5. The circled area in the Karnaugh map



is a 2×4 block representing z. Notice that in a typical term w' x' y' z in the block, the variable names w, x and y may change inside the circled block, and are thus not present in the final product term which becomes z.

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(v) Circle any blocks of 4 (containing at least one uncircled 1) such that the newly circled 1's can't be contained in a block of 8. Use as few circles as possible.

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- Then the simplified expression can be read off the Karnaugh map with each circle representing one product term.
- The term that changes, from x to x' say, within a circle is the one that "disappears".
- The final result is a minimal representation of the Boolean expression.

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Solution.

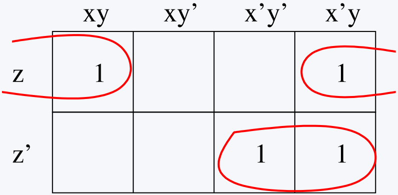
(a) $F = xyz + x'yz + x'yz' + x'y'z'$ is the canonical form.

(b)

	xy	xy'	$x'y'$	$x'y$
z	1			1
z'			1	1

(c) No isolated 1's .

(d) No blocks of 4. But there are pairing choices. We select *least* number of pairs.



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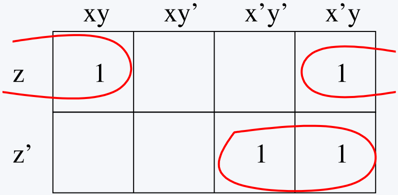

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In the circle in the first row x changes to x' so it "disappears", leaving yz.

In the other circle, y changes, so it goes leaving $x'z'$.

Thus the minimal representation can be read off as $yz + x'z'$.

Example

7. Suppose a Boolean expression can be represented by the following Karnaugh map

	wx	wx'	$w'x'$	$w'x$
yz	1	1	1	1
yz'	1	1		
$y'z'$		1	1	1
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yz	1	1	1	1
yz'	1	1		
$y'z'$		1	1	1
$y'z$			1	1

Solution. Then the simplification procedure is as follows.

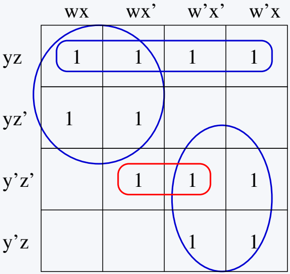
(a) No isolated 1's.

(b) Only 2 possible pairs exist, that can't be included in a block of 4 (1's). Since a 2nd pair will not cover any uncircled 1's that will not be covered by blocks of 4, we need to pick just 1 pair.

(c) No blocks of 8. But there are 3 blocks of 4, each covering new 1's. The Karnaugh map now looks like

	wx	wx'	w'x'	w'x
yz	1	1	1	1
yz'	1	1		
y'z'		1	1	1
y'z			1	1

(d) Since no uncircled 1's are left over the procedure is completed.



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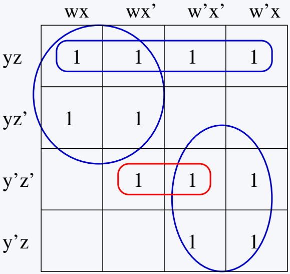

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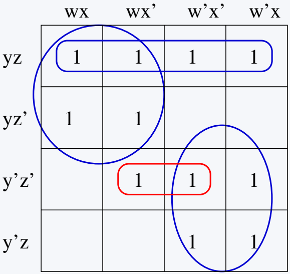

- In the circle in the first row both w and x change so they "disappear", leaving yz.
- In the circle in the top left corner, x and z change so they go, leaving wy.

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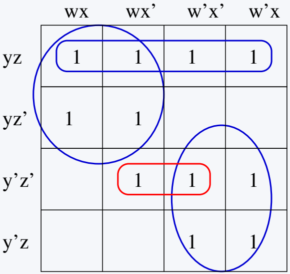

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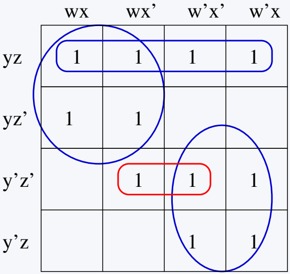

- In the circle in the first row both w and x change so they "disappear", leaving yz.
- In the circle in the top left corner, x and z change so they go, leaving wy.
- And so on.
- The minimal representation is thus read off as $wy + yz + w'y' + x'y'z'$

Example

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For instance, with bigger block first, we would have to circle in the Karnaugh map below in the following order

	wx	wx'	w'x'	w'x
yz			1	
yz'	1	1	1	
y'z'		1	1	1
y'z		1		

then

	wx	wx'	w'x'	w'x
yz			1	
yz'	1	1	1	
y'z'		1	1	1
y'z		1		

which gives 5 product terms (corresponding to 5 circles).
However, if we drop the block of 4 circle, we still have all the 1's circled.
But the new and equivalent expression will only have 4 product terms.

With our normal procedure, nevertheless, we'll arrive at the correct answer represented by